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PROGRESS ON THE AUSLANDER–REITEN CONJECTURE

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Abstract

Let R be a commutative Gorenstein ring. A result of Araya reduces the Auslander–Reiten conjecture on the vanishing of self-extensions to the case where R has Krull dimension at most one. In this paper we extend Araya's result to certain R-algebras. As a consequence of our argument, we obtain examples of bound quiver algebras that satisfy the Auslander–Reiten conjecture.

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1. Introduction

The *Auslander–Reiten conjecture* [7] says that, over an Artin algebra Λ, if M is a finitely generated Λ-module such that $\mathrm{Ext}^i_\Lambda(M, M \oplus \Lambda) = 0$ for all $i > 0$, then M is projective. This long-standing and celebrated conjecture in representation theory holds for several classes of algebras, including algebras of finite representation type [7] and symmetric Artin algebras with radical cube zero [15]. The Auslander–Reiten conjecture is rooted in Nakayama's 1958 conjecture on rings of infinite dominant dimension [19] and the following conjecture known as the *generalised Nakayama conjecture* [7]: for an Artin algebra Λ, every indecomposable injective Λ-module appears as a direct summand in the minimal injective resolution of Λ. The Auslander–Reiten conjecture is also called the *Gorenstein projective conjecture*, if the module M (in the formulation of the conjecture) is assumed in addition to be Gorenstein projective over Λ (see [18]). It is proved in [21] that the Gorenstein projective conjecture holds for algebras of finite Cohen–Macaulay type; however, the validity of the Auslander–Reiten conjecture for algebras of finite Cohen–Macaulay type remains unknown. Recall that an Artin algebra Λ is said to be of finite Cohen–Macaulay type provided that there are only finitely many isomorphism classes of indecomposable finitely generated Gorenstein projective Λ-modules. The Auslander–Reiten conjecture actually makes sense for any Noetherian ring. In fact, there are already some results in the study of the Auslander–Reiten conjecture in the setting of commutative algebra (see, for example, [10, 12, 16, 20]). In particular, Auslander *et al.* in [6] studied the following condition for commutative Noetherian rings which are not necessarily Artin algebras.

CONJECTURE 1.1 (ARC). *Let R be a commutative Noetherian ring and M a finitely generated R-module. If $\mathrm{Ext}_R^i(M, M \oplus R) = 0$ for all $i > 0$, then M is projective.*

Commutative Noetherian local rings known to satisfy (ARC) include Gorenstein local rings of codimension at most four [20] and excellent Cohen–Macaulay normal domains containing the rational numbers [16]. The main result of Araya indicates that the validity of (ARC) for the class of commutative Gorenstein rings depends on its validity for such rings of dimension at most one (see [1, Theorem 3]). This result has been extended in [9] to left Gorenstein R-algebras Λ, whenever R is a commutative Gorenstein ring. The aim of this paper is to extend Araya's result to certain R-algebras and well-behaved modules. Moreover, our result is effective in the sense that we can specify how many Ext functors must vanish to give the conclusion of (ARC). Precisely, our main result in this paper is as follows.

THEOREM 1.2. *Let R be a commutative Gorenstein local ring with $\dim R = d \geq 2$ and let Λ be an R-algebra which is a finitely generated free R-module. Suppose that every Λ-module which is projective as an R-module has finite projective dimension over Λ. Assume that M is a finitely generated Gorenstein projective Λ-module which is locally projective on the punctured spectrum of R. If $\mathrm{Ext}_\Lambda^{d-1}(M, M) = 0 = \mathrm{Ext}_\Lambda^d(M, M)$, then M is a projective Λ-module.*

We recall that a Λ-module M is locally projective on the punctured spectrum of R, provided $M_\mathfrak{p}$ is projective over $\Lambda_\mathfrak{p}$, for all nonmaximal prime ideals $\mathfrak{p}$ of R.

The above theorem leads to the following result, which extends Araya's main result to certain finitely generated free algebras.

COROLLARY 1.3. *Let R be a commutative Gorenstein ring and let Λ be an R-algebra which is a finitely generated free R-module. Suppose that every Λ-module which is projective as an R-module has finite projective dimension over Λ. Assume that M is a finitely generated Gorenstein projective Λ-module such that $\mathrm{Ext}_\Lambda^i(M, M) = 0$ for all $i > 0$. If $M_\mathfrak{p}$ is a projective $\Lambda_\mathfrak{p}$-module, for all prime ideals $\mathfrak{p}$ of height at most one, then M is projective over Λ.*

The above results enable us to provide examples of bound quiver algebras satisfying (ARC) (see Corollary 3.3).

For these results, without loss of generality, we may assume further that R is a local ring. So, throughout the paper, R denotes a d-dimensional commutative Noetherian local ring and Λ is an R-algebra which is finitely generated and free as an R-module. Unless otherwise specified, all modules are assumed to be finitely generated left modules.

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2. Preliminaries

2.1. Gorenstein modules. A Λ-module M is said to be Gorenstein projective if it is a syzygy of some exact sequence of projective Λ-modules

$$\mathbf{T}_\bullet : \cdots \longrightarrow T_2 \longrightarrow T_1 \longrightarrow T_0 \longrightarrow T_{-1} \longrightarrow \cdots ,$$

which remains exact after applying the functor $\mathrm{Hom}_\Lambda(-, P)$ for any projective Λ-module P. The exact sequence $\mathbf{T}_\bullet$ is called a totally acyclic complex of projectives. We refer the reader to [14] for the basic properties of these modules.

Gorenstein projective modules, which are a refinement of projective modules, were defined by Enochs and Jenda in [13]. This concept even goes back to Auslander and Bridger [5], who introduced the Gorenstein dimension of a finitely generated module M over a two-sided Noetherian ring. Then Avramov, Martisinkovsky and Reiten proved that M is Gorenstein projective if and only if the Gorenstein dimension of M is zero (see also the remark following Theorem 4.2.6 in [11]).

2.2. Syzygies. Let M be a Λ-module. For an integer $n > 0$, we denote by $\Omega^n_\Lambda M$ the nth syzygy of M. If N is a Gorenstein projective Λ-module, then, for a given Λ-module M and nonnegative integers n and i with $i < n$, one has the isomorphism

$$\mathrm{Ext}^n_\Lambda(N, \Omega^i M) \cong \mathrm{Ext}^{n-i}_\Lambda(N, M). \tag{2.1}$$

Remark 2.1. Let R be a Gorenstein ring and M an arbitrary Gorenstein projective Λ-module. Since the R-algebra Λ is a finitely generated free R-module, M is Gorenstein projective over R as well. Moreover, the natural R-isomorphism $M \cong M^{**}$ is also an isomorphism of Λ-modules, where $(-)^* = \mathrm{Hom}_R(-, R)$. For a given Λ-module X which is projective over R, its R-dual, X^*, is a Λ^{op}-module which is also a projective R-module (see [4, Lemma 1.1]).

Lemma 2.2. *Let R be a Gorenstein ring and let M be a Gorenstein projective Λ-module which is locally projective on the punctured spectrum of R. Then* $\mathrm{Tor}^\Lambda_i(\Lambda^*, M) = 0$ *for all $i > 0$.*

Proof. Take a free resolution of the Λ-module M, $\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$. Let $\mathbf{F}_\bullet : \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$ and apply the functor $\Lambda^* \otimes_\Lambda -$ to this sequence, giving the complex

$$\Lambda^* \otimes_\Lambda \mathbf{F}_\bullet : \cdots \longrightarrow \Lambda^* \otimes_\Lambda F_1 \longrightarrow \Lambda^* \otimes_\Lambda F_0 \longrightarrow 0.$$

In order to obtain the desired result, we only need to show that the homology of this complex at $\Lambda^* \otimes_\Lambda F_i$ is zero for all $i > 0$.

Assume on the contrary that there is an integer $t \geq 1$ for which $\mathrm{H}_t(\Lambda^* \otimes_\Lambda \mathbf{F}_\bullet) \neq 0$. Suppose that $0 \longrightarrow R \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^d \longrightarrow 0$ is a minimal injective resolution of R. Since R is a commutative Gorenstein local ring, the last term E^d is isomorphic to $E(R/\mathfrak{m})$, the injective envelope of the residue field $R/\mathfrak{m}$, as an R-module and $E(R/\mathfrak{m})$ does not appear in E^j for each $j < d$. Thus, for any finite-length R-module N, $\mathrm{Hom}_R(N, E^i) = 0$ if and only if $i < d$ (see [3, Proposition 7.1(c)]

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and also the proof of [17, Lemma 12.15]). Because $\Lambda^* \otimes_\Lambda \mathbf{F}_\bullet$ is a complex of projective R-modules, applying the functor $\mathrm{Hom}_R(\Lambda^* \otimes_\Lambda \mathbf{F}_\bullet, -)$ to the minimal injective resolution of R gives the following exact sequence of complexes:

$$0 \to \mathrm{Hom}_R(\Lambda^* \otimes_\Lambda \mathbf{F}_\bullet, R) \to \mathrm{Hom}_R(\Lambda^* \otimes_\Lambda \mathbf{F}_\bullet, E^0) \to \cdots \to \mathrm{Hom}_R(\Lambda^* \otimes_\Lambda \mathbf{F}_\bullet, E^d) \to 0.$$

Since M is locally projective on the punctured spectrum of R, for any $i > 0$, the R-module $\mathrm{H}_i(\Lambda^* \otimes_\Lambda \mathbf{F}_\bullet)$ is of finite length. Consequently, for each $0 \leq j < d$,

$$\mathrm{H}^i(\mathrm{Hom}_R(\Lambda^* \otimes_\Lambda \mathbf{F}_\bullet, E^j)) \cong \mathrm{Hom}_R(\mathrm{H}_i(\Lambda^* \otimes_\Lambda \mathbf{F}_\bullet), E^j) = 0. \tag{2.2}$$

Putting, for simplicity, $\mathcal{L}_{j+1} = \mathrm{Im}(\mathrm{Hom}_R(\Lambda^* \otimes_\Lambda \mathbf{F}_\bullet, E^j) \to \mathrm{Hom}_R(\Lambda^* \otimes_\Lambda \mathbf{F}_\bullet, E^{j+1}))$ and $\mathcal{L}_0 = \mathrm{Hom}_R(\Lambda^* \otimes_\Lambda \mathbf{F}_\bullet, R)$ gives the short exact sequence of R-modules:

$$0 \longrightarrow \mathcal{L}_j \longrightarrow \mathrm{Hom}_R(\Lambda^* \otimes_\Lambda \mathbf{F}_\bullet, E^j) \longrightarrow \mathcal{L}_{j+1} \longrightarrow 0,$$

for each $0 \leq j < d$. Taking homologies gives the following long exact sequence:

$$\begin{aligned}\cdots \to \mathrm{H}^i(\mathrm{Hom}_R(\Lambda^* \otimes_\Lambda \mathbf{F}_\bullet, E^j)) \to \mathrm{H}^i(\mathcal{L}_{j+1}) \to \mathrm{H}^{i+1}(\mathcal{L}_j)&\\ \to \mathrm{H}^{i+1}(\mathrm{Hom}_R(\Lambda^* \otimes_\Lambda \mathbf{F}_\bullet, E^j)) \to \cdots.&\end{aligned}$$

So, by virtue of (2.2), $\mathrm{H}^i(\mathcal{L}_{j+1}) \cong \mathrm{H}^{i+1}(\mathcal{L}_j)$ for all integers i. Using this fact repeatedly gives the isomorphisms

$$\begin{aligned}\mathrm{H}^t(\mathrm{Hom}_R(\Lambda^* \otimes_\Lambda \mathbf{F}_\bullet, E^d)) = \mathrm{H}^t(\mathcal{L}_d) \cong \mathrm{H}^{t+1}(\mathcal{L}_{d-1}) \cong \cdots \cong \mathrm{H}^{t+d}(\mathcal{L}_0)&\\ = \mathrm{H}^{t+d}(\mathrm{Hom}_R(\Lambda^* \otimes_\Lambda \mathbf{F}_\bullet, R)).&\end{aligned}$$

By our assumption, $\mathrm{Hom}_R(\mathrm{H}_t(\Lambda^* \otimes_\Lambda \mathbf{F}_\bullet), E^d)$ and so also $\mathrm{H}^t(\mathrm{Hom}_R(\Lambda^* \otimes_\Lambda \mathbf{F}_\bullet, E^d))$ are nonzero. Hence, the preceding isomorphisms yield $\mathrm{H}^{t+d}(\mathrm{Hom}_R(\Lambda^* \otimes_\Lambda \mathbf{F}_\bullet, R)) \neq 0$. Now consider the isomorphisms

$$\mathrm{H}^{t+d}(\mathrm{Hom}_R(\Lambda^* \otimes_\Lambda \mathbf{F}_\bullet, R)) \cong \mathrm{H}^{d+t}(\mathrm{Hom}_\Lambda(\mathbf{F}_\bullet, \mathrm{Hom}_R(\Lambda^*, R))) \cong \mathrm{Ext}_\Lambda^{d+t}(M, \Lambda),$$

where the first isomorphism follows from the adjointness of Hom and $\otimes$ and the second comes from the Λ-isomorphism $\mathrm{Hom}_R(\Lambda^*, R) = \Lambda^{**} \cong \Lambda$. Since the left-hand side does not vanish, the same is true for the right-hand side, contradicting the fact that M is a Gorenstein projective Λ-module. This completes the proof. □

Corollary 2.3. *Let M be a Gorenstein projective Λ-module which is locally projective on the punctured spectrum of R. Then, for a given free Λ-module F, $F^* \otimes_\Lambda M$ is Gorenstein projective over R.*

Proof. Take a totally acyclic complex of projective Λ-modules $\mathbf{P}_\bullet$ in which M is its zeroth syzygy. It is evident that the property of being locally projective on the punctured spectrum M is inherited by any syzygy of $\mathbf{P}_\bullet$. So, Lemma 2.2 forces the complex of projective R-modules $F^* \otimes_\Lambda \mathbf{P}_\bullet$ to be exact. To obtain the desired result, one only needs to show that $F^* \otimes_\Lambda \mathbf{P}_\bullet$ is indeed a totally acyclic complex of projective R-modules. Since $F^{**} \cong F$ as Λ-modules, for any projective

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R-module Q, $\mathrm{Hom}_R(F^*, Q)$ is a projective Λ-module. This, in turn, implies that $\mathrm{Hom}_\Lambda(\mathbf{P}_\bullet, \mathrm{Hom}_R(F^*, Q))$ is exact, because $\mathbf{P}_\bullet$ is a totally acyclic complex of projective Λ-modules. Now the adjoint isomorphism

$$\mathrm{Hom}_R(F^* \otimes_\Lambda \mathbf{P}_\bullet, Q) \cong \mathrm{Hom}_\Lambda(\mathbf{P}_\bullet, \mathrm{Hom}_R(F^*, Q))$$

completes the proof. □

2.3. Exact sequences. Let M be a Gorenstein projective Λ-module. Since M is finitely generated and Λ is Noetherian, M admits a projective resolution of finitely generated Λ-modules. So, according to [8, Theorem 4.2(vi)], there exists a totally acyclic complex of (finitely generated) free Λ-modules

$$\mathbf{P}_\bullet : \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F_{-1} \longrightarrow F_{-2} \longrightarrow \cdots \tag{2.3}$$

such that $M = \mathrm{Ker}(F_{-1} \longrightarrow F_{-2})$.

In view of (2.3), there is the following exact sequence of Λ-modules:

$$0 \longrightarrow \Omega^2_\Lambda M \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F_{-1} \longrightarrow F_{-2} \longrightarrow \cdots,$$

where F_i, for each i, is a free Λ-module and $M = \mathrm{Ker}(F_{-1} \longrightarrow F_{-2})$. Since Λ is a free R-module, M is also Gorenstein projective over R and hence, applying the functor $(-)^\vee = \mathrm{Hom}_R(-, \Lambda)$ to the above exact sequence, induces the acyclic complex

$$\cdots \longrightarrow F_{-1}^\vee \longrightarrow F_0^\vee \longrightarrow F_1^\vee \longrightarrow (\Omega^2_\Lambda M)^\vee \longrightarrow 0.$$

Applying the functor $- \otimes_\Lambda M$ to this sequence gives the complex

$$\cdots \longrightarrow F_{-1}^\vee \otimes_\Lambda M \longrightarrow F_0^\vee \otimes_\Lambda M \longrightarrow F_1^\vee \otimes_\Lambda M \longrightarrow (\Omega^2_\Lambda M)^\vee \otimes_\Lambda M \longrightarrow 0.$$

The following result is crucial for the proof of Theorem 1.2.

Proposition 2.4. *Let M be a Gorenstein projective Λ-module which is locally projective on the punctured spectrum of R. Consider the complex*

$$\cdots \longrightarrow F_{-1}^\vee \otimes_\Lambda M \longrightarrow F_0^\vee \otimes_\Lambda M \longrightarrow F_1^\vee \otimes_\Lambda M \longrightarrow (\Omega^2_\Lambda M)^\vee \otimes_\Lambda M \longrightarrow 0.$$

Assume that every Λ-module which is projective as an R-module has finite projective dimension over Λ. If the homologies at $F_0^\vee \otimes_\Lambda M$ and $F_{-1}^\vee \otimes_\Lambda M$ are zero, then M is projective.

Proof. The hypothesis imposed on the homologies at $F_0^\vee \otimes_\Lambda M$ and $F_{-1}^\vee \otimes_\Lambda M$ gives the exact sequence

$$0 \longrightarrow M^\vee \otimes_\Lambda M \longrightarrow F_0^\vee \otimes_\Lambda M \longrightarrow F_1^\vee \otimes_\Lambda M \longrightarrow (\Omega^2_\Lambda M)^\vee \otimes_\Lambda M \longrightarrow 0.$$

Note that the assignment $\phi \otimes y \mapsto \phi(x)y$ makes a functorial homomorphism $\theta_{M,N} : M^\vee \otimes_\Lambda N \longrightarrow \mathrm{Hom}_R(M, N)$. Clearly, $\theta_{M,N}$ is an isomorphism if M or N is a free Λ-module. So, one has the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 \to & M^\vee \otimes_\Lambda M & \to & F_0^\vee \otimes_\Lambda M & \to & F_1^\vee \otimes_\Lambda M & \to & (\Omega^2_\Lambda M)^\vee \otimes_\Lambda M \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \\
0 \to & \mathrm{Hom}_R(M, M) & \to & \mathrm{Hom}_R(F_0, M) & \to & \mathrm{Hom}_R(F_1, M) & &
\end{array}$$

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in which the last two vertical maps are isomorphisms ensuring that the first induced vertical map is also an isomorphism. Next, take the following commutative diagram of R-modules:

$$\begin{array}{ccccccc}
\cdots \to & M^{\vee} \otimes_{\Lambda} F_1 & \to & M^{\vee} \otimes_{\Lambda} F_0 & \to & M^{\vee} \otimes_{\Lambda} M & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & \mathrm{Hom}_R(M, F_1) & \to & \mathrm{Hom}_R(M, F_0) & \to & \mathrm{Hom}_R(M, M). &
\end{array}$$

Since the vertical maps are isomorphisms, the map $\mathrm{Hom}_R(M, F_0) \longrightarrow \mathrm{Hom}_R(M, M)$ is an epimorphism, implying that M is a projective R-module. Hence, the hypothesis made on Λ together with Gorenstein projectivity of M yields that M is projective over Λ, as desired. □

3. Proofs

Proof of Theorem 1.2. Since M is a Gorenstein projective Λ-module, $\Omega^2_{\Lambda} M$ is Gorenstein projective as well. Assume that

$$\mathbf{F}_{\bullet} : \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F_{-1} \longrightarrow F_{-2} \longrightarrow \cdots$$

is a totally acyclic complex of (finitely generated) free Λ-modules in which $\Omega^2_{\Lambda} M$ is its zeroth syzygy. From Remark 2.1, $\Omega^2_{\Lambda} M$ is also Gorenstein projective over R, so $\mathbf{F}^*_{\bullet}$ is an acyclic complex of Λ^{op}-modules. By Corollary 2.3, $\mathbf{F}^*_{\bullet} \otimes_{\Lambda} M$ is a complex of Gorenstein projective R-modules. Hence, applying the functor $\mathrm{Hom}_R(\mathbf{F}^*_{\bullet} \otimes_{\Lambda} M, -)$ to a minimal injective resolution of R, $0 \longrightarrow R \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^d \longrightarrow 0$, gives the following exact sequence of complexes:

$$0 \to \mathrm{Hom}_R(\mathbf{F}_{\bullet}{}^* \otimes_{\Lambda} M, R) \to \mathrm{Hom}_R(\mathbf{F}_{\bullet}{}^* \otimes_{\Lambda} M, E^0) \to \cdots \to \mathrm{Hom}_R(\mathbf{F}_{\bullet}{}^* \otimes_{\Lambda} M, E^d) \to 0.$$

By the argument in the proof of Lemma 2.2, for any $i > 0$, there is an isomorphism $\mathrm{H}^i(\mathrm{Hom}_R(\mathbf{F}_{\bullet}{}^* \otimes_{\Lambda} M, E^d)) \cong \mathrm{H}^{i+d}(\mathrm{Hom}_R(\mathbf{F}_{\bullet}{}^* \otimes_{\Lambda} M, R))$. Therefore, for any integer $i > 0$,

$$\begin{aligned}
\mathrm{Hom}_R(\mathrm{H}_i(\mathbf{F}_{\bullet}{}^* \otimes_{\Lambda} M), E^d) &\cong \mathrm{H}^i(\mathrm{Hom}_R(\mathbf{F}_{\bullet}{}^* \otimes_{\Lambda} M, E^d)) \\
&\cong \mathrm{H}^{d+i}(\mathrm{Hom}_R(\mathbf{F}_{\bullet}{}^* \otimes_{\Lambda} M, R)) \\
&\cong \mathrm{H}^{d+i}(\mathrm{Hom}_{\Lambda}(M, \mathrm{Hom}_R(\mathbf{F}_{\bullet}{}^*, R))) \\
&\cong \mathrm{Ext}^{d+i}_{\Lambda}(M, (\Omega^2_{\Lambda} M)^{**}) \\
&\cong \mathrm{Ext}^{d+i}_{\Lambda}(M, (\Omega^2_{\Lambda} M)) \\
&\cong \mathrm{Ext}^{d+i-2}_{\Lambda}(M, M).
\end{aligned}$$

The first isomorphism holds trivially and the validity of the second isomorphism has been mentioned just above. The third isomorphism uses the adjointness of Hom and $\otimes$, whereas the fourth one follows from the fact that $\mathrm{Hom}_R(\mathbf{F}_{\bullet}{}^*, R)$ is isomorphic to $\mathbf{F}_{\bullet}$ as Λ-modules and M is Gorenstein projective over Λ. The fifth isomorphism again comes from the Λ-isomorphism $\Omega^2_{\Lambda} M \cong (\Omega^2_{\Lambda} M)^{**}$. Finally, the last isomorphism follows from (2.1).

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Now assume that M is not projective over Λ. By Proposition 2.4, $\mathrm{H}_i(\mathbf{F}_{\bullet}{}^{*} \otimes_{\Lambda} M)$ does not vanish for $i = 1$ or 2. This implies that $\mathrm{Hom}_R(\mathrm{H}_i(\mathbf{F}_{\bullet}{}^{*} \otimes_{\Lambda} M), E^d) \neq 0$ for $i = 1$ or 2. Consequently, from the above isomorphisms, $\mathrm{Ext}^i_{\Lambda}(M, M) \neq 0$ for $i = d - 1$ or d , which is a contradiction. This completes the proof of Theorem 1.2. □

The proof of Corollary 1.3 follows the same lines as the proof of [1, Theorem 3].

Proof of Corollary 1.3. We show that $M_{\mathfrak{p}}$ is projective over $\Lambda_{\mathfrak{p}}$ for all nonmaximal prime ideals $\mathfrak{p}$ of R. Assume on the contrary that the set of all prime ideals $\mathfrak{p}$ in which $M_{\mathfrak{p}}$ is not projective as a $\Lambda_{\mathfrak{p}}$-module is nonempty. Letting $\mathfrak{q}$ be a minimal object of such ideals and replacing R and Λ with $R_{\mathfrak{q}}$ and $\Lambda_{\mathfrak{q}}$, respectively, we may assume that R is a Gorenstein local ring with dimension $d \geq 2$ such that M is not a projective Λ-module and $M_{\mathfrak{p}}$ is a projective $\Lambda_{\mathfrak{p}}$-module for all nonmaximal prime ideals $\mathfrak{p}$ of R. Hence, by [1, Theorem 3], $\mathrm{Ext}^i_{\Lambda}(M, M) \neq 0$ for $i = d - 1$ or d, which is a contradiction. Therefore, M is locally projective on the punctured spectrum of R. Now one may apply Theorem 1.2 and conclude that M is projective over Λ. □

We say that Λ is an isolated singularity if, for any nonmaximal prime ideal $\mathfrak{p}$ of R, the left global dimension of $\Lambda_{\mathfrak{p}}$ is finite. Theorem 1.2 leads to the following result.

Corollary 3.1. *Let R be a Gorenstein ring with* $\dim R = d \geq 2$ *and let Λ be an isolated singularity. Assume that every Λ-module which is projective as an R-module has finite projective dimension over Λ. Then Λ satisfies the Gorenstein projective conjecture.*

Proof. Let M be a Gorenstein projective Λ-module such that $\mathrm{Ext}^i_{\Lambda}(M, M) = 0$ for all $i > 0$. Since Λ is an isolated singularity, M will be locally projective on the punctured spectrum of R. So, Theorem 1.2 forces M to be projective over Λ. □

Example 3.2. Let R be a commutative ring. Let Q be a finite acyclic quiver and let $\mathcal{I}$ be an admissible ideal of the path algebra RQ. It is known that the bound quiver algebra $RQ/\mathcal{I}$ is a finitely generated free module over R. By the argument in the proof of [2, Theorem 1.6], $(\mathrm{mod}\, R)Q/\mathcal{I}$ is equivalent to $\mathrm{rep}_R(Q, \mathcal{I})$, the category of representations of Q bound by $\mathcal{I}$ in $(\mathrm{mod}\, R)$. This fact, together with [9, Proposition 2.10], shows that any $RQ/\mathcal{I}$-module M which is projective over R has finite projective dimension.

Applying Theorem 1.2 to this example yields the following result.

Corollary 3.3. *Let R be a commutative local Gorenstein ring with* $\dim R = d \geq 2$ *and let Q and $\mathcal{I}$ be as in Example 3.2. Assume that M is a finitely generated Gorenstein projective $RQ/\mathcal{I}$-module which is locally projective on the punctured spectrum of R. If* $\mathrm{Ext}^{d-1}_{RQ/\mathcal{I}}(M, M) = 0 = \mathrm{Ext}^{d}_{RQ/\mathcal{I}}(M, M)$*, then M is a projective $RQ/\mathcal{I}$-module.*

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