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Duals of Banach spaces which admit nontrivial smooth functions

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If a Banach space X admits a continuously Fréchet differentiable function with bounded nonempty support, then X^* admits a projectional resolution of identity and a continuous linear one-to-one map into $c_0(\Gamma)$.

1. Introduction

There are two difficulties in building up the projectional resolution of identity in nonseparable Banach spaces; such a resolution was originally constructed by Amir and Lindenstrauss ([*1*]) for spaces which are generated by a weakly compact set. First we need a compactness argument to ensure the existence of limit points for certain nets of operators and second we need to be able to ensure that the limit point is a projection. The first one can be overcome in any dual space. Tacon showed in ([*4*]) that also the second difficulty can be overcome in duals of spaces with Fréchet smooth norm. His argument relies on the uniqueness of Hahn-Banach extensions. Here we show that the projectional resolution of identity in X^* exists under the hypotheses in the abstract. This is done by basing the proof on the existence of differentials of certain functions constructed by Leduc ([*2*], [*3*]).

2. Notations and definitions

We will work in real Banach spaces. The norm $|\cdot|$ of a Banach space X is rotund if whenever $|x+y| = 2$, $|x| = |y| = 1$, then $x = y$. If X

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is a Banach space, then, following [4], X^{α} is the Banach space of all bounded homogeneous functionals on X with the sup-norm over the unit ball of X. If C is a subspace of X and $T : C^* \to X^*$ is a bounded linear map, then $\tilde{T} : X^* \to X^*$ is defined as $\tilde{T}f = TRf$, where R means the restriction-map to C^*. dens$X$ is the smallest cardinality of a dense subset of a Banach space X. The symbol cl$M$ denotes the norm closure of M in X.

3. Main result

THEOREM 1. *Let X be a Banach space which admits a continuously Fréchet differentiable function with bounded nonempty support. Let μ be the first ordinal of cardinality* dens$X$. *Then for every $0 \le \alpha \le \mu$ there is a subspace X_α of X and a linear operator $T : X^*_\alpha \to X^*$ such that $P = \tilde{T}_\alpha$ is a linear projection with $X_\alpha \subset X_\beta$ if $\alpha < \beta$ and $X_\mu = X$, and*

1. $|P_\alpha| = 1$ *for* $\alpha > 0$, $P_0 = 0$,
2. $P_\alpha X^*$ *is linearly isometric to* X^*_α, densX_α $\left(= \text{dens}X^*_\alpha\right) \le \overline{\overline{\alpha}}$ *for infinite* α,
3. $P_\alpha P_\beta = P_\beta P_\alpha = P_\beta$, *where* $\beta < \alpha$,
4. $\bigcup_{\beta<\gamma} P_{\beta+1}X^*$ *is norm dense in* $P_\gamma X^*$, *or equivalently*
5. *for every* $x^* \in X^*$, $P_\alpha x^*$ *is norm continuous on ordinals.*

COROLLARY. *If a Banach space X admits a continuously Fréchet differentiable function with bounded nonempty support, then X^* admits a bounded linear one-to-one map into $c_0(\Gamma)$. Thus X^* has an equivalent rotund norm.*

4. Proof of the main result

We need the following result of Amir and Lindenstrauss.

LEMMA 1 (see [1], Lemma 2). *Assume X is a normed linear space. Then given $\varepsilon > 0$, an integer $n > 0$, m elements $f_1, \ldots, f_m$ of X^* and any finite dimensional subspace $B \subset X$, there is an $\aleph_0$-dimensional*

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subspace $C \subset X$ *containing* B *such that, for every subspace* Z *of* X *with* $Z \supset B$ *and* $\dim Z/B = n$ *, there is a linear operator* $T : Z \to C$ *with* $|T| \le 1 + \varepsilon$ *,* $Tb = b$ *for every* $b \in B$ *and* $|f_k(z) - f_k(Tz)| \le \varepsilon|z|$ *for every* $z \in Z$ *and* $k = 1, 2, \ldots, m$ *.*

Also we need the following result of Leduc.

LEMMA 2 (see [2], Theorem 3 and [3], Corollary 1). *If* f *is a continuously Fréchet differentiable real valued function on a Banach space* X *with bounded nonempty support (we may assume* $f(0) > 0$ *and* $0 \le f \le 1$*), then the gauge of* f *defined by the formula*

$$\nu(x) = \left\{\int_{-\infty}^{+\infty} f(tx)dt\right\}^{-1} , \quad x \ne 0 ,$$

is continuously Fréchet differentiable, $\nu'(x) \ne 0$ *and*

$$\mathrm{cl}\left\{\nu'(x)\cdot|\nu'(x)|^{-1},\ |x| = 1\right\} = \{f \in X^*,\ |f| = 1\} .$$

LEMMA 3. *Let* X *be a Banach space,* B *a finite dimensional subspace of* X *,* $f_1, \ldots, f_m \in X^*$ *. Then there is a separable subspace* C *of* X *and a linear operator* $T : C^* \to X^*$ *such that* $|T| = 1$ *and* $\tilde{T}^*x = x$ *for all* $x \in B$ *,* $\tilde{T}f_i = f_i$ *,* $i = 1, 2, \ldots, m$ *.*

Proof. Let $C_n \supset B$, $n = 1, 2, \ldots$, be the $\aleph_0$-dimensional subspaces of X given by Lemma 1 for $\varepsilon = 1/n$, and let $C = \overline{\mathrm{sp}}\left(\bigcup_n C_n\right)$.

If E is a subspace of X , $E \supset B$, $\dim E/B = n$, then there is a linear operator $T_E : E \to C$ such that $|T_E| \le 1 + 1/n$, $T_E x = x$ for $x \in B$, $|f_k(T_E\ ) - f_k(z)| \le \varepsilon|z|$, $z \in E$, $k = 1, 2, \ldots, m$. We extend T_E to a homogeneous map $T'_E : X \to C$ by $T'_E x = 0$ if $x \in X\backslash E$. We consider $T'^* : C^* \to X^\alpha$ where in the space of bounded linear maps $C^* \to X^\alpha$ we consider the pointwise topology and on X^α the X-topology. By the Tychonoff Theorem, the net T'_E has a limit point $T : C^* \to X^*$ and if $x \in X$, then

$$\left(\tilde{T}f_j\right)(x) = (TR)f_j(x) =$$

$$= \lim\left(T'_E Rf_j\right)(x) = \lim\left(T^*_E Rf_j\right)(x) = \lim\left(Rf_j\right)T_E x = f_j(x) .$$

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Similarly $\tilde{T}^*x = x$ for $x \in B$.

LEMMA 4. *Let* X *be a Banach space,* f *a continuously Fréchet differentiable function on* X *with bounded support such that* $0 \le f \le 1$ *and* $f(0) > 0$. *Let* ν *be the gauge of* f *defined in Lemma 2,* $\aleph$ *an infinite cardinal number. Assume* Z, W *are subspaces of* X, X^* *respectively,* $\mathrm{dens} Z, \mathrm{dens} W \le \aleph$. *Then there is a subspace* $C \subset X$, $\mathrm{dens} C \le \aleph$, $C \supset Z$, *and a linear operator* $T : C^* \to X^*$ *with* $|T| = 1$, $TRg = g$ *for* $g \in W$, $TRd = d$ *for all differentials* d *of* ν *at all points of* $C \backslash \{0\}$ *and* $(TR)^*x = x$ *for* $x \in C$ *and such that*

$$TC^* = \mathrm{cl}\{\lambda d, \ \lambda \ge 0, \ d \ \text{differentials of } \nu \text{ at all points of } C\backslash\{0\}\} .$$

Then $P = TR$ *is a projection on* X^* , $|P| = 1$ *such that* $Pg = g$ *for* $g \in W$, $P^*x = x$ *for* $x \in C$. *Furthermore,* $R : PX^* \to C^*$ *is an isometry onto* C^* .

Proof. By transfinite induction on $\aleph$. If $\aleph = \aleph_0$ and x_j, f_j , $j = 1, 2, \ldots$, are dense in Z, W respectively, then there exist, by Lemma 3, separable subspaces $C_n \subset X$, $n = 1, 2, \ldots$, and linear operators $T_n : C_n^* \to X^*$ with $|T_n| = 1$, $\tilde{T}_n^* x_i = x_i$, $i = 1, 2, \ldots, n$, and $\tilde{T}_n^* x_i^k = x_i^k$, $1 \le i \le n$, $1 \le k \le n-1$, where $0 \ne x_i^k$, $i = 1, 2, \ldots$, is dense in C_k , $T_n f_i = f_i$, $i = 1, 2, \ldots, n$, $T_n d = d$ for all differentials d of ν at x_i^k , $1 \le i \le n$, $1 \le k \le n-1$. Let us put $C = \mathrm{cl} \bigcup_n C_n$. If R_n is the restriction map of C^* to C_n^* , then the limit point T in the X-operator topology of the net $\{T_n R_n\}_n$ is seen by the arguments used in Lemma 3 to satisfy that if $P = \tilde{T}$, then $|P| = 1$, P is linear, $P^* x_i^k = x_i^k$ for $i, k = 1, 2, \ldots$, so that $P^*x = x$ for all $x \in C$ and similarly $Pf = f$ for all $f \in W$, $Pd = d$ for all differentials d of ν at all $x \in C$, $x \ne 0$. Here we use the continuous Fréchet differentiability of ν on $X\backslash\{0\}$. It remains to prove that P is a projection; that is, $P^2 = P$. To show this it clearly suffices to prove that

$$PX = \mathrm{cl}\{\lambda d, \ d \ \text{differential of } \nu \text{ at a nonzero point of } C, \ \lambda \ge 0\} \equiv D .$$

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If $d \in D$, then for some sequence $\lambda_i \geq 0$, d_i differentials of ν at $C\setminus\{0\}$, $\lim \lambda_i d_i = d$. Then $Pd = P(\lim \lambda_i d_i) = \lim \lambda_i d_i = d$, so $D \subset PX^*$. If $x^* = Tc^*$, $c^* \in C^*$, then Lemma 2 used for C gives the existence of differentials d_i of ν at the points of $C\setminus\{0\}$ and $\lambda_i \geq 0$ such that $\lim \lambda_i Rd_i = c^*$, where Rd_i is the restriction of d_i to C. So, $Tc^* = T(\lim \lambda_i Rd_i) = \lim TR(\lambda_i d_i) = \lim \lambda_i d_i$, showing that $PX^* \subset D$.

Now we show that the restriction $R : PX^* \to C^*$ is an isometry onto. For if $c^* \in C$, $|c^*| = 1$, $\varepsilon > 0$, then there is a $c \in C$, $|c| = 1$, such that $|c^*(c)-1| < \varepsilon$. So if $x^* \in X^*$, $c^* = Rx^*$, then $Px^* = Tc^*$ and $(Px^*)(c) = c^*(P^*c) = c^*(c)$. From the last fact and from $|P| = 1$ easily follows that R is an isometry. Furthermore $RPX^* \supset RD$, so R is onto C^* by use of Lemma 2. If the lemma holds for all cardinals less than $\aleph$ and μ is the first ordinal of $\overline{\overline{\mu}} = \aleph$, then obviously there are subspaces $Z_\alpha \subset Z$, $W_\alpha \subset W$, $\alpha < \mu$ such that $Z_\alpha \subset Z_\beta$, $W_\alpha \subset W_\beta$ if $\alpha < \beta$ with $\operatorname{dens} Z_\alpha, \operatorname{dens} W_\alpha \leq \overline{\overline{\alpha}}$ and $Z = \operatorname{cl} \bigcup_{\alpha<\mu} Z_\alpha$, $W = \operatorname{cl} \bigcup_{\alpha<\mu} W_\alpha$. By the induction hypothesis, we construct for every $\alpha < \mu$, a subspace $C_\alpha \subset X$ with $\operatorname{dens} C_\alpha \leq \overline{\overline{\alpha}}$ and such that $C_\alpha \supset Z_\alpha \cup \bigcup_{\beta<\alpha} C_\beta$ together with a linear operator $T_\alpha : C_\alpha^* \to X^*$ such that $P = \tilde{T}_\alpha$ satisfies $|P_\alpha| = 1$, $P_\alpha^* x = x$ for $x \in C_\alpha$, $P_\alpha f = f$ for $f \in W_\alpha$,

$$P_\alpha X^* = \operatorname{cl}\{\lambda d,\ \lambda \geq 0,\ d \text{ differentials of } \nu \text{ at the points of } C_\alpha\setminus\{0\}\}\ .$$

We put $C = \operatorname{cl} \bigcup_{\alpha<\mu} C_\alpha$ and consider the extensions of T_α, $\tilde{T}_\alpha : C^* \to X^*$. Again for T we take a limit point in the X-operator topology of $\tilde{T}_\alpha$, $\alpha < \mu$ and see that

$$TX^* = \{\lambda d,\ d \text{ differentials of } \nu \text{ at nonzero points of } C,\ \lambda \geq 0\}$$

and $P = \tilde{T}$ satisfies our requirements.

Proof of Theorem. From Lemma 5 and the arguments developed in [1], [4], the theorem follows.

Proof of Corollary. It is the same as the proof of Theorem 1 and its corollary in [4].

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References

[1] D. Amir and J. Lindenstrauss, "The structure of weakly compact sets in Banach spaces", *Ann. of Math.* (2) **88** (1968), 35-46.

[2] Michel Leduc, "Jauges différentiables et partitions de l'unité", *Séminaire Choquet, Initiation à l'analyse*, 4e année, 1964/65, 12-01 - 12-10 (Mathematica Seminosa, 1. Secrétariat Mathématique, Paris, 1965).

[3] Michel Leduc, "Densité de certaines familles d'hyperplans tangents", *C.R. Acad. Sci. Paris Sér. A* **270** (1970), 326-328.

[4] D.C. Tacon, "The conjugate of a smooth Banach space", *Bull. Austral. Math. Soc.* **2** (1970), 415-425.

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