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The Solution of the Biquadratic $x^4+px^3+qx^2+rx+s=0$.

§ 1. Descartes in his *Geometria* shows how the solution may be made to depend on the intersections of two conics C and C'; an account of this method is given in the Algebra of Maclaurin. Thus the roots of $x^4+qx^2+rx+s=0$ are the x-coordinates of the points common to the parabola $y=x^2$ and the circle $x^2+y^2+rx+(q-1)y+s=0$.

By using the discriminant of the pencil of conics $C+\lambda C'=0$, which is of the third degree in λ, and considering the corresponding line pair $L_1 L_2=0$, Professor Chrystal used to shew to his students how any root of the λ-resolvent would enable one to complete the solution by the use of quadratic irrationals, viz., by solving $C=0$ and $L_1=0$ or $L_2=0$. Thus, if the equation is

$$x^4+px^3+qx^2+rx+s=0, \quad \text{.......................}(1)$$

a suitable pencil of conics of which one is $y-x^2=0$ is

$$y^2+pxy+qy+rx+s+\lambda(y-x^2)=0. \quad \text{..............} (2)$$

Or if we put $y=x+\dfrac{1}{x}$ and write (1) as $x^2+px+q+\dfrac{r}{x}+\dfrac{s}{x^2}=0$, we obtain

$$x^2+px+q+r(y-x)+s(y-x)^2+\lambda(x^2-xy+1)=0. \quad \text{......}(3)$$

These may be readily extended to the cases

$$y=ax^2+bx+c \quad (\text{or } y-bx-c=ax^2)$$

and $$y=\frac{ax^2+bx+c}{x+d} \quad \left(\text{or } y-mx-n=\frac{A}{x+d}\right).$$

When one of the conics C of the pencil is given, a suitable pencil may be found, but the calculations may be troublesome. It is in general possible to find the different pencils corresponding to C as a member. Let the roots of (1) be x_1, x_2, x_3, x_4. Let the lines $x=x_1$, $x=x_2$, $x=x_3$, $x=x_4$ cut C in P_1, P_2; Q_1, Q_2; R_1, R_2;

S_1, S_2. There are 16 different sets of four points $PQRS$ forming the base points of such a pencil. If, however, the members of the pencil are to have real coefficients, *i.e.* if C' is to have real coefficients, certain limitations are to be observed.

§2. We note first that since the equation in λ is of the third degree, at least one line pair of the pencil is real for such a pencil. Again, no line $ax+by+c=0$ can have one coordinate of a point on it real and the other complex (save when a or $b=0$). Finally, the line joining $(\alpha+\beta i, \gamma+\delta i)$, $(\alpha-\beta i, \gamma-\delta i)$ is real. It is the only real line through either point.

Case (i). Roots of (1) real and all points $PQRS$ real. All the pencils are real.

Case (ii). Let $x=x_1$ be any real root of (1) cutting C in two imaginary points. There is no real line pair and no real pencil, if x_1, x_2, x_3, x_4 are distinct.

Case (iii). Two roots of (1) real x_1 and x_2 cutting C in real points, but $x_3=\alpha+\beta i$; $x_4=\alpha-\beta i$. Let $x=x_3$ and $C=0$ give $y_3=\gamma+\delta i$, and $y_3'=\lambda+\mu i$, then $x_4=\alpha-\beta i$ and $C=0$ furnish $y_4=\gamma-\delta i$, and $y_4'=\lambda-\mu i$. There are therefore in general four real pencils, at least.

Case (iv). If all the roots of (1) are imaginary, then C may be any conic whatsoever with real coefficients (and need not have real points on it). There are two sets of two pairs of points as in (iii) and four real pencils, at least.

Cases of contact and of points at infinity also arise.

The choice of a pencil in which one member is a circle and another a parabola has always been of special interest, and is always possible. Any real pencil $C+\lambda C'=0$ contains two parabolas which are both real if the four base points are concyclic. Hence to get a solution, let α and β be upper and lower limits of the real roots of (1), and take any circle cutting $x=\alpha$ and $x=\beta$ in real points. Of course, when all the roots of (1) are imaginary, any circle may be taken.

§3. The connection with the algebraic theory of resolvents is easily traced.

Thus, when the conic C' is given by $y=x^2$, a line pair is given by

$$y-x_1^2-\frac{x_2^2-x_1^2}{x_2-x_1}(x-x_1)=0$$

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or $$y - x_1^2 - (x_2 + x_1)(x - x_1) = 0,$$
and $$y - x_3^2 - (x_4 + x_3)(x - x_3) = 0.$$

By taking this product and comparing with

$$y^2 + pxy + qy + rx + s + \lambda(x^2 - y) = 0,$$

we find $$\lambda = (x_4 + x_3)(x_2 + x_1), \qquad (4)$$

or if we take the pencil as

$$y^2 + pxy + qx^2 + rx + s + \lambda(y - x^2) = 0,$$

we find $$q - \lambda = (x_1 + x_2)(x_3 + x_4)$$

and $$\lambda = x_1 x_2 + x_3 x_4. \qquad (5)$$

When C' is taken to be $y = x + \frac{1}{x}$ and the pencil as

$$x^2 + px + q + r(y - x) + s(y - x)^2 + \lambda(x^2 - xy + 1) = 0,$$

we find $$\left(1 - \frac{1}{x_1 x_2}\right)\left(1 - \frac{1}{x_3 x_4}\right) = \frac{1 + s + \lambda}{s}$$

or $$\frac{1}{x_1 x_2} + \frac{1}{x_3 x_4} = \frac{-\lambda}{s},$$

so that $$-\lambda = x_1 x_2 + x_3 x_4. \qquad (6)$$

More generally, let (1) be found by eliminating y between

$$C = (a\,b\,c\,f\,g\,h)(x\,y\,1)^2 = 0$$

and $$C' = (a'b'c'f'g'h')(x\,y\,1)^2 = 0,$$

the four base points being $(x_1, y_1) \dots (x_4, y_4)$.

By solving $C = 0$ and $C' = 0$ as linear equations in y and y^2, we obtain y as a rational function of x, which by (1) can be transformed into an integral cubic function of x, say $\phi(x)$.

A pair of lines is given by

$$y - y_1 = \frac{\phi(x_2) - \phi(x_1)}{x_2 - x_1}(x - x_1)$$

$$y - y_3 = \frac{\phi(x_4) - \phi(x_3)}{x_4 - x_3}(x - x_3).$$

Hence $$\frac{a + \lambda a'}{b + \lambda b'} = \frac{\phi(x_2) - \phi(x_1)}{x_2 - x_1} \times \frac{\phi(x_4) - \phi(x_3)}{x_4 - x_3}$$

$$= \psi(x_1, x_2) \times \psi(x_3, x_4), \qquad (7)$$

where ψ is a quadratic symmetric function in its variables.

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