BULL. AUSTRAL. MATH. SOC. MOS 3040, 3406

VOL. 1 (1969), 183-194.

Analytic iterations on Riemann surfaces

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A complex analytic family of mappings $P \to M(a,P)$ from an abstract Riemann surface (analytic manifold) into itself is studied. The mapping $M(a,P)$ is assumed to satisfy in local coordinates the autonomous differential equation $\frac{dw}{da} = L(w)$, and the condition $M(0,P) = P$. Under certain assumptions of regularity of the reciprocal differential L in a domain $D \subset S$, we prove that for every fixed a , $|a| < \alpha_0$, the mapping $M(a,P)$ is conformal and one to one in D . Moreover, it is shown that the family of mappings $M(a,P)$ satisfies the iteration equation $M[a,M(b,P)] = M(a+b,P)$ and hence is an analytic group (analytic iteration).

1. Let $w(a,z)$ be analytic and regular in a and z for $|a| < a_0$ $(a_0 > 0)$ and $z \in D$, where D is a given domain in the complex plane. If $w(a,z)$ satisfies the iteration equation

(1) $$w[a,w(b,z)] = w(a+b,z)$$

and the initial condition

(2) $$w(0,z) = z$$

for $z \in D$ and $|a|$, $|b|$, $|a+b| < a_0$, then $w(a,z)$ is called an *analytic*

Received 21 March 1969. Received by J. Austral. Math. Soc. 18 September 1967. Revised 28 October 1968. Communicated by E. Strzelecki. Research sponsored by Army Research Office Grant No. DA-ARO-D-31-124-G951 and Air Force Office of Scientific Research Grant AF-AFOSR-62-414. The author is deeply grateful to the referee for his valuable comments and helpful suggestions, which led to the present form of this paper.

iteration. Setting

$$\left.\frac{\partial w(a,z)}{\partial a}\right|_{a=0} = L(z) , \tag{3}$$

it is well known [2], [3], that an analytic iteration $w(a,z)$ satisfies the differential equation

$$\frac{\partial w(a,z)}{\partial a} = L[w(a,z)] . \tag{4}$$

Assuming that it is $L(z)$, rather than $w(a,z)$, that is given, we propose to use the differential equation (4) with the initial condition (2), in order to generate analytic iterations. The case when $L(z)$ is a *single-valued* analytic function in the domain D of the complex plane, has already been studied in an earlier paper and the following theorem was established [4, Th. 1].

PRELIMINARY THEOREM *Let $L(z)$ be a regular single-valued function in the closure of the bounded domain D of the complex plane. Then there exists a unique function $w(a,z)$ with the following properties.*

(i) There exists a number $\alpha_0(D) > 0$, such that for any fixed z in D, $w(a,z)$ is regular in a for $|a| < \alpha_0(D)$, and satisfies the differential equation (4) and the initial condition (2).

(ii) For any fixed a, $|a| < \alpha_0(D)$, $w(a,z)$ maps D conformally and univalently onto the domain D_a.

(iii) For any fixed $z \in D$, $w(a,z)$ satisfies the iteration equation (1), whenever both sides of (1) can be defined.

Note that the single-valuedness of $L(z)$ is essential, because if $L(z)$ is multiple-valued in the domain D of the complex plane, so is the solution $w(a,z)$ of the system (4) and (2). In this case, it is not clear what significance has the equality sign in (1).

In this paper we study the case of multiple-valued $L(z)$. We apply a technique frequently used in the theory of functions, namely: we embed the domain D, and consequently the domains D_a, in a Riemann surface S. By doing so, we are led to a study of the differential system (4) and (2), and the iteration equation (1) on a Riemann surface S.

https://doi.org/10.1017/S0004972700041435 Published online by Cambridge University Press

Though our original plan was to study the differential system (4) and (2) when the domain D is embedded in the Riemann covering surface of L, it was pointed out to us by the referee that the proofs go through on a general Riemann surface. Thus, we consider now the more general problem of the existence and the properties of a solution of the differential system (4) and (2) when the domain D is embedded in a general Riemann surface (complex analytic manifold). Throughout the paper we use the definitions and the terminology of [5]. We assume now that S is a given Riemann surface, namely: S is a connected topological Hausdorff space with a given analytic structure defined by the collection $\{\Omega_i, \Phi_i\}_{i\in I}$ (I an index set), where $\{\Omega_i\}_{i\in I}$ is an open covering of S, and Φ_i is a homeomorphism of Ω_i onto an open set in the complex plane [5, p. 59]. We state now

2. MAIN THEOREM *Let* D *be a domain included in a compact connected set* C, *where* $C \subset S$, *and* S *is a given Riemann surface. Let* Λ *be a first order differential defined in a region* G, $G \supset C \supset D$, *which in any local coordinate* $w = \Phi_i(P)$ *has the form*

$$\Lambda(P) = \Lambda[\Phi_i^{-1}(w)] = \frac{dw}{L_i(w)}, \qquad P \in \Omega_i \cap G. \tag{5}$$

Assume that for every $P_0 \in C$, *and any local coordinate* $w = \Phi_i(P)$, *the function* $L_i(w)$ *is regular at* $w^0 = \Phi_i(P_0)$. *Then there exists a unique family of mappings* $M(a,P)$ *with the following properties.*

(i) *There exists a number* $\alpha_0 > 0$, *such that for any fixed* $P \in D$, $M(a,P)$ *is regular in* a *for* $|a| < \alpha_0$, *and satisfies the initial condition*

$$M(0,P) = P, \tag{2'}$$

and the differential equation (4); i.e. when expressed in terms of a local coordinate, $w(a) = \Phi_i[M(a,P)]$ *is regular in* a *for* $|a| < \alpha_0$, *and satisfies the differential system*

$$\frac{dw(a)}{da} = L_i[w(a)], \quad w(0) = \Phi_i(P). \tag{6}$$

(ii) *For any fixed* a, $|a| < \alpha_0$, $M(a,P)$ *maps* D *conformally and*

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univalently onto the domain $D_a \subset S$ $(D_0 = D)$.

(iii) For any fixed $P \in D$, $M(a,P)$ *satisfies the iteration equation*

$$M[a,M(b,P)] = M(a+b,P) \,, \tag{7}$$

whenever both sides of (7) are defined.

We first make the following remark. Without loss of generality, we may assume [5, p. 60] that the analytic structure of S is such that for every $P_0 \in S$, there exists an open neighborhood Ω_i $(i \in I)$ and a homeomorphism Φ_i , such that $\Phi_i(P_0) = 0$ and $\Omega_i = \{P : P = \Phi_i^{-1}(w) \,, \, |w| < 1\}$. Ω_i is then a coordinate disk with center at P_0 , and $w = \Phi_i(P)$ assigns a local coordinate to every $P \in \Omega_i$. Moreover, if $w_1 = \Phi_1(P)$ and $w_2 = \Phi_2(P)$ are local coordinates in Ω_1 and Ω_2 respectively, and if $\Omega_1 \cap \Omega_2 \neq \emptyset$, then $\Phi_2\Phi_1^{-1}$ is a conformal one-to-one mapping of $\Phi_1(\Omega_1 \cap \Omega_2)$ onto $\Phi_2(\Omega_1 \cap \Omega_2)$, i.e.

$$w_2 = \Phi_2\Phi_1^{-1}(w_1) = f(w_1) \tag{8}$$

is a regular univalent function in $\Phi_1(U_1 \cap U_2)$ and $f'(w_1) \neq 0$.

In order that the system (6) will be satisfied by any local coordinate defined at P , it is necessary to take L as a reciprocal differential rather than a function. Hence for $P \in \Omega_1 \cap \Omega_2 \cap G$, we have

$$\Lambda(P) = \Lambda[\Phi_1^{-1}(w_1)] = \frac{dw_1}{L_1(w_1)} \,,$$

$$\Lambda(P) = \Lambda[\Phi_2^{-1}(w_2)] = \frac{dw_2}{L_2(w_2)} \,,$$

and the relation between $L_1(w_1)$ and $L_2(w_2)$ is given by

$$L_2(w_2) = L_2[f(w_1)] = L_1(w_1)\frac{dw_2}{dw_1} = L_1(w_1)f'(w_1) \,. \tag{9}$$

From (9) and from the fact that $f(w_1)$ is regular and $f'(w_1) \neq 0$, it follows that regularity of $L_1(w_1)$ at w_1^0 implies regularity of $L_2(w_2)$ at the corresponding point $w_2^0 = f(w_1^0)$, and vice versa.

Let $P_0 \in C$, and let $w = \Phi_i(P)$, $\Phi_i(P_0) = 0$, be the local coordinate in the coordinate disk Ω_i . If $\Lambda(P)$ is given by (5), then by our assumptions $L_i(w)$ is regular at $w = 0$, and thus there exists a number $r_i > 0$, such that $L_i(w)$ is regular for $|w| < r_i$. We associate now two open neighborhoods with every $P_0 \in C$, namely;

$$U_i = U(P_0) = \left\{P : P = \Phi_i^{-1}(w) , \quad |w| < r_i\right\} ,$$

and

$$V_i = V(P_0) = \left\{P : P = \Phi_i^{-1}(w) , \quad |w| < \frac{r_i}{2}\right\} .$$

Since $P_0 = \Phi_i^{-1}(0)$, evidently $P_0 \in V(P_0) \subset U(P_0) \subset \Omega_i$.

The collection $\{V(P_0)\}_{P_0 \in C}$ is an open covering of the compact set C . Hence, there exists a finite number of points $P_1 = P_{01}$, $P_2 = P_{02}$, ..., $P_n = P_{0n}$, $P_j \in C$, $j = 1,2,\ldots,n$, such that $\bigcup_{j=1}^{n} V(P_j) \supset C$. For $1 \leqq j \leqq n$, let $V_j = V(P_j)$ and $U_j = U(P_j)$; then $P_j \in V_j \subset U_j$ and $w = \Phi_j(P)$ maps P_j to the origin, V_j onto $|w| < \frac{r_j}{2}$ and U_j onto $|w| < r_j$. For $P \in U_j$, $1 \leqq j \leqq n$, $\Lambda[\Phi_j^{-1}(w)] = L_j^{-1}(w)dw$, and $L_j(w)$ is regular for $|w| < r_j$.

Consider now the differential system

$$(10) \qquad \frac{dw}{da} = L_j(w) , \quad w(0) = z = \Phi_j(P) , \quad P \in U_j , \quad 1 \leqq j \leqq n ,$$

where $L_j(w)$ is regular for $|w| < r_j$, and $z \in \Phi_j(U_j)$, i.e. $|z| < r_j$. Given the initial value $z \in \Phi_j(U_j)$, there exists, by the existence and uniqueness theorem [1, p. 1-5], a unique solution $w_j(a,z) = w_j[a,\Phi_j(P)]$ of the system (10) which is regular in a for $|a| < a_0(z)$, for some $a_0(z) > 0$. Moreover, $|w_j[a,\Phi_j(P)]| < r_j$, for $|a| < a_0(z) = a_0[\Phi_j(P)]$.

We define now a mapping $M_j(a,P)$ for every $P \in U_j$, in the following way.

(11) $$M_j(a,P) = \Phi_j^{-1}[w_j[a,\Phi_j(P)]] , \quad |a| < a_0[\Phi_j(P)] , \quad P \in U_j ,$$

where $w_j[a,\Phi_j(P)]$ is the solution of the system (10). Hence for any given $P \in U_j$, $M_j(a,P)$ is regular in a for $|a| < a_0[\Phi_j(P)]$, and $M_j(a,P) \in U_j$.

We set now:

(12) $$M(a,P) = M_j(a,P) , \quad P \in U_j , \quad 1 \leqq j \leqq n , \quad |a| < a_0[\Phi_j(P)] .$$

In order to show that (12) defines $M(a,P)$ uniquely for $|a| < a_m(P)$, where

$$a_m(P) = \max a_0[\Phi_j(P)]$$

and the maximum is taken over all indices j, such that $P \in U_j$, $1 \leqq j \leqq n$, we require the following lemma.

3. LEMMA *Given* $Q_1 \in U_1$ *and* $Q_2 \in U_2$, *(*Q_1 *and* Q_2 *are not necessarily distinct), let* $M_1(a,Q_1)$ *and* $M_2(a,Q_2)$ *be defined by (11). If both* $M_1(a,Q_1)$ *and* $M_2(a,Q_2)$ *are regular in* a *for* $|a| < t_0 = \min\{a_0[\Phi_1(Q_1)] , a_0[\Phi_2(Q_2)]\}$, *and if* $|a^*| < t_0$, *then*

$$M_1(a^*,Q_1) = M_2(a^*,Q_2)$$

if and only if $Q_1 = Q_2$.

Proof Let A be the open disk defined by

$$A = \{a : |a| < t_0\} ,$$

and let

$$B_1 = \{a : a \in A , M_1(a,Q_1) = M_2(a,Q_2)\}$$

and

$$B_2 = \{a : a \in A , M_1(a,Q_1) \neq M_2(a,Q_2)\} .$$

Evidently $B_1 \cap B_2 = \emptyset$, and $A = B_1 \cup B_2$. We shall prove now that B_1 and B_2 are open sets.

Suppose $a^* \in B_1$, i.e. $M_1(a^*,Q_1) = M_2(a^*,Q_2) = R$, then $R \in U_1 \cap U_2$, and by the analytic structure of S, (8) maps $\Phi_1(U_1 \cap U_2)$

onto $\Phi_2(U_1 \cap U_2)$. Moreover, $L_1(w_1)$ is regular in $\Phi_1(U_1 \cap U_2)$, and $L_2(w_2)$ is regular in $\Phi_2(U_1 \cap U_2)$ and (9) holds.

Consider now the differential system

$$\frac{dw}{da} = L_1(w) , \qquad w(a^*) = w_1^* = \Phi_1(R) . \tag{13}$$

Since $w_1^* \in \Phi_1(U_1 \cap U_2)$, there exists a number $\rho > 0$, such that $|w-w_1^*| < \rho$ is included in the open set $\Phi_1(U_1 \cap U_2)$. It follows now from the existence and uniqueness theorem [1], that the system (13) has a unique analytic solution $w(a)$. More specifically, there exists a number $\gamma_1 > 0$, such that for $|a-a^*| < \gamma_1$, the solution $w(a)$ of (13) satisfies $|w(a) - w_1^*| < \rho$. Observe now that if $w(a)$ satisfies the system (13) for $|a-a^*| < \gamma_1$, then $\tilde{w}(a) = f[w(a)]$, where $f = \Phi_2\Phi_1^{-1}$, satisfies the system

$$\frac{d\tilde{w}}{da} = L_2(\tilde{w}) , \qquad \tilde{w}(a^*) = \Phi_2(R) . \tag{14}$$

Indeed, $w(a) \in \Phi_1(U_1 \cap U_2)$ for $|a-a^*| < \gamma_1$, and therefore $\tilde{w}(a) = f[w(a)]$ is defined and regular. From (9) and (13) it follows now that

$$\frac{d\tilde{w}}{da} = f'[w(a)]\frac{dw}{da} = f'(w)L_1(w) = L_2[f(w)] = L_2(\tilde{w}) ,$$

and

$$\tilde{w}(a^*) = f[w(a^*)] = f[\Phi_1(R)] = \Phi_2(R) .$$

But similarly to (13), the system (14) has also a unique analytic solution for $|a-a^*| < \gamma_2$ for some $\gamma_2 > 0$. It is easily confirmed by (11) that $\Phi_1[M_1(a,Q_1)]$ is the unique analytic solution of the system (13) for $|a-a^*| < \gamma_1$, and $\Phi_2[M_2(a,Q_2)]$ is the unique solution of the system (14) for $|a-a^*| < \gamma_2$. On the other hand, by the observation made above $f(\Phi_1[M_1(a,Q_1)])$ satisfies the system (14) for $|a-a^*| < \gamma_1$. Hence, by the uniqueness of the solution of the system (14), it follows that

$$\Phi_2[M_2(a,Q_2)] = f(\Phi_1[M_1(a,Q_1)]) = \Phi_2[M_1(a,Q_1)] , \quad |a-a^*| < \gamma_0 = \mathrm{Min}(\gamma_1,\gamma_2) .$$

Hence

$$M_1(a,Q_1) = M_2(a,Q_2) , \quad |a-a^*| < \gamma_0 ,$$

and we conclude that if $a^* \in B_1$, then all the points in the disk

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$|a-a^*| < \gamma_0$ also belong to B_1, and B_1 is an open set.

Next we prove that B_2 is open. Suppose now that $a^* \in B_2$, i.e. $|a^*| < t_0$ and

$$R_1 = M_1(a^*,Q_1) \neq M_2(a^*,Q_2) = R_2 .$$

Since a Riemann surface is a Hausdorff space, there exist two disjoint open sets $\tilde{U}_1$ and $\tilde{U}_2$, such that $R_1 \in \tilde{U}_1 \subset U_1$ and $R_2 \in \tilde{U}_2 \subset U_2$.

Consider now the differential systems

$$(13') \qquad \frac{dw}{da} = L_1(w) , \qquad w(a^*) = \Phi_1(R_1) ,$$

and

$$(14') \qquad \frac{dw}{da} = L_2(w) , \qquad w(a^*) = \Phi_2(R_2) .$$

$L_1(w_1)$ is regular in $\Phi_1(U_1) \supset \Phi_1(\tilde{U}_1)$, and therefore there exists a number $\delta_1 > 0$ such that for $|a-a^*| < \delta_1$, the unique solution of (13'), namely $w(a) = \Phi_1[M_1(a,Q_1)]$ satisfies $w(a) \in \Phi_1(\tilde{U}_1)$, i.e. $M_1(a,Q_1) \in U_1$ for $|a-a^*| < \delta_1$. By the same argument there exists a number $\delta_2 > 0$, such that $M_2(a,Q_2) \in \tilde{U}_2$ for $|a-a^*| < \delta_2$. Since $\tilde{U}_1 \cap \tilde{U}_2 = \emptyset$, it follows that

$$M_1(a,Q_1) \neq M_2(a,Q_2) , \qquad |a-a^*| < \delta_0 = \min(\delta_1,\delta_2) .$$

Hence, if $a^* \in B_2$, then all the points of the disk $|a-a^*| < \delta_0$ also belong to B_2.

Since A is a connected open set, and $A = B_1 \cup B_2$ where B_1 and B_2 are disjoint open sets, then either B_1 or B_2 is empty. Assume now that $Q_1 = Q_2$, then $Q_1 = M_1(0,Q_1) = M_2(0,Q_2) = Q_2$, and $a = 0$ belongs to B_1. Consequently, B_2 is empty and $B_1 = A$, i.e. $M_1(a,Q_1) = M_2(a,Q_2)$ for $|a| < t_0$. In the other case, if $Q_1 \neq Q_2$ then $a = 0$ belongs to B_2, which implies that B_1 is empty, i.e. $M_1(a,Q_1) \neq M_2(a,Q_2)$ for $|a| < t_0$. This completes the proof of the lemma and we proceed now with the proof of the theorem.

4. It follows from the lemma that for $P \in U_j \cap U_k$, $j \neq k$, $1 \leqq j$, $k \leqq n$, we have

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$$M_j(a,P) = M_k(a,P) , \quad |a| < \min\{a_0[\Phi_j(P)] , \quad a_0[\Phi_k(P)]\} .$$

Hence, $M(a,P)$ is uniquely defined by (12) for $P \in \bigcup_{j=1}^{n} U_j$ and $|a| < a_m(P) = \max\{a_0[\Phi_j(P)]\}$. Furthermore, $M(a,P)$ depends on the analytic structure of S, and on the reciprocal differential L, but not on the choice of the finite collection of open sets $U_1, \ldots, U_n$. Because suppose $U_1^*, \ldots, U_m^*$ is a different finite collection of open sets such that $P \in \bigcup_{k=1}^{m} U_k^*$, then $U_1, \ldots, U_n , U_1^*, \ldots, U_m^*$ is also a finite collection of open sets, and by the lemma $M(a,P)$ is not changed.

We turn now to the mapping properties of $M(a,P)$. We restrict now the set of initial values in (10), and consider only $P \in V_j$. Consequently, for $z = \Phi_j(P)$, $P \in V_j$, we have $|z| < \frac{r_j}{2}$, and we may apply the preliminary theorem to the system (10). Thus, there exists a number $\alpha_j = \alpha_0[\Phi_j(V_j)] > 0$, such that for any fixed $z \in \Phi_j(V_j)$, the solution $w_j(a,z) = w_j[a,\Phi_j(P)]$ is regular in a for $|a| < \alpha_j$. Moreover, for any fixed $|a| < \alpha_j$, $w_j(a,z)$ maps the disk $|z| < \frac{r_j}{2}$ conformally and univalently into the disk $|w_j| < r_j$. Hence, for any fixed a, $|a| < \alpha_j$, $M_j(a,P) = \Phi_j^{-1}[w_j(a,\Phi_j(P))]$ is a conformal one to one mapping from V_j into U_j. Let

$$\alpha_0 = \min\{\alpha_1,\ldots,\alpha_n\} ,$$

then for any fixed a, $|a| < \alpha_0$, the mapping $M(a,P)$, defined by (12), is conformal and locally univalent in $\bigcup_{j=1}^{n} V_j$. But by the lemma, if $Q_1 \in V_1$, $Q_2 \in V_2$ and $Q_1 \neq Q_2$, then

$$M(a,Q_1) = M_1(a,Q_1) \neq M_2(a,Q_2) = M(a,Q_2) , \quad |a| < \alpha_0 .$$

Hence, for $|a| < \alpha_0$, $M(a,P)$ maps $D \subset \bigcup_{j=1}^{n} V_j$ conformally and univalently onto $D_a \subset \bigcup_{j=1}^{n} U_j$.

Finally, we show that for a given $P \in D$, $M(a,P)$ satisfies the iteration equation (7) whenever both sides can be defined. More specifically, for any given $P \in D$, $M(b,P)$ is defined at least for $|b| < a_m(P)$. Let $M(b,P) = R$, where b is fixed and $|b| < a_m(P)$; then $M(a,R) = M[a,M(b,P)]$ is defined at least for a such that $|a| < a_m(R)$. On the other hand, $M(a+b,P)$ is defined at least for $|a+b| < a_m(P)$. Let

$$A_b = \left\{a : |a| < a_m[M(b,P)] , \quad |a+b| < a_m(P) , \quad |b| < a_m(P)\right\} ,$$

then we claim that for a given point $P \in D$, (7) holds at least for $|b| < a_m(P)$ and $a \in A_b$.

Assume now that

$$a_m(P) = \max\{a_0[\Phi_j(P)]\} = a_0[\Phi_1(P)] ,$$

and therefore

$$M(c,P) = M_1(c,P) , \quad |c| < a_m(P) = a_0[\Phi_1(P)] .$$

We distinguish two cases. In the first case

$$a_m(R) = a_m[M(b,P)] = a_0[\Phi_1(R)]$$

and

$$M(a,R) = M_1(a,R) , \quad |a| < a_m(R) .$$

Thus, $M[a,M(b,P)] = M_1[a,M_1(b,P)]$, and $M(a+b,P) = M_1(a+b,P)$ and equation (7) reduces to

(7') $$M_1[a,M_1(b,P)] = M_1(a+b,P) .$$

The fact that $M_1(a,P)$ satisfies equation (7') follows from the uniqueness of the solution of the system

(15) $$\frac{dw}{da} = L_1(w) , \quad w(0) = \Phi_1(R) = \Phi_1[M_1(b,P)] .$$

Indeed, $\Phi_1[M_1(a,R)]$ is regular in a for $|a| < a_m(R) = a_0[\Phi_1(R)]$ and satisfies the system (15), while $\Phi_1[M_1(a+b,P)]$ is regular in a for

$|a+b| < a_m(P)$ and also satisfies (15). Hence

$$\Phi_1[M_1(a,R)] = \Phi_1[M_1(a+b,P)] \;, \quad a \in A_b \;.$$

Therefore, (7') holds for $P \in D$, $|b| < a_m(P)$ and $a \in A_b$. Note that this result follows also directly from the preliminary theorem.

In the second case

$$a_m(R) = a_m[M(b,P)] = a_0[\Phi_k(R)] \;, \quad k \neq 1 \;, \quad 2 \leq k \leq n \;.$$

Without loss of generality we assume now that $k = 2$. Thus, $R = M(b,P) = M_1(b,P)$ and

$$M[a,M(b,P)] = M(a,R) = M_2(a,R) = M_2[a,M_1(b,P)] \;, \quad |a| < a_m(R) \;,$$

and equation (7) reduces in this case to

$$(7'') \qquad M_2[a,M_1(b,P)] = M_1(a+b,P) \;.$$

Note that for $|a| < a_0[\Phi_1(R)] < a_0[\Phi_2(R)]$, it follows from the lemma that $M_2(a,R) = M_1(a,R)$, and (7") reduces to (7'). To establish (7") for $a \in A_b$, namely for a such that $|a| < a_0[\Phi_2(R)]$, $|a+b| < a_0[\Phi_1(P)]$, we consider the two sets

$$B_1^* = \{a : a \in A_b \;, \quad M_2[a,M_1(b,P)] = M_1(a+b,P)\}$$

and

$$B_2^* = \{a : a \in A_b \;, \quad M_2[a,M_1(b,P)] \neq M_1(a+b,P)\} \;.$$

By similar arguments to those used in the lemma, it follows that B_1^* and B_2^* are both open. Since A_b is an open connected set, and $A_b = B_1^* \cup B_2^*$ where B_1^* and B_2^* are disjoint open sets, then either B_1^* or B_2^* is empty. But B_1^* is not empty, because it includes the point $a = 0$, therefore $B_1^* = A_b$. Hence (7") holds for $P \in D$, $|b| < a_m(P)$ and $a \in A_b$. This completes the proof of our theorem.

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https://doi.org/10.1017/S0004972700041435 Published online by Cambridge University Press