BULL. AUSTRAL. MATH. SOC. 26A54, 20M20
VOL. 18 (1978), 37-43.

Differentiable retracts and a modified inverse function theorem

W. Barit and G.R. Wood

A lemma is presented which is a weak version of the inverse function theorem, in that differentiability is assumed instead of continuous differentiability. The result holds only for finite dimensional spaces; a counter-example is given for the infinite dimensional analogue. The lemma is used to answer a question posed by Nadler concerning differentiable retracts.

Terminology

Denote n-dimensional euclidean space by $\mathbb{R}^n$ and the Hilbert space of square summable real sequences by l_2. The origin is denoted $\bar{0}$ and $\| \|$ is the usual norm. A Fréchet differentiable function, f, is one that has the appropriate linear approximation at each point a, denoted $Df(a)$. The class of functions satisfying the stronger condition of being n times continuously differentiable is represented by C^n. A map r which is the identity on its image, or equivalently, with $r \circ r = r$, is called a retraction.

Introduction

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be C^1. An intuitive version of the inverse function theorem is: if $Df(a) = \text{id}$, then f "behaves" like the identity near a. More precisely, there are open neighbourhoods U of a and V of $f(a)$ where $f|_U$ is a homeomorphism from U onto V. That is, $f|_U$

Received 13 October 1977. The authors wish to thank Dr Sadayuki Yamamuro for his helpful comments and references.

satisfies these four conditions:

(i) one-to-one,

(ii) onto V ,

(iii) continuous, and

(iv) open.

It is easy to see that the C^1 hypothesis for f is necessary [2, p. 269]. The lemma presented here shows what modifications of properties (i) to (iv) are needed when the hypothesis is only that f be differentiable. The proof of the lemma uses some algebraic topology valid only in finite dimensions. The analogue of the lemma for infinite dimensions is shown to be false. The lemma is used to prove that a differentiable retraction, $r : \mathbb{R}^n \to \mathbb{R}^n$, which is different from the identity, has a nowhere dense image.

LEMMA. *Let* $f : \mathbb{R}^n \to \mathbb{R}^n$ *be continuous, and let* f *be differentiable at* a *with* $Df(a) = \mathrm{id}$. *There exist open neighbourhoods* U *of* a *and* V *of* $f(a)$ *such that* $f|_U$ *is:*

(i) weakly one-to-one at a *(that is, for any* $b \in U$, $b \neq a \Rightarrow f(b) \neq f(a)$ *);*

(ii) onto V *;*

(iii) continuous;

(iv) weakly open at a *(that is, for each open set* $W \subset U$, $a \in W \Rightarrow f(a) \in$ *interior of* $f(W)$ *).*

In this lemma global properties *(i)* and *(iv)* for the inverse function theorem become "localized" at a .

Proof. Since f may be composed with a translation without affecting the relevant properties, we may assume $f(a) = a$. An exercise in Dieudonné [2, p. 269] shows *(i)*, and *(iii)* follows from the hypothesis. Differentiability of f at a means

$$\frac{\|f(p)-f(a)-Df(a)(p-a)\|}{\|p-a\|} \to 0 \quad \text{as} \quad \|p-a\| \to 0 \, .$$

Since $Df(a) = \mathrm{id}$, the following inequality can be verified.

Fix a positive $\varepsilon < 1$. Then there exists $\delta > 0$ such that for each t, $0 \leq t \leq 1$,

$$1 - \varepsilon \leq \frac{\|tf(p)+(1-t)p-f(a)\|}{\|p-a\|} \leq 1 + \varepsilon ,$$

whenever $\|p-a\| < \delta$.

This inequality has a clear geometric meaning. Let $r < \delta$. Consider S_r the sphere of radius r about a and $A_{r,\varepsilon}$ the annulus obtained by thickening S_r by $\pm\varepsilon r$ (Figure 1). Let $F_t = tf|_{S_r} + (1-t)\mathrm{id}$. This is a linear homotopy from $f|_{S_r}$ to the identity on S_r. The inequality guarantees that the image of S_r under F_t is always inside $A_{r,\varepsilon}$.

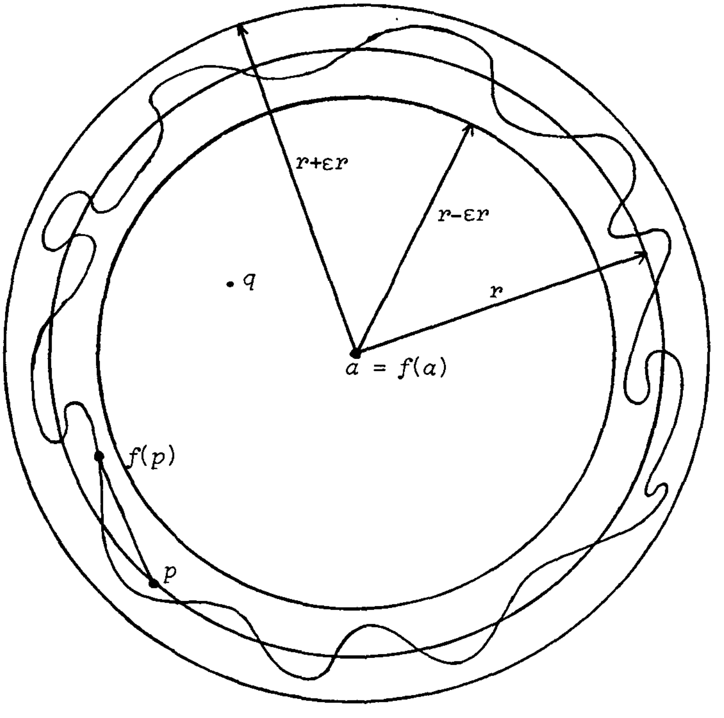

FIGURE 1

The existence of open U and V with properties *(ii)* and *(iv)* will now be established. Let B_r and $B_{r-\varepsilon r}$ be the balls about a and $f(a)$

https://doi.org/10.1017/S0004972700007784 Published online by Cambridge University Press

with radii indicated by subscript (Figure 1). It is necessary to show that the image of B_r under f contains $B_{r-\varepsilon r}$. Suppose the contrary. Let q be a point of $B_{r-r\varepsilon}$ not in $f(B_r)$, and let π denote the projection of $R^n \backslash q$ onto S_r (Figure 1). Note that π is the identity of S_r, and $A_{r,\varepsilon}$ is in the domain of π. Now $\pi \circ F_t$ is a homotopy of $\pi \circ f|_{S_r}$ to the identity of S_r.

Diagrammatically we have $S_r \xrightarrow{i} B_r \xrightarrow{f} f(B_r) \xrightarrow{\pi} S_r$ being homotopic to the identity. Taking $n-1$ homology we have that $Z \to 0 \to H_{n-1}(f(B_r)) \to Z$ must be the identity. This is a contradiction, so $(f(B_r))$ contains $B_{r-\varepsilon r}$.

Take for V the open ball of radius $(r-\varepsilon r)/2$ about $f(a)$. The above result shows the image of the open r-ball contains V. Let U be the intersection of $f^{-1}(V)$ with the open r-ball. The above results also confirm *(iv)* for small open balls about a, and hence *(iv)* holds for all open W in U.

The algebraic technique of this argument is used in a different setting by Munkres [4, pp. 36-37]. Granas [3, pp. 77-78] proves a similar lemma in the more general context of compact displacements in Banach spaces. His argument uses essential mappings of spheres. A straightforward inequality about f deduced from $Df(a) = \text{id}$ together with Granas' lemma provide an alternative proof of the lemma here.

Infinite dimensions

The corresponding statement for l_2 is false (the same holds for most infinite dimensional spaces where differentiability can be defined). This is shown by constructing an example of a differentiable function $f : l_2 \to l_2$ with $f(\overline{0}) = \overline{0}$, $Df(\overline{0}) = \text{id}$, yet $\overline{0}$ is not in the interior of $f(l_2)$. Perhaps this should not seem strange when one notes that the algebraic results concerning finite dimensional spheres do not generalize. In infinite dimensions the unit sphere is a retract of the unit ball [4].

Consider the following scheme for constructing a map f with $Df(\overline{0}) = \text{id}$ (this works in finite dimensions as well). Let B_i be a sequence of disjoint closed balls whose centres c_i approach $\overline{0}$ and whose radii, r_i, satisfy $\frac{r_i}{\|c_i\|} \to 0$, and $r_i < \frac{1}{2}\|c_i\|$. Observe that any function f which is the identity outside the B_i's and has $f(B_i) \subset B_i$, is differentiable at $\overline{0}$ with $Df(\overline{0}) = \text{id}$. Of course it will generally not be C^1. To construct the counter example in l_2 we use the following remarkable result.

THEOREM. *There exists a C^∞ diffeomorphism of l_2 onto $l_2 \backslash \overline{0}$ which is the identity outside the unit ball* [1].

By using affine adjustment we can get the required f which is the identity outside all the B_i and is onto $l_2 \backslash \{c_1, c_2, c_3, \ldots\}$ (that is, pushed off the centre of each B_i). This is C^∞ except at $\overline{0}$ where it is only differentiable with $Df(\overline{0}) = \text{id}$.

Differentiable retracts

Nadler [5] explores C^1 retracts on Banach spaces. He proves that if such a retract is different from the identity, then its image is nowhere dense. Yamamuro [6] proves a similar result using the inverse function theorem. Nadler remarked that for retracts of $\mathbb{R}$ the assumption of differentiability is all that is needed and asked if this was so for $\mathbb{R}^n$. The lemma is used to provide an affirmative answer.

THEOREM. *Let $r : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable retract. If $r \neq \text{id}$, then $r(\mathbb{R}^n)$ is nowhere dense.*

Proof. To show $r(\mathbb{R}^n)$ is closed, let y be a limit point, and let $r(x_i)$ be a sequence approaching it. It follows that $r \circ r(x_i)$ approaches $r(y)$, so $y = r(y)$ and $y \in r(\mathbb{R}^n)$. To show the interior of $r(\mathbb{R}^n)$ is empty use contradiction. Suppose $x \in \text{int}\, r(\mathbb{R}^n)$ (Figure 2).

https://doi.org/10.1017/S0004972700007784 Published online by Cambridge University Press

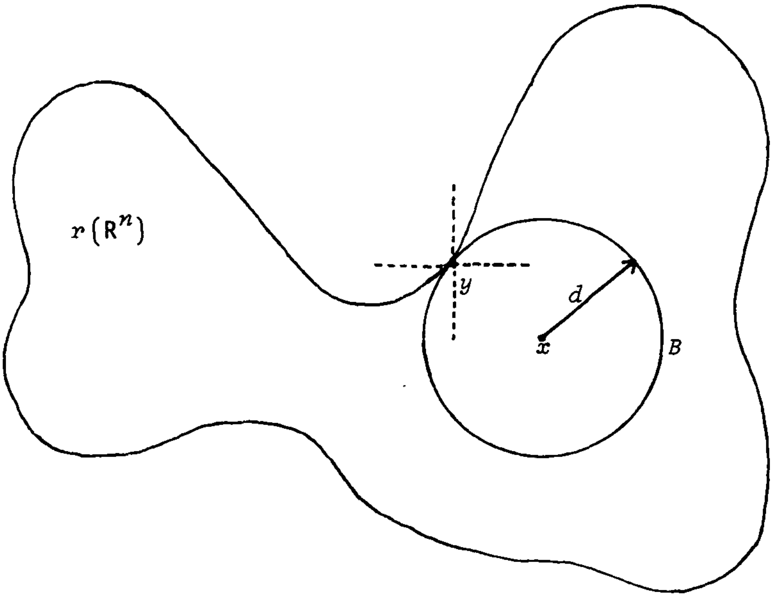

FIGURE 2

Consider open balls about x which are contained in $r(\mathrm{R}^n)$. Let d be the supremum of their radii. Let B be the open ball of radius d about x . B is inside int $r(\mathrm{R}^n)$, but its boundary, a sphere of radius d about x , must meet $r(\mathrm{R}^n)\backslash\mathrm{int}\, r(\mathrm{R}^n)$. Here the compactness of the unit sphere is used. Let y be such a meeting point. Since r is the identity inside B , there are n mutually perpendicular directions to approach y within B , and as r is differentiable at y , it can be deduced that $Dr(y) = \mathrm{id}$. The lemma implies that y must be interior to $r(\mathrm{R}^n)$. This cannot be, so $\mathrm{int}(\mathrm{R}^n) = \emptyset$.

Open questions

The two major facts in the above proof, the lemma and compactness of the sphere, do not hold in infinite dimensional Banach spaces. Granas' result most likely could extend the situation a little for these, but the question about differentiable retracts in such spaces remains open.

https://doi.org/10.1017/S0004972700007784 Published online by Cambridge University Press

References

[1] Dan Burghelea and Nicolaas H. Kuiper, "Hilbert manifolds", *Ann. of Math.* (2) 90 (1969), 379-417.

[2] J. Dieudonné, *Foundations of modern analysis* (Pure and Applied Mathematics, 10. Academic Press, New York and London, 1960).

[3] Andrzej Granas, *Introduction to topology of functional spaces* (Lecture Notes, 501. University of Chicago, Chicago, 1960).

[4] James R. Munkres, *Elementary differential topology*, revised edition (Annals of Mathematics Studies, 54. Princeton University Press, Princeton, New Jersey, 1966).

[5] Sam B. Nadler, Jr., "Differentiable retractions in Banach spaces", *Tôhoku Math. J.* 19 (1967), 400-405.

[6] S. Yamamuro, *Differential calculus in topological linear spaces* (Lecture Notes in Mathematics, 374. Springer-Verlag, Berlin, Heidelberg, New York, 1974).

Department of Mathematics,
University of Canterbury,
Christchurch,
New Zealand.