Multiple Periodic Variable Stars
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CEPHEID PULSATION THEORY AND MULTIPERIODIC CEPHEID VARIABLES*

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In this review of the situation with regard to the multiperiodic Cepheid variables, our subject matter is divided into four parts. The first discusses general causes of pulsation of Cepheids and other variable stars, and their locations on the H-R diagram. For this section we draw upon the work during the past 10-15 years of J. P. Cox, Baker, Kippenhahn, A. N. Cox, King, Christy, Castor, Stobie, Stellingwerf, Davey, Iben, and Tuggle, mostly with the small amplitude linear nonadiabatic radial pulsation theory. In the second section we review the linear adiabatic and non-adiabatic theory calculation of radial pulsation periods and their application to the problem of masses of double-mode Cepheids. Contributions discussed are by Cogan, J. P. Cox, King, Stellingwerf, Petersen, Hansen, and Ross. Periodic solutions, and their stability, of the nonlinear radial pulsation equations for Cepheids and RR Lyrae stars are considered in the third section. This research has been done by Stellingwerf with previous development of methods by Baker and von Sengbusch and current work by A. N. Cox and Davey at Los Alamos. In the last section we give the latest results on nonlinear, nonperiodic, radial pulsations for Cepheids and RR Lyrae stars. This work has been done by Stellingwerf, King, A. N. Cox, J. P. Cox, and Davey.

References to the first three sections are given by Ledoux and

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Walraven (1958) and by J. P. Cox (1974).

The basic causes of pulsation in most common types of variable stars are now reasonably well understood. These stars include the classical Cepheids, the RR Lyrae variables, the W Virginis variables, and the dwarf Cepheids and Delta Scuti variables. These stars lie in the Hertzsprung-Russell diagram in a long, narrow, almost vertical region above and to the right of the main sequence--roughly, from a few to a few hundred thousand solar luminosities and from some 5000 K to 8000 K. This region shown in Fig. 1 is sometimes called the Cepheid instability strip, or simply the instability strip. Even the basic destabilizing mechanism of the red variables such as the Mira variables which have cooler surface temperatures than those in the instability strip is probably fairly well understood.

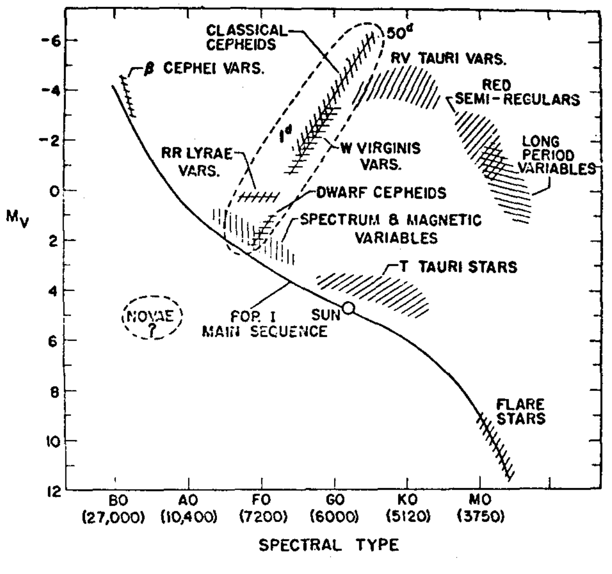

Fig. 1. The Hertzsprung-Russell diagram showing regions where variable stars are found. The double-mode Cepheids are found near the red edge of the Cepheid instability strip.

The instability mechanisms referred to above may be called envelope ionization mechanisms. They can produce pulsational instability because

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an abundant constituent of the star, such as hydrogen and/or helium, is in the midstages of ionization at a critical depth below the stellar photosphere. This partial ionization of a dominant constituent prevents the temperature in the regions of partial ionization from varying very much relative to the temperature in surrounding regions during the pulsation. This effect is called the gamma mechanism. In particular, the temperature in these regions does not increase very much during the compression phase of the pulsation cycle. This relative coolness during the compression phase, for example, has two consequences: first, it tends to impede the radiation flow out of these regions at this phase. Second, this partial ionization and relative coolness tend to cause the opacity of these regions to increase upon compression (the kappa mechanism), and thus, again, to impede the flow of radiation out of these regions at this phase. The net result of both of these factors is to prevent the escape of radiation, or to "trap" the radiation, during the compression phase. The trapped radiation then heats the relevant regions, and causes the pressure here to be somewhat larger, during the expansion phase than during the contraction phase. Hence, the pulsations tend to be pumped up, or excited, and thus to increase in amplitude with time, at least until nonlinear processes limit the amplitude to some finite value, presumably, to the observed values.

On the other hand, the deeper stellar layers, below the ionization regions, tend to behave in just the opposite manner and so tend to damp the pulsations. This effect of the deeper regions is sometimes called radiative damping.

Detailed calculations show that it is mainly the He^{+} - He^{++} -- the second helium ionization region -- which is responsible for the pulsations of stars in the instability strip, with a much smaller, though nonnegligible, contribution from hydrogen ionization. On the other hand, H ionization seems to be mainly responsible for the pulsations of the red variables.

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Envelope ionization mechanisms have well accounted for most of the qualitative features of the above regions on the H-R diagram where many types of variable stars are found. For example, these mechanisms restrict most variable stars to relatively low effective temperatures. Also, the condition that the ionization regions lie at a great enough depth below the stellar surface implies that the hot, or blue, edges of instability regions should be fairly sharp and nearly vertical on an H-R diagram, as observed. This condition follows because this depth is considerably more sensitive to a star's effective temperature than to any other stellar parameter. The present general theories do not well account for the termination of instability on the red, or cool, edge of the instability strip. It is generally believed that this termination is caused by the onset of effective convection in the relevant parts of the stellar envelope, thus throttling the action of the above destabilizing mechanisms. However, the quantitative details of this throttling action still remain to be worked out.

In the linear theory of Cepheid pulsation with the destabilizing region near the surface, several modes of radial pulsation may be simultaneously excited. In most cases, observation and nonlinear theory show that only a single mode actually occurs at large, observable amplitudes. For these cases, or for the double-mode case which is occasionally also observed, what mechanism decides the actual behavior, and is there a chance to predict this behavior by the simpler, more efficient linear theory? For prediction of actual light and velocity curves, and the modal behavior, it seems that we must use nonlinear theory.

We now discuss linear theory results for periods.

It had been shown from the results of Cogan (1970) and of J. P. Cox, King, and Stellingwerf (1972) that the pulsation constant Q for a given mode depended mostly on the ratio M/R. These data from Cox, King and

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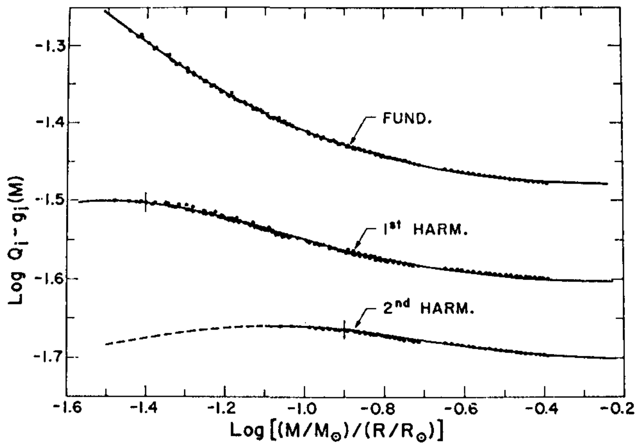
Stellingwerf are in a plot as Fig. 2. Since Q is also proportional to the quantity $\sqrt{M/R^3}$ (M and R denote, respectively, stellar mass and radius) multiplied by the period, given a period and a radius (from an effective temperature and luminosity) a pulsation mass can be derived.

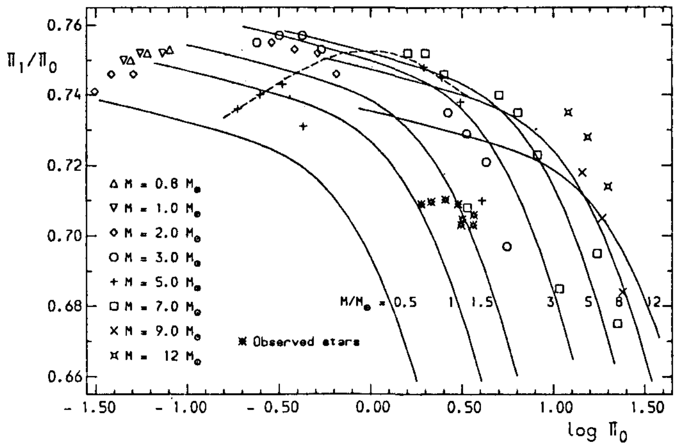
Petersen (1973) published an important paper in which he determined the masses and radii of eight double-mode Cepheids. He showed that period ratios are also related to M/R. Thus with two quantities, Π_0 and Π_1/Π_0 (Π = period, subscripts 0 and 1 denote, respectively, fundamental and first overtone), the double-mode fundamental period and the ratio of the two decomposed periods (presumed to be the fundamental and first overtone), the unknowns M and R can be found.



—Log $Q_i - g_i(\mathfrak{M})$ *versus* log $(\mathfrak{M}/R)$ for the Population I models, where the $g_i(\mathfrak{M})$ $(i = 0, 1, 2)$ are defined in eqs. (2)–(4), which are also plotted (*solid* and *dashed lines*). Solar units are used for $\mathfrak{M}$ and R, and the Q_i are in days. A few overlapping points have been omitted for clarity. No models lying to the left of the two vertical ticks are unstable in the 1H or 2H, respectively. The dashed part of the curve for the 2H indicates that this curve may not be a very accurate fit in this $\mathfrak{M}/R$ range.

Fig. 2. The pulsation constant Q for the lowest three radial pulsation modes as a function of M/R according to models of Cox, King, and Stellingwerf.

Results are shown in Fig. 3 which plots Π_1/Π_0 versus log Π_0 for various masses from linear theory data of Cox, King, and Stellingwerf and of Cogan. Observed stars are seen to have double-mode masses even lower than the possibly already low pulsation masses compared to evolutionary theory.



Period ratio as function of the period of the fundamental mode for an extreme Population I composition. Full curves are based upon the fitting formulae given by Cox *et al.* (1972); the number at each curve gives the mass in solar units. Individual points are taken from Cogan (1970), and the dashed curve is a roughly estimated mean curve for Cogan's models of 5 solar masses (see text). The variables given in Table 1 are also plotted in the figure

Fig. 3. The first overtone to fundamental period ratio as a function of fundamental period for various masses according to data collected by Petersen.

Further discussion by King, Hansen, Ross, and J. P. Cox (1975) showed that composition did not seem to affect greatly the Petersen argument. The same eight stars were redone with two more compositions and the results are given in Fig. 4. Periods in days are given in the second and third columns and masses and radii in solar units are given next for all three compositions -- the Kippenhahn (with hydrogen mass fraction X close to 0.6), King IVa, and King Va mixtures. Masses range from 1.04 to 2.16 solar masses -- not 3 to 5 as expected from evolutionary theory.

These authors then discuss UTrA in detail, assuming that its mass is 1.60 solar masses as given by the most reasonable population I composition and using the observed color. Results are in Fig. 5. For the four models considered, it appeared that luminosities like $M_{bol} = -1.4$ to -1.9 and T_e values approximately 5500 to 6100 K were needed to give the proper

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radius and simultaneous linear theory instability in both fundamental and first harmonic modes. These models are near the red edge of the instability strip.

MASSES AND RADII OF THE DOUBLE MODE CEPHEIDS

Star	Composition		CKS		X = 0.7		X = 0.8	
	Π_0	Π_1	M	R	M	R	M	R
V439 Oph	1.89	1.34	1.04	13.0	1.24	14.0	1.34	14.4
TU Cas	2.14	1.52	1.16	14.6	1.37	15.7	1.48	16.2
U TrA	2.57	1.83	1.13	17.3	1.60	18.6	1.74	19.2
VX Pup	3.01	2.14	1.50	19.8	1.76	21.2	1.92	21.9
AP Vel	3.13	2.20	1.40	19.6	1.64	21.0	1.81	21.8
BK Cen	3.17	2.24	1.46	20.1	1.70	21.5	1.88	22.4
Y Car	3.64	2.56	1.59	22.6	1.85	24.1	2.05	25.1
AX Vel	3.67	2.59	1.68	23.1	1.95	24.8	2.16	25.8

Fig. 4. Masses and radii of double-mode Cepheids for three compositions, Kippenhahn Ia, King IVa, and King Va, with, respectively X = 0.602, 0.7, and 0.8. Both equation of state and opacity are from tables computed at Los Alamos.

MODELS OF THE DOUBLE MODE CEPHEID U TrA

Mass	M_{bol}	T_{eff} (°K)	Π_1/Π_0	R	η_1/η_0
1.60	-1.4	5500	0.72	19.0	1
1.60	-1.6	5800	0.72	18.8	2
1.60	-1.9	6100	0.72	19.5	2
1.60	-2.2	6600	0.72	19.1	*

*For this model both fundamental and first harmonic modes were stable.

Fig. 5. Models of the double-mode Cepheid UTrA for 1.6 $M_{\odot}$ with the fundamental and first overtone periods held fixed at 2.57 and 1.83 days.

Moving now to periodic solutions of the nonlinear pulsation equations, we review briefly some results recently completed by Stellingwerf (1975). His method (1974) has been described as a Henyey method where the pulsation period and all starting values of temperature, radius, and velocity for Lagrangian mass zones at an arbitrary phase are iteratively adjusted so that conditions are exactly reproduced at the end of the true period. Stellingwerf has discussed mostly RR Lyrae stars, but he has also done work on two mixed-mode models for UTrA at 5800 and 6100 K--two models discussed by King, Hansen, Ross, and Cox.

Figure 6 gives the relevant results for the modal behavior of the two 1.6 $M_\odot$, 19 $R_\odot$, King IVa composition models. Obviously much more than the two models is needed for all the details of these curves which actually are based on Stellingwerf's (1975) RR Lyrae work also. Plotted are linear kinetic energy growth rates for the fundamental, first overtone and second overtone pulsations as solid lines. Blue edges are where growth rates become positive, indicating an increase in amplitude with time. A linear analysis of the stability of the nonlinear periodic pulsations shows that the first overtone will grow out of a pure fundamental pulsation between 6400 and 6500 K. At about 6500 K, the fundamental blue edge, the fundamental pulsation is just stable and so the linear growth of the first overtone from the fundamental is really the same as the growth of the first overtone from a static model in linear theory. No mode switching is predicted between about 6350 K and 6200 K, but at temperatures down to about 5800 K there at least is a tendency for the first overtone to switch to the fundamental. At and below about 5800 K mixed-mode behavior is expected.

Thus, proceeding from hot to cooler effective temperatures, there are four regions of distinct large-amplitude pulsation behavior: stable only (neither mode exists), first overtone only present, either mode occurring,

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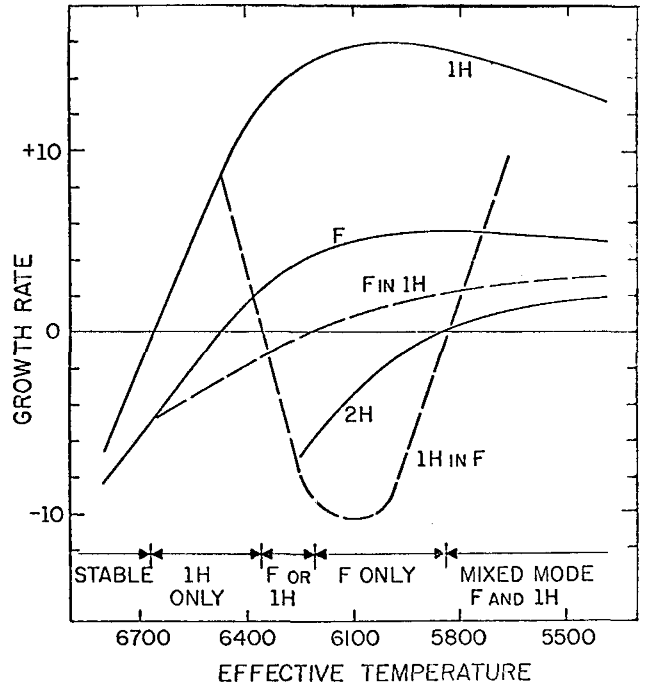
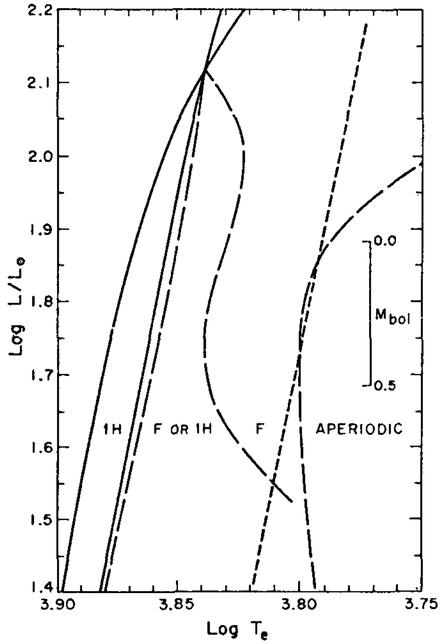


Fig. 6. Linear theory radial pulsation growth rates from static models and from nonlinear periodic solutions for the fundamental and first overtone modes. The Stellingwerf data are for UTrA models with 1.6 $M_{\odot}$ and 19 $R_{\odot}$ giving a constant period of 2.6 days.

fundamental only present, and then mixed fundamental and first overtone mode pulsations. There are no indications of switching from the fundamental or first overtone to the second overtone, which has a blue edge for this mass and luminosity at about 5800 K.

It is interesting to see what the stability analysis of the nonlinear pulsations gives for the RR Lyrae stars in the H-R diagram. This is shown in Fig. 7. The instability diagram just shown was for a line of constant radius, 19 $R_{\odot}$, and constant period, 2.6 days, in the H-R diagram. The same regions discussed before appear for these lower luminosities and lower masses. Note that the aperiodic or mixed mode region may lie entirely in the stable region to the red of the red edge. This may explain the very few (or zero) mixed mode RR Lyrae stars and in the case of higher masses the unlikelihood of finding multiperiodic Cepheids.



—H-R diagram showing the overall stability results for the main survey, $M = 0.578\ M_\odot$, $X = 0.7$, $Y = 0.299$, $Z = 0.001$. *Solid lines*, linear blue edges; *dashed lines*, nonlinear transition lines (type of behavior indicated); *dotted line*, estimated red edge.

Fig. 7. The instability strip in the region of the RR Lyrae stars on the theoretical Hertzsprung-Russell diagram. Modal behavior for 0.578 $M_\odot$ stars of the King Ia composition are indicated.

Figure 8 shows the only successful nonlinear, nonperiodic calculation, here actually for the case of the Stellingwerf RR Lyrae star model 2.6 with the fundamental period of 0.96 days and perhaps redward of the red edge. This case has been compared to the single observed mixed-mode RR Lyrae variable AC Andromedae with Π_o = 0.711 day. His shorter period models do not show aperiodic behavior. Fitch, however, suggests a much larger mass for AC And. Thus there may be no double-mode low-mass RR Lyrae stars. We should point out that Stobie and King, J. P. Cox, Eilers, and Davey have mentioned mixed-mode behavior but these cases have not been well documented

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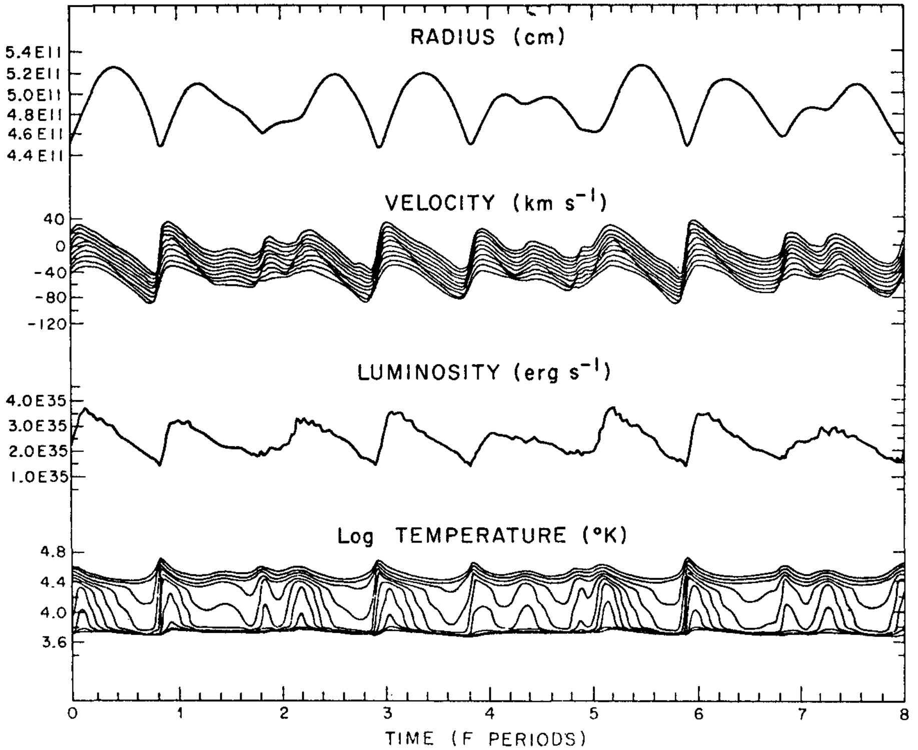
and may not be a permanent mixture of modes. It should also be pointed out that now at Los Alamos we are not able to reproduce this model 2.6 result with direct integrations through time.

King, A. N. Cox, Eilers, and J. P. Cox (1975) have attempted the direct initial value problem with integrations through time for the case of the mixed-mode Cepheid UTrA. As seen in Fig. 7, near the red edge, a star should show double-mode behavior. For $M_{bol} = -1.6$ and $T_e = 5600$ K, the complete time behavior calculated at Los Alamos is given in the next four figures. Figure 9 shows the growth of the pure first overtone mode to its limiting amplitude. No hint of any mixed mode appears even though a perturbation kinetic energy growth rate to the fundamental of several percent each period is expected according to Stellingwerf. The fundamental pulsations were followed in Fig. 10 with similar pure-mode behavior. This last result is greatly unexpected because, according to Stellingwerf's models and calculations, the linear growth rate from this fundamental mode toward the first overtone is over ten percent in kinetic energy per period. Also shown on this plot is a deliberate artificial perturbation, mixing the full amplitude fundamental velocities with three-quarters amplitude first overtone velocities. While there is mixed-mode behavior for a time, it all damps out to give a pure fundamental after about 100 periods. Figure 11 shows this artificial multiperiodic behavior in the velocity curve after six fundamental periods. Unfortunately after 250 periods the behavior given in Fig. 12 is indistinguishable from the pure fundamental before the perturbation was artificially imposed.

We note the following results to this date:

1. The linear theory gives periods and period ratios which can be interpreted in terms of very low masses for these observed stars, compared to evolutionary theory masses.



—Radius, velocity, luminosity, and log temperature (units and scales as indicated) for eight fundamental periods of the mixed-mode model described in the text. Successive velocity curves are shifted by 6 km s^{-1}.

Fig. 8. The time behavior of the Stellingwerf model 2.6. This calculation was made with an artificial advancing technique based on the stability analysis of the fundamental periodic limit cycle. The normal initial value technique gives identical results.

2. Certain nonlinear calculations predict only the permanent mixed-mode behavior for pulsators near the red edge of the instability strip. These calculations are based on a linear stability analysis of the full amplitude nonlinear periodic pulsation.

3. Direct initial value nonlinear calculations for RR Lyrae and Cepheid pulsations do not give any stable double-mode behavior except for the Stellingwerf case for his model 2.6 which cannot be confirmed by two distinct Los Alamos codes.

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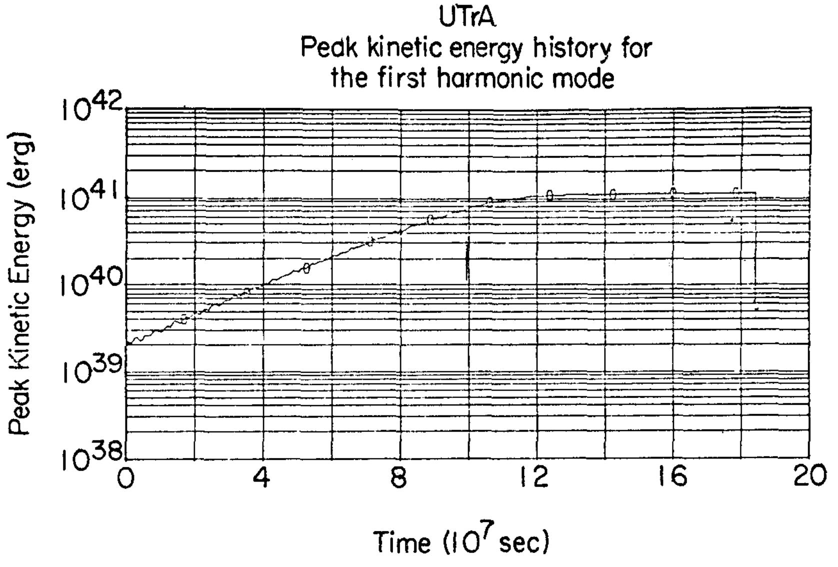

Fig. 9. The peak kinetic energy growth for the first overtone mode of a 1.6 $M_\odot$ UTrA model. An approximate velocity distribution is imposed on a hydrostatic structure.

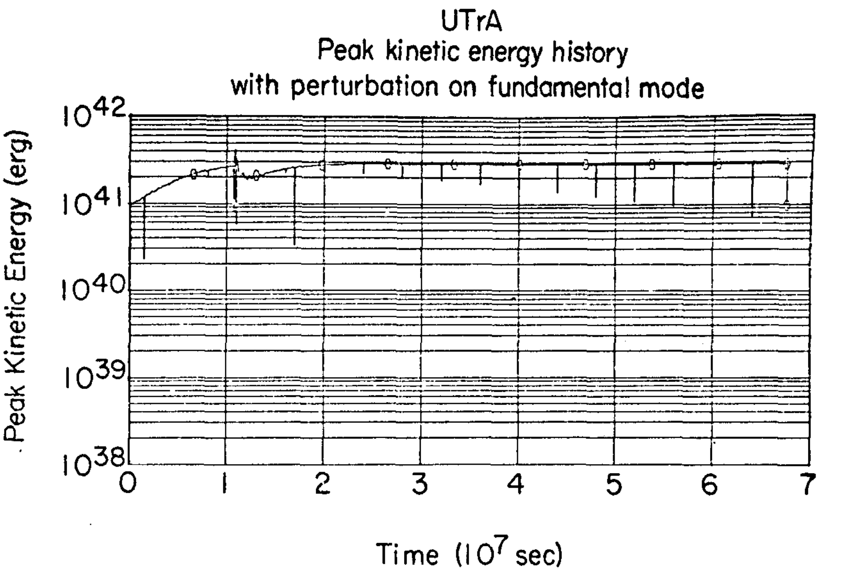

Fig. 10. The peak kinetic energy growth for the fundamental mode of a 1.6 $M_\odot$ UTrA model. After reaching the limiting level an artificial perturbation was introduced to induce double-mode behavior. After less than 50 periods the motion reverts to the pure fundamental mode.

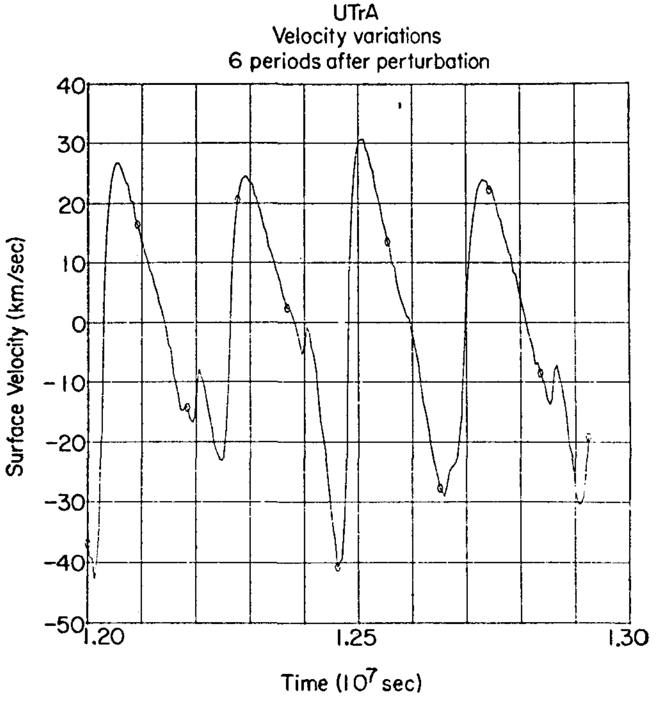

Fig. 11. Velocity curve history for a 1.6 $M_\odot$ UTrA model after an artificial first overtone perturbation of 15 percent of the total kinetic energy is put onto a pure fundamental mode.

The situation is confused. We observed mixed-mode Cepheids but we cannot explain their behavior. Observationally and theoretically they must be near the red edge, but are they not low mass? The evidence is strong that they are indeed low mass stars because periods are changed only a few percent by nonlinear theory and period ratios are changed even less. Luminosities are not greatly in error if the radius and effective temperature are fixed. What causes mixed-mode behavior observed in some stars?

Assuming the reality of the low masses, we conjecture that compositions appreciably different from those used heretofore may affect the results of the nonlinear calculations. Composition changes do not greatly affect periods or period ratios, and therefore, masses and radii. The internal composition may affect, however, the detailed nonlinear effects and give

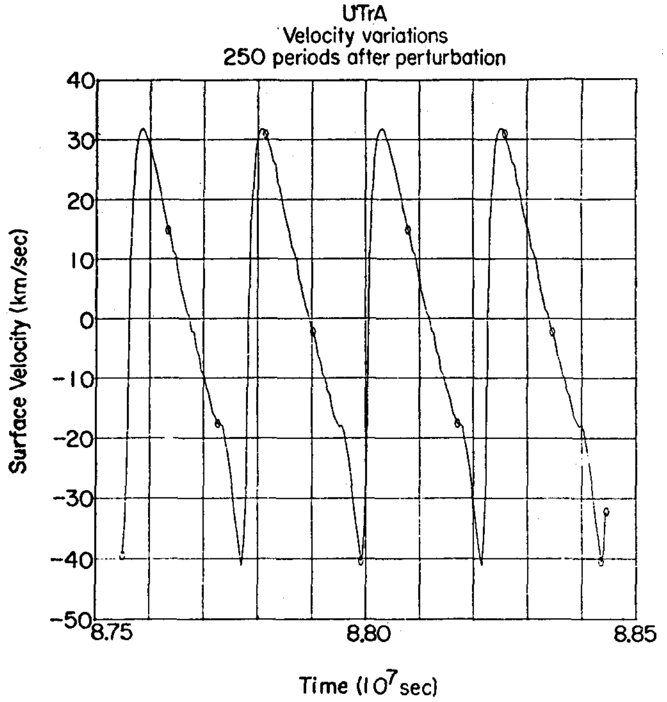

Fig. 12. Velocity curve history for a 1.6 $M_{\odot}$ UTrA model 250 periods after an artificial perturbation. All traces of the first overtone mode have disappeared.

multiperiodic behavior.

Finally we only mention convection. The energy flow by convection is significant near the red edge of the instability strip, but has been neglected in all studies to date. This, as always, needs immediate attention.

Our conclusion is that we do not yet understand all the parameters of multiperiodic Cepheids, but their masses and radii given by Petersen and more recently by King, Hansen, Ross, and Cox do not seem to be greatly in error. Prediction of observed multiperiodic pulsations has not yet been achieved for Cepheids.

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Discussion to the paper of COX and COX

VAN HORN: I would like to understand more about your comments on convection. First, do I understand correctly that convection has not been included in any of the nonlinear hydrodynamic calculations? Second, I thought Stellingwerf's method allowed him to include convection. Has he not done this?

COX: Stellingwerf has not so far included any allowance for convection, either in establishing the model or in the variation of the flux of energy from the central regions. Actually, Baker and Kippenhahn in 1963 and Iben in 1973 considered convection in establishing the model and considered that the convective flux was not modulated by the gamma and kappa effects which operate on radiation flow and cause the instability. No time dependent convection has ever been used in calculations of model Cepheids.

FITCH: If you have a pulsating star with both P_1 and P_2 excited, but not P_0, would you expect it to be near the red side of the strip?

COX: At present, theory cannot predict where such a star might be found.

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