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ON THE ITERATIONS AND THE ARGUMENT DISTRIBUTION OF MEROMORPHIC FUNCTIONS

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Abstract

This paper consists of two parts. The first is to study the existence of a point a at the intersection of the Julia set and the escaping set such that a goes to infinity under iterates along Julia directions or Borel directions. Additionally, we find such points that approximate all Borel directions to escape if the meromorphic functions have positive lower order. We confirm the existence of such slowly escaping points under a weaker growth condition. The second is to study the connection between the Fatou set and argument distribution. In view of the filling disks, we show nonexistence of multiply connected Fatou components if an entire function satisfies a weaker growth condition. We prove that the absence of singular directions implies the nonexistence of large annuli in the Fatou set.

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1. Introduction and main results

It is well known that iteration and argument distribution of transcendental meromorphic functions basically belong to different topics in the theory of meromorphic functions. The main objects studied in the iteration theory of meromorphic functions are the Fatou set and Julia set, while those in the argument distribution are the singular directions and filling disks. It would be interesting to explore their connections. We try to do that in this paper. So let us begin with the basic knowledge and notation from these two topics.

For a transcendental meromorphic function f, denote by

$$f^n := \underbrace{f \circ \cdots \circ f}_{n}$$

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the n th iterate of f for $n \in \mathbb{N}$. The Fatou set $F(f)$ of f is the set of points in $\mathbb{C}$ each of which has a neighborhood where $\{f^n\}$ is well defined and forms a normal family in the sense of Montel (or, equivalently, it is equicontinuous). The complement $J(f)$ of $F(f)$ with respect to $\hat{\mathbb{C}}$ is called the Julia set of f. In the study of iterates of meromorphic functions, escaping sets have been extensively investigated in recent years and continue to attract a lot of interest. The escaping set $I(f)$ of a meromorphic function f is defined by

$$I(f) = \{z : \ f^n(z) \to \infty \ (n \to \infty)\}.$$

The escaping set is first introduced and investigated by Eremenko [14] for transcendental entire functions and by Dominguez [13] for meromorphic functions. Many important dynamical behaviors of the escaping set have been revealed in the literature, for example, refer to [22, 23, 25, 27]; here we just mention some of them.

Let f be a transcendental meromorphic function. Define

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta,$$

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} \, dt + n(0, f) \log r,$$

where $n(r, f = a)$ denotes the number of a-points of f counted with multiplicities in $\{z : |z| < r\}$, and briefly write $n(r, f)$ for $n(r, f = \infty)$ and

$$T(r, f) = m(r, f) + N(r, f).$$

Here, $T(r, f)$ is known as the Nevanlinna characteristic of f. Then the growth order $\rho(f)$ and lower order $\lambda(f)$ of f are, respectively, defined as

$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \lambda(f) = \underline{\lim_{r\to\infty}} \frac{\log T(r, f)}{\log r}.$$

The filling disks are one of the main objects of study in the argument distribution of meromorphic functions, which was first introduced by Milloux [18] in 1928. Here and henceforth, by $\chi(a, b)$, we denote the spherical distance between a and b on $\hat{\mathbb{C}}$. By $B_\chi(a, r)$, we denote the spherical disk $\{z \in \hat{\mathbb{C}} : \chi(z, a) < r\}$ centered at a with spherical radius r and $B(a, r)$ the disk $\{z \in \mathbb{C} : |z - a| < r\}$ centered at a with (Euclidean) radius r.

Definition. A disk $B(z_0, \varepsilon|z_0|) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon|z_0|\}$ is called a *filling disk*, or *circle de remplissage*, of f with index m if f takes all values at least m times on $B(z_0, \varepsilon|z_0|)$, possibly except those values contained in two spherical disks with spherical radius at most e^{-m}.

The following is the first main result of this paper, which reveals the connections among the filling disks, the escaping sets, and Julia sets.

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THEOREM 1.1. *Let f be a transcendental meromorphic function with*

$$\lim_{r\to\infty} \frac{T(r,f)}{(\log r)^5} = \infty. \tag{1-1}$$

Then there exist a sequence of increasing positive numbers R_n tending to infinity and a point $a \in I(f) \cap J(f)$ such that for each $n \in \mathbb{N}$,

$$\Gamma_n := \left\{ z : |z - f^n(a)| < \frac{16\pi}{\log\log R_n} |f^n(a)| \right\}$$

is a filling disk of f with index

$$m_n = c^* \frac{T(R_n, f)}{(\log\log R_n)^2 (\log R_n)^3}, \tag{1-2}$$

where c^* is an absolute constant.

For a domain Ω, by $n(r, \Omega, f = a)$, we denote the number of roots of $f = a$ counted with multiplicities in $\Omega \cap \{z : |z| < r\}$. Set

$$N(r, \Omega, f = a) = \int_1^r \frac{n(t, \Omega, f = a)}{t}\, dt.$$

Closely related to filling disks are singular directions, whose existence was established by Julia in 1924 and Valiron in 1928.

DEFINITION. A ray $\arg z = \theta$ is called a Borel direction of order $\rho \in (0, \infty]$ of a meromorphic function f if for arbitrary $\varepsilon > 0$ and any $c \in \hat{\mathbb{C}}$, with at most two exceptions,

$$\limsup_{r\to\infty} \frac{\log^+ N(r, Z_\varepsilon(\theta), f = c)}{\log r} \geq \rho, \tag{1-3}$$

where $Z_\varepsilon(\theta) = \{z : \theta - \varepsilon < \arg z < \theta + \varepsilon\}$; a ray $\arg z = \theta$ is called a Julia direction of f if (1-3) is replaced by

$$\lim_{r\to\infty} n(r, Z_\varepsilon(\theta), f = c) = \infty. \tag{1-4}$$

In this paper, by singular directions, we mean Borel directions or Julia directions. We note that singular directions determine the argument distribution of c-points, while a sequence of filling disks makes a further refinement of the location of c-points. In Definition 1, the order ρ is allowed to be ∞, that is, a Borel direction may be of infinite order. For background and knowledge about singular directions and filling disks, please refer to [15, Ch. 5] and [30, Ch. 3].

To clarify the statements of the results dealing with singular directions, let us introduce the notation as below. We denote by E' the limit set of subset E of $[0, 2\pi)$ and set

$$J_{\text{direct}}(f) = \{\theta \in [0, 2\pi] : \text{the ray } \arg z = \theta \text{ is a Julia direction of } f\}$$

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and

$$B_{\text{direct}}(f) = \{\theta \in [0, 2\pi] : \text{the ray } \arg z = \theta \text{ is a Borel direction of } f\}.$$

If we want to stress the order ρ of Borel directions, we write $B_{\text{direct}}(f;\rho)$. Additionally, $S_{\text{direct}}(f)$ is the set of arguments of all singular directions of *f*. We say that a point a goes under iterates to infinity (approximately) along singular directions if $\{\arg f^n(a)\}' \subseteq S_{\text{direct}}(f)$ and so on.

The following result was proved by Qiao [21], but is stated in our way.

THEOREM A. *Let f be a transcendental meromorphic function of lower order* $\lambda(f) \in (0, \infty)$. *If*

$$\mathcal{K} := \overline{\lim_{r\to\infty}} \frac{\log T(2r, f)}{\log T(r, f)} < \infty, \tag{1-5}$$

then there exists a point $a \in I(f) \cap J(f)$ *that goes under iterates along Borel directions of order* $\lambda(f)$, *that is,* $\{\arg f^n(a)\}' \subseteq B_{\text{direct}}(f;\lambda(f))$.

However, it is easy from Theorem 1.1 to obtain a corollary which generalizes Theorem A without the Assumption (1-5), that is to say, the condition in (1-5) is not necessary for Theorem A and deals with the case when $\lambda(f) = 0$ or ∞.

COROLLARY 1.2. *Let f be a transcendental meromorphic function. Then there is a dense subset I of* $I(f) \cap J(f)$ *such that for every point* $a \in I$:

(1) *if* $\lambda(f) = 0$ *and (1-1) holds, then* $\{\arg f^n(a)\}' \subseteq J_{\text{direct}}(f)$;
(2) *if* $0 < \lambda(f) < \infty$, *then* $\{\arg f^n(a)\}' \subseteq B_{\text{direct}}(f;\lambda(f))$;
(3) *if* $\lambda(f) = \infty$, *then* $\{\arg f^n(a)\}' \subseteq B_{\text{direct}}(f;\infty)$.

When $0 < \lambda(f) \le \infty$, (1-1) immediately holds and if (1-1) is only assumed, f may be of zero lower order. Therefore, we can directly use the result of Theorem 1.1 under the assumptions of Corollary 1.2. The quantity defined by the upper limit in (1-5) is not larger than $\rho(f)/\lambda(f)$. Therefore, if $0 < \lambda(f) \le \rho(f) < +\infty$, then (1-5) holds. However, basically, a meromorphic function f with infinite order $\rho(f) = \infty$ does not satisfy (1-5).

The case that a escapes along Borel directions of order $\rho(f)$ is not considered in the conclusions obtained above. The second main result is to determine the existence of escaping points which approximate all singular directions to infinity under iterates.

THEOREM 1.3. *Let f be a transcendental meromorphic function with* $0 < \lambda(f) \le \rho(f) \le +\infty$. *Then there is a dense subset I of* $I(f) \cap J(f)$ *such that for every point* $a \in I$,

$$\{\arg f^n(a)\}' = B_{\text{direct}}(f).$$

Actually, the set of such a is dense in $J(f)$. Theorem 1.3 tells us that we can determine locations of all Borel directions in terms of the orbit of some escaping points. Let us mention that Theorem 1.3 includes the cases when $\rho(f) = \infty$ or $\lambda(f) = \infty$. When $\lambda(f) = \infty$, f has only Borel directions of infinite order.

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In [26], Rippon and Stallard defined slow escaping points and proved their existence for a transcendental meromorphic function. The next result confirms the existence of slow escaping points which go to infinity along Borel directions of $\rho(f)$.

THEOREM 1.4. *Let f be a transcendental meromorphic function with* $\rho(f) > 0$ *satisfying for* $0 < c < 1$ *and all sufficiently large r,*

$$T(er, f) > \left(1 + \frac{1}{(\log r)^c}\right)T(r, f). \tag{1-6}$$

Then for any increasing positive sequence $\{a_n\}$ *tending to* ∞*, there exists a point* $a \in I(f) \cap J(f)$ *such that* $|f^n(a)| \leq a_n$ *and*

$$\{\arg f^n(a)\}' \subseteq B_{\text{direct}}(f; \rho(f)).$$

We wonder if the condition in (1-6) is necessary.

The second main purpose of this paper is to investigate the nonexistence of round annuli around the origin in the Fatou sets of meromorphic functions under the condition of argument distribution. It is often meaningful to judge the nonexistence of multiply connected wandering domains in the study of transcendental dynamics; see [5, 6, 8]. In terms of the distribution of filling disks, we establish the following.

THEOREM 1.5. *Let f be a transcendental entire function. If for* $0 < c < 1$*, (1-6) holds, then f has no multiply connected Fatou components.*

Theorem 1.5 is an improvement of [32, Corollary 5] and [34, Theorem 1.3] where, instead of (1-6), the condition that $T(er, f) > dT(r, f)$ with some $d > 1$ is assumed. Meromorphic functions satisfying such a condition are said to be of regular growth.

Theorem 1.5 need not hold if f has many poles, but under (1-6), we can still confirm the nonexistence of large annuli in the Fatou set (see the proof of Theorem 1.5). However, we need stronger conditions to set around narrow annuli in the Fatou set if the poles are not severely restricted.

Throughout this article, denote by $M(r, f)$ and $L(r, f)$ the maximum modulus and minimum modulus of f on circle $\{z : |z| = r\}$, respectively. By $A(r, R)$, we denote the annulus $\{z : \ r < |z| < R\}$.

THEOREM 1.6. *Let f be a transcendental meromorphic function with the lower order* $\lambda(f) = \infty$ *and*

$$\beta(f, \infty) := \varliminf_{r\to\infty} \frac{\log M(r, f)}{T(r, f)} > 0. \tag{1-7}$$

If there is a ray $\arg z = \theta$ *which is not a Borel direction of infinite order, then there exist* $\varepsilon(r) \to 0^+ (r \to \infty)$ *and* $R_0 > 0$ *such that* $F(f)$ *contains no annulus* $A(r, R)$ *with* $r > R_0$ *and* $R \geq (1 + \varepsilon(r))r$.

We make a remark on (1-7). Here, $\beta(f, \infty)$ is the quantity which was first introduced and studied by Petrenko [20] and others extensively. The Nevanlinna deficiency $\delta(\infty, f)$ of f at ∞ is

$$\delta(\infty, f) = \varliminf_{r\to\infty} \frac{m(r, f)}{T(r, f)}$$

and for $a \in \mathbb{C}$, $\delta(a, f) = \delta(\infty, 1/(f - a))$. Here, $a \in \hat{\mathbb{C}}$ is called a deficient value of f if $\delta(a, f) > 0$. The deficiency and deficient value are the main objects studied in the modulo distribution of the Nevanlinna theory. It is easily seen that $\beta(f, \infty) \geq \delta(\infty, f) = \lim_{\overline{r\to\infty}}(m(r, f)/T(r, f))$, and so if $\delta(\infty, f) > 0$, then (1-7) holds.

In fact, we establish the following general result, while Theorem 1.6 is a consequence of it for $\lambda = \infty$.

THEOREM 1.7. *Let f be a transcendental meromorphic function with the lower order λ and suppose that (1-7) holds. Assume that there are two distinct values a and b, and an angle $\Omega = \Omega(\alpha, \beta) = \{z : \alpha < \arg z < \beta\}$ such that $\lambda > \pi/(\beta - \alpha) = \omega$ and*

$$\varlimsup_{r\to\infty} \frac{\log(N(r, \Omega, f = a) + N(r, \Omega, f = b))}{\log T(r, f)} < 1 - \frac{\omega}{\lambda}. \tag{1-8}$$

Let $\phi(r)$ be a positive function in $[1, \infty)$ such that $\phi(r) \to \infty$ and $\phi(r)/\log T(r, f) \to 0$ as $r \to \infty$ and $\phi(r)r/\log T(r, f) > 2 \inf_{z\in J(f)} |z|$ for $r \geq 1$.

Then there exists $R_0 > 0$ such that the Fatou set $F(f)$ contains no round annulus $A(r, R)$ with $r > R_0$ and $R > (1 + \phi(r)/\log T(r, f))r$.

We make remarks on (1-8). It is easy to see that (1-8) follows from

$$\varlimsup_{r\to\infty} \frac{\log(N(r, \Omega, f = a) + N(r, \Omega, f = b))}{\log r} < \lambda - \omega. \tag{1-9}$$

This means that the convergence exponents of a-points and b-points in Ω are smaller than $\lambda - \omega$. However, (1-8) does not exclude the possibility that the convergence exponent equals to $\lambda - \omega$, that is, if $\lambda = \infty$, the exponent is allowed to be ∞. Additionally, the condition in (1-9) has something to do with the singular directions of meromorphic functions. If $\arg z = \theta$ is not a Borel direction of f with the order $\lambda - \omega$, then there exist an angle Ω containing the ray, and two values a and b such that (1-9) holds, but we do not know the size of the opening angle of Ω. However, we do not need to be concerned with the size for $\lambda(f) = \infty$. Therefore, Theorem 1.6 follows from Theorem 1.7. In view of a result of Valiron (see [33, Theorem 2.7.5]), if f has no Borel directions of order $\lambda - \omega$ in Ω, then (1-9) holds for some a and b. Hence, (1-8) can be replaced by the assumption that f has no Borel directions of order $\lambda - \omega$ in Ω.

For completeness, let us give an example to show that the condition in (1-7) is necessary.

THEOREM 1.8. *For any given $\lambda \in (1, \infty]$, there exists a meromorphic function of order and lower order equal to λ such that (1-8) holds on the upper half plane and lower half plane, and its Julia set lies on the real axis and its Fatou set contains a sequence of annuli $A(r_n, dr_n)$ with $r_n \to \infty$ as $n \to \infty$ for some $d > 1$.*

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The organization of this paper is the following. In Section 2, we make further remarks on the results stated above. In Section 3, we complete proofs of Theorems 1.1–1.5 in terms of knowledge of filling disks and an application of the considerable covering theorem of annuli. We provide proofs of Theorems 1.7 and 1.8 in Section 4 in terms of the Nevanlinna theory of angular domains and that covering theorem together with the hyperbolic metric.

2. Remarks on the results

This section is mainly devoted to further interpretation of our basic notation and the results obtained in this paper. Both the Fatou set and Julia set of a meromorphic function f are completely invariant under f, that is, $z \in F(f)$ (respectively $J(f)$) if and only if $f(z) \in F(f)$ (respectively $J(f)$). Let U be a connected component of $F(f)$, then $f^n(U)$ is contained in a component of $F(f)$, denoted by U_n. If for some positive integer p, $f^p(U) \subseteq U_p = U$, then U is called a periodic Fatou component of f and the smallest such integer p is the period of U. If for some $n > 0$, U_n is periodic, but U is not periodic, then U is called pre-periodic. If it is neither periodic nor pre-periodic, that is, $U_n \neq U_m$ for all pairs $n \neq m$, then U is called a wandering domain of f.

An introduction to the basic properties of these sets for a rational function can be found in [3, 19] and for transcendental meromorphic functions in the survey [4] or book [31].

Borel directions of zero order make no sense and so the Borel directions we mention in this paper have positive order or infinite order. For this case of Borel directions of zero order, actually, what we are talking about is the Julia directions instead or we give a more precise expression than (1-4). It is easily seen that a Borel direction must be a Julia direction. In 1924, Julia showed the existence of Julia directions for all entire functions and most meromorphic functions. If $T(r, f) \neq O((\log r)^2)$ $(r \to \infty)$, then f has at least one Julia direction (see [30, Theorem 3.6]). However, Ostrowski [15] found a meromorphic function with $T(r, f) = O((\log r)^2)$ $(r \to \infty)$ which has no Julia directions and hence no filling disks. The existence of Borel directions was first shown by Valiron in 1928. Borel directions stem from the Borel theorem (see [30, Theorem 1.8]) and Julia directions from the Picard theorem (see [30, Theorem 1.7]).

It is clear that a sequence of filling disks $B(z_n, \varepsilon_n|z_n|)$ with index m_n determines singular directions $\arg z = \theta$ whenever $\theta \in \{\arg z_n\}'$ if $z_n \to \infty$, $m_n \to \infty$, and $\varepsilon_n \to 0$ as $n \to \infty$; if $\overline{\lim}_{n\to\infty}(\log m_n/\log|z_n|) > 0$, then we obtain Borel directions with the order between $\underline{\lim}_{n\to\infty}(\log m_n/\log|z_n|)$ and $\overline{\lim}_{n\to\infty}(\log m_n/\log|z_n|)$, and at least one of the Borel directions decided has the order $\overline{\lim}_{n\to\infty}(\log m_n/\log|z_n|)$. Conversely, a Borel direction determines a sequence of filling disks with centers in the direction, which was shown by Rauch (see [30, Theorem 3.11]).

We make remarks on Theorem 1.1.

(A) As we know (see [30, Theorem 3.6]), the condition $T(r, f) \neq O((\log r)^2)$ $(r \to \infty)$ implies the existence of a sequence of filling disks with centers tending to infinity. Then we wonder if 5 in (1-1) could be replaced by 2.

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(B) We can require that $R_{n+1} > R_n^3$, $3R_n \geq |f^n(a)|$. From these together with m_n, we can say something about the rate of $f^n(a)$ escaping to infinity in Theorem 1.1.

(C) For any domain U intersecting $J(f)$ and any compact set K in $\mathbb{C}$ which contains no Fatou exceptional values (any meromorphic function has at most two Fatou exceptional values), there exists $m \in \mathbb{N}$ such that $K \subset f^m(U)$, which is a simple consequence of the Montel theorem (see [4]). Then we can choose in U the escaping point a in Theorem 1.1 and the set of all such a is dense in $J(f)$. We wonder about the size of the set, for example, its Hausdorff dimension.

(D) A meromorphic function f satisfying (1-1) may be of zero lower order, positive lower order, or infinite lower order. When $\lambda(f) = 0$ with (1-1), a goes under iterates to ∞ along Julia directions. When $\lambda(f) = \infty$, a goes under iterates to ∞ along Borel directions of infinite order. For these two cases, (1-2) offers more precise counting of c-points than (1-4) and (1-3) in Definition 1 of singular directions.

We proceed by remarking on the condition in (1-6). It is not essential that we choose the base e of natural logarithms in (1-6) and e can be replaced by any number greater than 1. Since $T(r, f)$ is nondecreasing and logarithmic convex, for large r, we always have

$$T(er, f) \geq \Big(1 + \frac{1}{\log r}\Big)T(r, f).$$

To some extent, this shows that (1-6) is not strong. It was proved in [26] that there exist slow escaping points for any transcendental meromorphic function. We wonder if the condition in (1-6) could be dropped.

The fast escaping set $A(f)$ of a transcendental entire function is introduced in [7] and can be defined in [25] by

$$A(f) = \{z \in I(f) : \text{there exists } L \in \mathbb{N} \text{ such that } |f^{n+L}(z)| \geq M_n(R, f)\},$$

where R is a fixed number and $R > \min\{|z| : z \in J(f)\}$ and $M_n(r, f)$ is the nth iterate of $M(r, f)$ with respect to r.

It is natural to ask whether there exists a fast escaping point which goes along singular directions. We guess that for an entire function with regular growth $\log M(er, f) > d \log M(r, f)$ and $d > 1$, such a fast escaping point may exist. For example, we consider the exponential function λe^z with $0 < \lambda < 1/e$. Its Julia set consists of uncountably many pairwise disjoint simple curves extending to ∞, called hairs (which was proved by Devaney and Krych [11]), and all points on the hairs possibly except their finite endpoints are fast escaping points (which was proved by Devaney and Tangerman [12] and Rempe *et al.* [24]). Therefore, a slow escaping point must be a finite endpoint of some hair, and the other points of the hair are fast escaping points and go to ∞ under iterates far away from the singular directions. Here, $f(z) = \lambda e^z$ has only two singular directions: the positive imaginary axis and the negative imaginary axis. For $-\pi/2 + \varepsilon < \arg z < \pi/2 - \varepsilon$ and $|z|$ large,

$$|f(z)| \geq \lambda e^{|z| \sin\varepsilon} = \lambda^{1-\sin\varepsilon} M(|z|, f)^{\sin\varepsilon} > M(|z|, f)^{1/2 \sin\varepsilon}.$$

Therefore, for a point $a \in I(f)$, if no limit points of $\{\arg f^n(a)\}$ are $\pm\frac{\pi}{2}$, in view of a result in [26], a must be in the fast escaping set of f. We can prove that there exist finite endpoints of hairs which are fast escaping to ∞ along the positive imaginary axis, that is, the argument of the iterate points tends to $\frac{\pi}{2}$. The Eremenko points [28] under iteration do not go along the singular directions and the maximally fast escaping points introduced by Sixsmith [29] go far away from the singular directions.

Finally, we remark on Theorem 1.5. In [1, 9, 32], for transcendental entire functions, it was proved that every multiply connected Fatou component U is wandering and for all sufficiently large n, $f^n(U)$ contains a round annulus $A(r_n, R_n)$ with $r_n \to \infty$, $R_n/r_n \to \infty$ as $n \to \infty$. This result was extended in [34] to some transcendental meromorphic functions with infinitely many poles. However, for a general meromorphic function, a multiply connected Fatou component may not be wandering. There exist meromorphic functions which have a sequence of large annuli in a periodic domain.

3. Proofs of Theorems 1.1–1.5

3.1. Some lemmas. The following result is natural; see [26, Lemma 1].

LEMMA 3.1. *Let f be a meromorphic function and let $\{E_n\}_{n=0}^{\infty}$ be a sequence of compact sets in $\mathbb{C}$. If*

$$E_{n+1} \subset f(E_n) \quad \text{for } n \geq 0,$$

then there exists ξ such that $f^n(\xi) \in E_n$ for $n \geq 0$. If $E_n \cap J(f) \neq \emptyset$ for $n \geq 0$, then ξ can be chosen to be in $J(f)$.

The following result can be extracted from the proof of [26, Lemmas 6 and 7].

LEMMA 3.2. *Let f be a meromorphic function. If there exist two sequences $\{B_m\}_{m=0}^{\infty}$ and $\{V_m\}_{m=0}^{\infty}$ of compact sets with $\operatorname{dist}(0, B_m) \to \infty$ and $\operatorname{dist}(0, V_m) \to \infty$ as $m \to \infty$, and a strictly increasing sequence of positive integers $\{m(k)\}$ such that*

$$B_{m+1} \subseteq f(B_m),\ B_{m(k)-p(k)} \subseteq f(V_k),\ V_k \subseteq f(B_{m(k)}), \tag{3-1}$$

where $0 \leq p(k) \leq M$ for a fixed integer $M > 0$, then for any increasing sequence of positive numbers $\{a_n\}$ with $a_n \to \infty (n \to \infty)$, there exists $\zeta \in I(f)$ such that for all sufficiently large n, $|f^n(\zeta)| \leq a_n$. If, in addition, $B_m \cap J(f) \neq \emptyset$ for all $m \geq 0$, then we can require $\zeta \in J(f)$.

PROOF. We choose a subsequence $\{a_{n(m)}\}$ of $\{a_n\}$ such that

$$B_p \subset B(0, a_{n(m)}) \quad \text{for } 0 \leq p \leq m, \text{ and } V_m \subset B(0, a_{n(m)}).$$

Inductively, we construct a sequence $\{s(k)\}$ of positive integers that is used to control the speed of iterates of f on B_m. Set $d(k) = s(k)p(k) + 2s(k)$ and $q(k) = d(0) + d(1) + d(2) + \cdots + d(k) = q(k-1) + d(k)$.

Define $s(0) = 0$. Suppose that we have $s(k-1)$, and so $d(k-1)$ and $q(k-1)$ are fixed. Take $s(k)$ such that

$$m(k) + q(k) > n(m(k+1)).$$

Let us construct a sequence $\{E_n\}$ of compact sets as follows:

$$\begin{aligned}
&E_0 = B_0,\ E_1 = B_1, \ldots, E_{m(1)} = B_{m(1)},\\
&E_{m(1)+1} = V_1,\\
&E_{m(1)+2} = B_{m(1)-p(1)}, \ldots,\ E_{m(1)+p(1)+2} = B_{m(1)},\\
&\ldots\ldots\\
&E_{m(1)+jp(1)+2j+1} = V_1,\\
&E_{m(1)+jp(1)+2j+2} = B_{m(1)-p(1)}, \ldots,\ E_{m(1)+(j+1)p(1)+2j+2} = B_{m(1)},\\
&E_{m(1)+(j+1)p(1)+2(j+1)+1} = V_1,\ 0 \le j \le s(1),\\
&E_{m(1)+q(1)} = B_{m(1)},\ q(1) = d(1) = s(1)p(1) + 2s(1),\\
&E_{m(1)+1+q(1)} = B_{m(1)+1}, \ldots,\ E_{m(2)+q(1)} = B_{m(2)},\\
&E_{m(2)+q(1)+1} = V_2,\\
&E_{m(2)+q(1)+2} = B_{m(2)-p(2)}, \ldots,\ E_{m(2)+q(1)+p(2)+2} = B_{m(2)},\\
&\ldots\ldots
\end{aligned}$$

that is to say, for $k \ge 0$,

$$E_n = \begin{cases} B_{n-q(k)}, & m(k) + q(k) \le n \le m(k+1) + q(k);\\ V_{k+1}, & n = m(k+1) + q(k) + jp(k+1) + 2j + 1;\\ B_{n-q(k)-(j+1)p(k+1)-2j-2}, & \begin{array}{l} m(k+1) + q(k) + jp(k+1) + 2j + 2\\ \le n < m(k+1) + q(k)\\ \quad + (j+1)p(k+1) + 2j + 1,\\ 0 \le j \le s(k+1). \end{array} \end{cases}$$

Then it is easy to see that $E_{n+1} \subseteq f(E_n)$. Since $m(k) - p(k) \to \infty$ $(k \to \infty)$ and $\operatorname{dist}(0, E_n) \to \infty$ $(n \to \infty)$, in view of Lemma 3.1, there exists a point $\zeta \in B_0 \cap I(f)$ such that $f^n(\zeta) \in E_n$.

For $m(k) + q(k) \le n \le m(k+1) + q(k)$,

$$E_n = B_{n-q(k)} \subset B(0, a_{n(m(k+1))}) \subset B(0, a_n)$$

by noting that $n(m(k+1)) < m(k) + q(k) \le n$. When $n = m(k+1) + q(k) + jp(k+1) + 2j + 1$, we have $n(k+1) < n(m(k+1)) < m(k) + q(k) < n$ and so

$$E_n = V_{k+1} \subset B(0, a_{n(k+1)}) \subset B(0, a_n).$$

For $m(k+1) + q(k) + jp(k+1) + 2j + 2 \le n < m(k+1) + q(k) + (j+1)p(k+1) + 2j + 1$, that is, $m(k+1) - p(k+1) \le n - q(k) - (j+1)p(k+1) - 2j - 2 \le m(k+1)$,

$$E_n = B_{n-q(k)-(j+1)p(k+1)-2j-2} \subset B(0, a_{n(m(k+1))}) \subset B(0, a_n).$$

We have proved that for all $n \ge m(1) + q(1)$, $E_n \subset B(0, a_n)$. □

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The result in Lemma 3.2 also holds if the condition in (3-1) is replaced by

$$B_{m+1} \subseteq f(B_m),\ B_{m(k)-p(k)} \subseteq f(B_{m(k)}).$$

To be able to find sequences $\{B_m\}$ and $\{V_m\}$ in Lemma 3.2, the following covering theorem of annulus plays a key role. For a hyperbolic domain U, by $\lambda_U(z)$, we denote the hyperbolic density of U at $z \in U$ and by $d_U(z_1, z_2)$, the hyperbolic distance between z_1 and z_2 in U.

LEMMA 3.3 [34, Theorem 2.2]. *Let f be analytic on a hyperbolic domain U with $0 \notin f(U)$. If there exist two distinct points z_1 and z_2 in U such that $|f(z_1)| > e^{\kappa\delta}|f(z_2)|$, where $\delta = d_U(z_1, z_2)$ and $\kappa = \Gamma(\frac{1}{4})^4/(4\pi)^2 = 4.376\,879\,6\ldots$, then there exists a point $\hat{z} \in U$ such that $|f(z_2)| \le |f(\hat{z})| \le |f(z_1)|$ and*

$$f(U) \supset A\bigg(e^{\kappa}\bigg(\frac{|f(z_2)|}{|f(z_1)|}\bigg)^{1/\delta}|f(\hat{z})|,\quad e^{-\kappa}\bigg(\frac{|f(z_1)|}{|f(z_2)|}\bigg)^{1/\delta}|f(\hat{z})|\bigg).$$

Moreover, if $|f(z_1)| \ge \exp(\kappa\delta/(1-\delta))|f(z_2)|$ and $0 < \delta < 1$, then

$$f(U) \supset A(|f(z_2)|, |f(z_1)|). \tag{3-2}$$

In particular, for $\delta \le \frac{1}{6}$ and $|f(z_1)| \ge e|f(z_2)|$, we have (3-2).

We need a result on the existence of filling disks.

LEMMA 3.4 (See [30, Lemma 3.4]). *Let f be a transcendental meromorphic function. Given arbitrarily $k > 1$, r, and R with $R > r$ satisfying*

$$T(R, f) \ge \max\bigg\{240, \frac{240\log(2R)}{\log k}, 12T(r, f), \frac{12T(kr, f)}{\log k}\log\frac{2R}{r}\bigg\},$$

then for large positive numbers q, there exists a point a with $r < |a| < 2R$ such that the disk

$$\Gamma : |z - a| < \frac{4\pi}{q}|a|$$

is a filling disk with index

$$m = c^*\frac{T(R, f)}{q^2(\log\frac{r}{R})^2}, \tag{3-3}$$

where $c^* > 0$ is an absolute constant.

There is $r_0 \ge 0$ such that $T(r, f) \equiv$ constant for $r \in [0, r_0)$ and $T(r, f)$ is strictly increasing in $[r_0, +\infty)$. Therefore, $T(r, f)$ is invertible in $[r_0, +\infty)$. We denote the inverse of $s = T(r, f)$ in $[r_0, +\infty)$ by $r = T^{-1}(s, f)$ with $s \ge T(r_0, f)$. For our purposes, we rewrite Lemma 3.4 as follows.

LEMMA 3.5. *Let f be a transcendental meromorphic function. Then there exists $R_0 > 0$ such that for $R > R_0$ and $q > R_0$, the annulus $A(r, 3R)$ contains a filling disk*

$$\Gamma : |z - a| < \frac{4\pi}{q}|a|$$

of f with index m given in (3-3) where

$$r = \frac{1}{2e}T^{-1}\Big(\frac{T(R, f)}{12\log R}, f\Big). \tag{3-4}$$

Since f is transcendental, it is clear that as $R \to \infty$, we have $(T(R, f)/12\log R) \to \infty$ so that r in (3-4) goes to ∞ as $R \to \infty$.

For the proof of Theorem 1.3, we need the following conclusion that asserts that a Borel direction decides the existence of a sequence of filling disks.

LEMMA 3.6. *Let f be a transcendental meromorphic function and $\arg z = \theta$ be a Borel direction of f of order μ with $0 < \mu \le \infty$. Then there exists a sequence of filling disks:*

$$\Gamma_1 := \{z : |z - z_j| < \varepsilon_j|z_j|\},\ z_j = |z_j|e^{i\theta},\ j = 1, 2, \ldots,$$

$$\lim_{j\to\infty} |z_j| = +\infty,\ \lim_{j\to\infty} \varepsilon_j = 0$$

with index $m_j = |z_j|^{\mu_j}$ for a sequence μ_j such that $\mu > \mu_j \to \mu - 0$ $(j \to \infty)$.

Lemma 3.6 is essentially due to Rauch; see [30, Theorem 3.11]. What we mention is that the original conclusion of Rauch does not deal with the case of infinite order $\mu = \infty$. However, we can get Lemma 3.6 without difficulty by making a small change of the proof of [30, Theorem 3.11].

Now we recall the Ahlfors–Shimizu characteristic of a meromorphic function; see [17]. By $f^{\#}(z)$, we mean the spherical derivative of f at z. For a closed domain D, define

$$\mathcal{A}(D, f) = \iint_D (f^{\#}(z))^2\, d\sigma(z),$$

where $\sigma(z)$ is the area element and write $\mathcal{A}(r, f)$ for $\mathcal{A}(\overline{B}(0, r), f)$. The Ahlfors–Shimizu characteristic of f is defined as

$$\mathcal{T}(r, f) = \int_0^r \frac{\mathcal{A}(t, f)}{t}\, dt.$$

Then,

$$|T(r, f) - \mathcal{T}(r, f) - \log^+ |f(0)|| \le \tfrac{1}{2}\log 2.$$

To confirm the existence of filling disks in view of Lemmas 3.4 and 3.5 in our proofs of theorems, we need to give a convenient version of (1-6).

LEMMA 3.7. *Let f be a transcendental meromorphic function. Assume that for a $0 < c < 1$ and all sufficiently large r, (1-6) holds. Then for large r,*

$$\frac{T(r^2, f)}{\log r^2} > 12\, T(er, f). \tag{3-5}$$

PROOF. Using the condition in (1-6) repeatedly, for $\sigma > 0$,

$$\begin{aligned} T(r^{1+\sigma}, f) = T(e^{\sigma \log r} r, f) &> \Big(1 + \frac{1}{((1+\sigma)\log r - 1)^c}\Big) T(e^{(1+\sigma)\log r - 1}, f) \\ &> \prod_{k=1}^{[\sigma \log r - 2]} \Big(1 + \frac{1}{((1+\sigma)\log r - k)^c}\Big) \cdot T(e^2 r, f) \\ &> \Big(1 + \frac{1}{((1+\sigma)\log r)^c}\Big)^{[\sigma \log r - 2]} T(e^2 r, f) \\ &> \exp \frac{[\sigma \log r - 2]}{((1+\sigma)\log r)^c + 1} \cdot T(e^2 r, f) \\ &> 12(1+\sigma)(\log r) T(e^2 r, f) \\ &> 12\, T(e^2 r, f), \end{aligned} \tag{3-6}$$

where $[x]$ denotes the integer part of x.

That is to say, when σ is chosen to be 1, we obtain (3-5) since $T(r, f)$ is increasing on r. □

Let us make a remark on (3-5). For an arbitrarily given large integer $N > 0$ and sufficiently large r, we have from (3-5) that

$$T(r, f) \geq \frac{6^N}{2^{2^N - 1}} (\log r)^N T(r^{1/2^N}, f).$$

Therefore,

$$\lim_{r \to \infty} \frac{T(r, f)}{(\log r)^N} = \infty. \tag{3-7}$$

The final lemma can be deduced using calculus.

LEMMA 3.8. *For $R > 0$, the function $\sqrt{(1 + x^2)/(1 + (x - R)^2)}$ is increasing in $[0, \frac{1}{2}(R + \sqrt{4 + R^2})]$ and decreasing in $[\frac{1}{2}(R + \sqrt{4 + R^2}), \infty)$ and*

$$\sqrt{\frac{1 + x^2}{1 + (x - R)^2}} \leq \frac{1}{2}(R + \sqrt{4 + R^2}) \quad \textit{for all } x > 0.$$

3.2. Proof of Theorem 1.1. Let R_0 be as in Lemma 3.5. Take R_1 with $q = \log \log R_1 > R_0$; in view of Lemma 3.5, there exists a filling disk Γ_1 in $A(r_1, 3R_1)$ with index

$$m_1 = c^* \frac{T(R_1, f)}{(\log\log R_1)^2(\log R_1)^2} > 6\log R_1$$

and r_1 defined by (3-4) with $R = R_1$. Note that $\chi(z,\infty) > \frac{3}{2}e^{-m_1}$ on $|z| = \frac{1}{2}e^{m_1}$ and the spherical diameter of the circle $|z| = \frac{1}{2}e^{m_1}$ is larger than $2e^{-m_1}$. According to the definition of filling disks, there exists a point $z_1 \in \Gamma_1$ such that $|f(z_1)| = \frac{1}{2}e^{m_1}$. Set

$$W_1 = \{z : 2 < |z| < \tfrac{1}{128}|f(z_1)|\}.$$

We need to treat three cases.

Case A. Assume that there exists a point $w_0 \in W_1$ such that $w_0 \notin f(5\Gamma_1)$ and f is analytic in $5\Gamma_1$. Here and henceforth, for a disk $\Gamma = B(a,r)$, we define $5\Gamma = B(a, 5r)$. Since $\operatorname{diam}_\chi(B(w_0, 2)) > 5e^{-m_1}$, there exists a point $z_0 \in \Gamma_1$ such that $|f(z_0) - w_0| \le 2$. Set $g(z) = f(z + w_0) - w_0$. Then $0 \notin g(H_1)$ with $H_1 = 5\Gamma_1 - w_0$. In view of Lemma 3.3, by noting that $d_{H_1}(z_0 - w_0, z_1 - w_0) = d_{5\Gamma_1}(z_0, z_1) \le \log 17/7 < 1$ and $|g(z_1 - w_0)| \ge |f(z_1)| - |w_0| \ge \frac{1}{2}|f(z_1)|$,

$$g(H_1) \supset A(2, |g(z_1 - w_0)|) \supset A(2, \tfrac{1}{2}|f(z_1)|).$$

Set $\hat{R}_2 = 1/32|f(z_1)|$. In view of Lemma 3.5, g has a filling disk $\hat{\Gamma}_2$ in $A(\hat{r}_2, 3\hat{R}_2)$ with index

$$\hat{m}_2 = c^* \frac{T(\hat{R}_2, g)}{(\log\log \hat{R}_2)^2(\log \hat{R}_2)^2}$$

and $\hat{r}_2 = 1/2eT^{-1}(T(\hat{R}_2, g)/12\log \hat{R}_2, g) > 32$. Thus, $5\hat{\Gamma}_2 \subset g(H_1)$.

By $\hat{\gamma}_{21}$ and $\hat{\gamma}_{22}$, we denote the two exceptional disks of g for the filling disk $\hat{\Gamma}_2$. Then $\gamma_{21} = \hat{\gamma}_{21} + w_0$ and $\gamma_{22} = \hat{\gamma}_{22} + w_0$ are the exceptional disks of f for $\Gamma_2 = \hat{\Gamma}_2 + w_0$. We can assume without any loss of generality that the center of γ_{21} is a finite number and ∞ is the center of γ_{22}. For any two points z_1 and z_2 in γ_{21}, set $\hat{z}_1 = z_1 - w_0$ and $\hat{z}_2 = z_2 - w_0$ in $\hat{\gamma}_{21}$. Then

$$\begin{aligned}
\chi(z_1, z_2) &= \frac{\sqrt{1+|\hat{z}_1|^2}\sqrt{1+|\hat{z}_2|^2}}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}\chi(\hat{z}_1, \hat{z}_2)\\
&\le \sqrt{\frac{1+|\hat{z}_1|^2}{1+(|\hat{z}_1| - |w_0|)^2}}\sqrt{\frac{1+|\hat{z}_2|^2}{1+(|\hat{z}_2| - |w_0|)^2}}\chi(\hat{z}_1, \hat{z}_2)\\
&\le \frac{1}{4}(|w_0| + \sqrt{4+|w_0|^2})^2\chi(\hat{z}_1, \hat{z}_2) < 3R_2^2\chi(\hat{z}_1, \hat{z}_2).
\end{aligned}$$

Additionally, for a point $\hat{z} \in \hat{\gamma}_{22}$, by noting that $\chi(\hat{z}, \infty) < e^{-\hat{m}_2}$, we have $|\hat{z}| > e^{\hat{m}_2} - 1 > R_2^3 > 2|w_0|$ and so for $z = \hat{z} + w_0 \in \gamma_{22}$,

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$$\chi(z,\infty) = \frac{\sqrt{1+|\hat{z}|^2}}{\sqrt{1+|z|^2}}\chi(\hat{z},\infty) \le \sqrt{\frac{1+|\hat{z}|^2}{1+(|\hat{z}|-|w_0|)^2}}\,\chi(\hat{z},\infty)$$
$$< \frac{|\hat{z}|}{|\hat{z}|-|w_0|}\chi(\hat{z},\infty) < 2\chi(\hat{z},\infty).$$

Set

$$m_2 = c^* \frac{T(R_2,f)}{5(\log\log R_2)^2(\log R_2)^3}, \tag{3-8}$$

with $R_2 = \frac{1}{4}\hat{R}_2$. Since $B(0,R_2) \subset B(0,\frac{1}{2}\hat{R}_2) + w_0$,

$$\mathcal{A}(\tfrac{1}{2}\hat{R}_2, f(z+w_0)) = \mathcal{A}(B(0,\tfrac{1}{2}\hat{R}_2) + w_0, f(z))$$
$$\ge \mathcal{A}(B(0,R_2), f(z)) = \mathcal{A}(R_2,f).$$

Therefore,

$$\begin{aligned}
T(\hat{R}_2,g) &= T(\hat{R}_2, f(z+w_0)-w_0)\\
&= N(\hat{R}_2, f(z+w_0)) + m(\hat{R}_2, f(z+w_0)-w_0)\\
&\ge T(\hat{R}_2, f(z+w_0)) - \log|w_0| - \log 2\\
&\ge T(\hat{R}_2, f(z+w_0)) - T\Big(\frac{1}{2}\hat{R}_2, f(z+w_0)\Big) - \log|w_0| - \log 2\\
&\ge \mathcal{T}(\hat{R}_2, f(z+w_0)) - \mathcal{T}\Big(\frac{1}{2}\hat{R}_2, f(z+w_0)\Big) - \log|w_0| - \frac{3}{2}\log 2\\
&= \int_{1/2\hat{R}_2}^{\hat{R}_2} \frac{\mathcal{A}(t, f(z+w_0))}{t}\,dt - \log|w_0| - \frac{3}{2}\log 2\\
&\ge \mathcal{A}\Big(\frac{1}{2}\hat{R}_2, f(z+w_0)\Big)\log 2 - \log|w_0| - \frac{3}{2}\log 2\\
&\ge \mathcal{A}(R_2,f)\log 2 - \log|w_0| - \frac{3}{2}\log 2\\
&\ge \frac{\log 2}{\log R_2}\int_1^{R_2} \frac{\mathcal{A}(t,f)}{t}\,dt - \log|w_0| - \frac{3}{2}\log 2\\
&= \frac{\log 2}{\log R_2} T(R_2,f) + O(1) - \log|w_0| - \frac{3}{2}\log 2\\
&\ge \frac{1}{2\log R_2} T(R_2,f).
\end{aligned}$$

This implies that $\hat{m}_2 > 2m_2$ and so $3R_2^2 e^{-\hat{m}_2} < 3R_2^2 e^{-2m_2} < e^{-m_2}$. Therefore, f has the filling disk Γ_2 with index m_2. Since $5\hat{\Gamma}_2 \subset g(H_1)$, we have $5\Gamma_2 - w_0 \subset f(5\Gamma_1) - w_0$ and so $5\Gamma_2 \subset f(5\Gamma_1)$.

Case B. Assume that $W_1 \subset f(5\Gamma_1)$ and f is analytic in $5\Gamma_1$. In view of Lemma 3.5, we have a filling disk Γ_2 of f in $A(r_2, 3R_2) \subset W_1$ with index m_2, where $R_2 = 1/640|f(z_1)|$, and $5\Gamma_2 \subset f(5\Gamma_1)$.

Case C. Assume that $5\Gamma_1$ contains a pole of f. Then for some $\hat{r}_2 > 0$, $\{z : |z| > \hat{r}_2\} \subset f(5\Gamma_1)$. Take a sufficiently large $R_2 > R_1^3$ such that $r_2 > 5\hat{r}_2$. In view of Lemma 3.5, there exists a filling disk Γ_2 of f in $A(r_2, 3R_2)$ with index m_2. Obviously, $5\Gamma_2 \subset A(r_2/5, 15R_2) \subset f(5\Gamma_1)$.

In one word, there exists a filling disk Γ_2 of f with index m_2 given in (3-8) where $R_2 > R_1^3$ and $5\Gamma_2 \subset f(5\Gamma_1)$. Proceeding step by step, we obtain a sequence of filling disks $\{\Gamma_n\}$ of f with index m_n given by (3-8) with R_2 replaced by R_n, and $R_n > R_{n-1}^3 \to \infty$ $(n \to \infty)$ and $5\Gamma_n \subset f(5\Gamma_{n-1})$. Since f is meromorphic on the complex plane, f takes any value at most finitely many times on any bounded subset of the complex plane and therefore, since $m_n \to \infty$ $(n \to \infty)$, we know that $\mathrm{dist}(\Gamma_n, 0) \to \infty$ $(n \to \infty)$. It is easily seen that for every n, $\Gamma_n \cap J(f) \neq \emptyset$. In view of Lemma 3.1, there exists a point $a \in I(f) \cap J(f)$ such that $f^n(a) \in 5\Gamma_n$. Of course, $5\Gamma_n$ is also a filling disk of f with index m_n.

We complete the proof of Theorem 1.1.

3.3. Proof of Theorem 1.3. From the proof of Theorem 1.1, we have a sequence of filling disks B_n such that $B_{n+1} \subset f(B_n)$ centered at $z_n \to \infty$ with index m_n, and having order greater than or equal to λ. Therefore, the spherical radius of the exceptional disks is $e^{-m_n} \to 0$ $(n \to \infty)$.

Noting that $B_{\mathrm{direct}}(f)$ is closed in $[0, 2\pi]$, we can choose a sequence of $\{\theta_p\}_{p=1}^N$ with $1 \leq N \leq +\infty$ such that the closure $\overline{\{\theta_p : p = 1, 2, \ldots, N\}} = B_{\mathrm{direct}}(f)$. In view of Lemma 3.6, for every θ_p, there exists a sequence of filling disks $\{A_{pj}\}_{j=1}^{\infty}$ with index m_{pj} centered at $z_{pj} = r_{pj}e^{i\theta_p}, j = 1, 2, \ldots$ with $r_{pj} \to \infty$ $(j \to \infty)$ and $m_{p(j+1)} > m_{pj}$. Additionally, we can require that $r_{p(j+1)} > 2r_{pj}$ and $r_{(p+1)1} > 2r_{p1} \to +\infty$ $(p \to \infty)$. Since the exceptional disks of A_{pj} have spherical radius at most $e^{-m_{pj}}$, choosing sufficiently large r_{p1}, we have that for every j, there exists a $B_{n(pj)}$ with $n(pj) \to \infty$ $(p \to \infty)$ such that

$$B_{n(pj)} \subset f(A_{pj}).$$

Now let us write A_{pj}, $j = 1, 2, \ldots, p = 1, 2, \ldots$ in the following order:

$$A_{11},\ A_{12},\ A_{21}, A_{13}, A_{22}, A_{31}, \ldots.$$

Take a sufficiently large n_0 and then one of A_{11}, A_{12}, A_{13} is in $f(B_{n_0})$, and denote it by C_{11}. Then $f(C_{11})$ contains one $B_{n_{11}}$. Take $B_{n_{11}+1}, \ldots, B_{n_{11}+p_{11}}$ such that $f(B_{n_{11}+p_{11}})$ contains one of A_{12}, A_{13}, A_{14}, and denote it by C_{12}. Then $f(C_{12})$ contains one B_{n_2}. Take $B_{n_{12}+1}, \ldots, B_{n_{12}+p_{12}}$ such that $f(B_{n_{12}+p_{12}})$ contains one of A_{21}, A_{22}, A_{23}, and denote it by C_{21}. We go on forever in this way to obtain a sequence of filling disks:

$$B_{n_0}$$
$$C_{11}, B_{n_{11}}, B_{n_{11}+1}, \ldots, B_{n_{11}+p_{11}},$$
$$\cdots$$
$$C_{sk}, B_{n_{sk}}, B_{n_{sk}+1}, \ldots, B_{n_{sk}+p_{sk}}$$
$$\ldots,$$

where C_{sk} is one of $A_{sk}, A_{s(k+1)}$, and $A_{s(k+2)}$. In view of Lemma 3.1, there exists a point $a \in I(f) \cap J(f)$ such that $f^n(a)$ goes along the sequence of the above filling disks. Then a satisfies our requirement.

3.4. Proof of Theorem 1.4. Under (1-6), we have (3-5). Take r_1 sufficiently large and set $R_1 = r_1^2$ such that

$$T(R_1, f) \geq \max\left\{240, \frac{240\log(2R_1)}{\log 2}, 12T(er_1, f)\log(2r_1)\right\},$$

$q_1 = \log r_1$ and, in view of (3-7),

$$c^* \frac{T(R_1, f)}{(\log r_1)^4} > 2\log(1 + 2R_1),$$

where c^* is the constant in Lemma 3.4. Applying Lemma 3.4, there exists a z_1 lying in annulus $\{z : r_1 < |z| < 2R_1\}$ such that

$$\Gamma_1 := \left\{z : |z - z_1| < \frac{4\pi}{q_1}|z_1|\right\}$$

is a filling disk of f with index

$$n_1 = c^* \frac{T(R_1, f)}{q_1^2(\log r_1)^2} = c^* \frac{T(R_1, f)}{(\log r_1)^4}.$$

Take r_2, $R_2 = r_2^2$ and $q_2 = \log r_2$ such that

$$r_2 > \frac{2R_1 + (1 + 2R_1)e^{-n_1}}{1 - e^{-n_1}(1 + 2R_1)}.$$

This implies that $\chi(r_2, 2R_1) > e^{-n_1}$, where $\chi(z, w)$ denotes the spherical distance between z and w. There exists a z_2 lying in the annulus $\{z : r_2 < |z| < 2R_2\}$ such that

$$\Gamma_2 := \left\{z : |z - z_2| < \frac{4\pi}{q_2}|z_2|\right\}$$

is a filling disk of f with index

$$n_2 = c^* \frac{T(R_2, f)}{(\log r_2)^4}.$$

Additionally, the spherical distance between Γ_1 and Γ_2 is at least $\chi(r_2, 2R_1) > e^{-n_1} > e^{-n_2}$.

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Using the same method, we have r_j, R_j, q_j, n_j and Γ_j $(j = 1, 2, \ldots, 5)$ such that the spherical distance between Γ_j and Γ_i with $i \neq j$ is larger than e^{-n_t} $(t = 1, 2, \ldots, 5)$.

Take $B_1 = \Gamma_1$. We can have $B_2 = \Gamma_i$ for some $2 \leq i \leq 4$ with $f(B_1) \supset B_2$ and then $B_3 = \Gamma_j$ for some $j \in \{2, 3, 4, 5\} \setminus \{i\}$ with $f(B_2) \supset B_3$. It is easy to see that $f(B_3) \supset B_1, B_2$ or B_3. Starting from B_3, we apply the same method to obtain B_4 and B_5 such that $f(B_j) \supset B_{j+1}$ with $j = 3, 4$ and $f(B_5) \supset B_3, B_4$, or B_5. Thus, we obtain a sequence of disks $\{B_n\}$ such that $f(B_n) \supset B_{n+1}$ and $f(B_{2n+1}) \supset B_{2n-1}, B_{2n}$, or B_{2n+1}.

We can take r_n such that $\lim_{n\to\infty}(\log T(R_n, f)/\log R_n) = \rho(f)$. Applying Lemma 3.2, we complete the proof of Theorem 1.4.

3.5. Proof of Theorem 1.5. It follows from (3-6) that

$$\frac{1}{2e}T^{-1}\Big(\frac{T(r^{1+\sigma}, f)}{12\log r^{1+\sigma}}, f\Big) > r.$$

In view of Lemma 3.5, for any $\sigma > 0$ and all sufficiently large r, the annulus $A(r^{1-\sigma}, r)$ contains a filling disk with index $m(r) \to \infty$ $(r \to \infty)$. Since $J(f)$ is nonempty and unbounded, $F(f)$ cannot contain any filling disks with large index so that given arbitrarily $0 < \sigma < 1$, for large r, under (1-6), $F(f)$ cannot contain any annulus $A(r^{1-\sigma}, r)$.

Suppose that f has a multiply connected Fatou component. Then there exists a sequence of annuli $\{A(r_n^{1-\sigma}, r_n)\}$ with $r_n \to \infty$ for some $0 < \sigma < 1$ in the Fatou set $F(f)$; see [9, Theorem 1.2] and [34, Theorem 1.1]. This derives a contradiction. Theorem 1.5 follows.

4. Proofs of Theorems 1.7 and 1.8

4.1. Some lemmas. We need the Nevanlinna characteristic in an angle; see [16, 33]. We set

$$\Omega(\alpha, \beta) = \{z : \alpha < \arg z < \beta\}$$

with $0 \leq \alpha < \beta \leq 2\pi$ and denote by $\overline{\Omega}(\alpha, \beta)$ the closure of $\Omega(\alpha, \beta)$. Let $f(z)$ be meromorphic on the angle $\overline{\Omega}(\alpha, \beta)$. We define

$$A_{\alpha,\beta}(r, f) = \frac{\omega}{\pi}\int_1^r \Big(\frac{1}{t^\omega} - \frac{t^\omega}{r^{2\omega}}\Big)\{\log^+|f(te^{i\alpha})| + \log^+|f(te^{i\beta})|\}\frac{dt}{t};$$

$$B_{\alpha,\beta}(r, f) = \frac{2\omega}{\pi r^w}\int_\alpha^\beta \log^+|f(re^{i\theta})|\sin\omega(\theta - \alpha)\,d\theta;$$

$$C_{\alpha,\beta}(r, f) = 2\sum_{1<|b_n|<r}\Big(\frac{1}{|b_n|^\omega} - \frac{|b_n|^\omega}{r^{2\omega}}\Big)\sin\omega(\beta_n - \alpha),$$

where $\omega = \pi/(\beta - \alpha)$ and $b_n = |b_n|e^{i\beta_n}$ are poles of $f(z)$ in $\overline{\Omega}(\alpha, \beta)$ appearing according to their multiplicities. Additionally, define $\overline{C}_{\alpha,\beta}(r, f)$ in the same form as $C_{\alpha,\beta}(r, f)$ for distinct poles b_n of $f(z)$, that is, ignoring their multiplicities. For $a \in \mathbb{C}$, we write $C_{\alpha,\beta}(r, f = a)$ for $C_{\alpha,\beta}(r, 1/(f - a))$. The Nevanlinna angular characteristic is defined as

$$S_{\alpha,\beta}(r, f) = A_{\alpha,\beta}(r, f) + B_{\alpha,\beta}(r, f) + C_{\alpha,\beta}(r, f).$$

LEMMA 4.1 [33, Lemma 2.2.2]. *Let $f(z)$ be a meromorphic function on $\overline{\Omega}(\alpha, \beta)$. Then we have the following:*

$$C_{\alpha,\beta}(r, f = a) \leq 4\omega \frac{N(r, \Omega, f = a)}{r^{\omega}} + 2\omega^2 \int_1^r \frac{N(t, \Omega, f = a)}{t^{\omega+1}}\, dt.$$

The inequality also holds for $a = \infty$.

LEMMA 4.2 [33, Equation (2.2.6) and Lemma 2.5.3]. *Let $f(z)$ be a meromorphic function. Then for any two distinct values a_1 and a_2 on $\mathbb{C}$,*

$$S_{\alpha,\beta}(r, f) \leq C_{\alpha,\beta}(r, f) + \sum_{\nu=1}^{2} \overline{C}_{\alpha,\beta}(r, f = a_\nu) + O(\log rT(r, f))$$

for all $r > 0$ with the possible exception of a finite-measure set of r.

We need the following lemma, which is established in terms of the hyperbolic metric.

LEMMA 4.3 [34, Theorem 2.4]. *Let $h(z)$ be an analytic function on the annulus $A(r, R) = \{z : r < |z| < R\}$ with $0 < r < R < \infty$ such that $|h(z)| > 1$ on $A(r, R)$. Then*

$$\log L(\rho, h) \geq \exp\Big(-\frac{\pi^2}{2} \max\Big\{\frac{1}{\log \frac{R}{\rho}}, \frac{1}{\log \frac{\rho}{r}}\Big\}\Big) \log M(\rho, h),$$

where $\rho \in (r, R)$ and $L(\rho, h) = \min\{|h(z)| : |z| = \rho\}$.

In Lemma 4.3, when $\rho = \sqrt{rR}$,

$$\log M(\rho, h) \leq \exp\Big(\frac{\pi^2}{\log \frac{R}{r}}\Big) \log L(\rho, h).$$

4.2. Proof of Theorem 1.7. In contrast, suppose that there exists a sequence of annuli $A_n = \{z : r_n < |z| < R_n\}$ in $F(f)$ with $R_n \geq (1 + \phi(r_n)/\log T(r_n, f))r_n$, $r_{n+1} > r_n$ and $r_n \to \infty$ $(n \to \infty)$. We treat two cases.

Case A. There exists a subsequence of A_n such that $f(A_n) \subset \{|z| > 1\}$. Without loss of generality, we assume that for all n, $f(A_n) \subset \{|z| > 1\}$. Set $\rho_n = \sqrt{R_n r_n}$. Since $|f(z)| > 1$ on A_n, using Lemma 4.3, we have

$$\log L(\rho_n, f) \geq \lambda_n \log M(\rho_n, f),$$

where $\lambda_n = \exp(-\pi^2/\log(R_n/r_n))$.

For the angular domain $\Omega(\alpha,\beta)$, according to the definition of $B_{\alpha,\beta}(\rho_n, f)$,

$$\begin{aligned}\rho_n^{\omega} B_{\alpha,\beta}(\rho_n, f) &= \frac{2\omega}{\pi}\int_{\alpha}^{\beta} \log^+ |f(\rho_n e^{i\phi})| \sin(\omega(\phi-\alpha))\, d\phi \\ &\geq \frac{2\omega}{\pi}\log L(\rho_n, f)\int_{\alpha}^{\beta} \sin(\omega(\phi-\alpha))\, d\phi \\ &= \frac{4}{\pi}\log L(\rho_n, f) \geq \frac{4\lambda_n}{\pi}\log M(\rho_n, f). \qquad (4\text{-}1)\end{aligned}$$

However, for any two distinct complex numbers a_v ($v = 1, 2$), we use Lemmas 4.1 and 4.2 in turn to obtain that

$$\begin{aligned}\rho_n^{\omega} B_{\alpha,\beta}(\rho_n, f) &\leq \rho_n^{\omega}(C_{\alpha,\beta}(\rho_n, f = a_1) + C_{\alpha,\beta}(\rho_n, f = a_2)) + O(\rho_n^{\omega}\log \rho_n T(\rho_n, f)) \\ &\leq 4\omega N(\rho_n) + 2\omega^2\rho_n^{\omega}\int_1^{\rho_n}\frac{N(t)}{t^{\omega+1}}\,dt + O(\rho_n^{\omega}\log\rho_n T(\rho_n, f)) \\ &\leq 4\omega N(\rho_n) + 2\omega\rho_n^{\omega}N(\rho_n) + O(\rho_n^{\omega}\log\rho_n T(\rho_n, f)) \\ &= (4\omega + 2\omega\rho_n^{\omega})N(\rho_n) + O(\rho_n^{\omega}\log\rho_n T(\rho_n, f)), \qquad (4\text{-}2)\end{aligned}$$

where $N(\rho_n) = N(\rho_n, \Omega, f = a_1) + N(\rho_n, \Omega, f = a_2)$. Combining (4-1), (4-2), and (1-7) yields

$$(4\omega + 2\omega\rho_n^{\omega})N(\rho_n) + O(\rho_n^{\omega}\log\rho_n T(\rho_n, f)) \geq \lambda_n \log M(\rho_n, f)) \geq K\lambda_n T(\rho_n, f)$$

for some positive constant K. Thus,

$$K\lambda_n T(\rho_n, f) \leq 2\max\{(4\omega + 2\omega\rho_n^{\omega})N(\rho_n), O(\rho_n^{\omega}\log\rho_n T(\rho_n, f))\}. \qquad (4\text{-}3)$$

It follows from the definition of lower order that

$$\lim_{n\to\infty}\frac{\log T(\rho_n, f)}{\log \rho_n} \geq \lambda$$

for the above sequence $\{\rho_n\}$. By noting that

$$\begin{aligned}\log\lambda_n &= -\frac{\pi^2}{\log(R_n/r_n)} \geq -\frac{\pi^2}{\log(1+\phi(r_n)/\log T(r_n, f))} \\ &\sim -\frac{\pi^2}{\phi(r_n)}\log T(r_n, f) \geq -\frac{\pi^2}{\phi(r_n)}\log T(\rho_n, f),\end{aligned}$$

we have

$$\underline{\lim}_{n\to\infty}\frac{\log \lambda_n}{\log T(\rho_n, f)} \geq 0.$$

In view of (4-3),

$$\lim_{n\to\infty}\frac{\log N(\rho_n)}{\log T(\rho_n, f)} \geq 1 - \frac{\omega}{\lambda}.$$

Then we get a contradiction for a and b.

Case B. Set

$$c_n = 1 + \frac{\phi(r_n)}{12 \log T(r_n, f)}.$$

When $c_n \leq 2$, we have $c_n^3 \leq 1 + \phi(r_n)/\log T(r_n, f)$. Consider the annulus $B_n = A(r_n, c_n^3 r_n) \subset A_n$ and $C_n = A(c_n r_n, c_n^2 r_n)$. In view of the implication in Case A, we can assume that for all n, $f(C_n) \cap \{|z| \leq 1\} \neq \emptyset$. Then there exist two points $z_0 \in C_n$ and z_n with $|z_n| = \rho_n = c_n^{3/2} r_n$, in view of (1-7), such that

$$|f(z_0)| = 1 + |c| \quad \text{and} \quad \log|f(z_n)| \geq KT(\rho_n, f) > \log \rho_n \quad \text{for } n \geq N,$$

where $c \in J(f)$ with $|c| = \min_{z \in J(f)} |z|$, K is a positive number, and N is a positive integer. Thus, for $n \geq N$ and $N \leq k \leq n$, $A_k \cap f(A_n) \neq \emptyset$. This implies that $\bigcup_{n=N}^{\infty} A_n \subset U$, where U is a Fatou component of f with $f(U) \subseteq U$.

A simple calculation yields

$$\lambda_{B_n}(z) \leq \frac{2\sqrt{3}\pi}{9 \log c_n} \frac{1}{|z|}, \quad z \in C_n \quad \text{and} \quad \lambda_{B_n}(z) = \frac{\pi}{3 \log c_n} \frac{1}{|z|}, \ |z| = \rho_n.$$

Then

$$d_{B_n}(z_0, z_n) \leq \int_{c_n r_n}^{\rho_n} \frac{2\sqrt{3}\pi}{9 \log c_n} \frac{dt}{t} + \int_0^{\pi} \frac{\pi}{3 \log c_n} \, d\theta \leq \frac{\sqrt{3}\pi}{9} + \frac{\pi^2}{3 \log c_n}$$

and

$$\frac{1}{\delta_n} \geq \frac{9 \log c_n}{\sqrt{3}\pi \log c_n + 3\pi^2} \geq \frac{\log c_n}{4 + \log c_n},$$

where $\delta_n = d_{B_n}(z_0, z_n)$. It follows that for sufficiently large n,

$$|f(z_n)| \geq \exp(KT(\rho_n, f)) > e^{\kappa \delta_n}(1 + 2|c|) + |c| = e^{\kappa \delta_n}(|f(z_0)| + |c|) + |c|.$$

Set $g(z) = f(z) - c$. Then $0 \notin g(B_n)$ and $|g(z_n)| \geq e^{\kappa \delta_n} |g(z_0)|$. In view of Lemma 3.3, we have $g(B_n) \supseteq A(d_n^{-1} t_n, d_n t_n)$ with $t_n \geq |g(z_0)| \geq 1$ and

$$d_n := e^{-\kappa} \Big(\frac{|g(z_n)|}{|g(z_0)|} \Big)^{1/\delta_n} \geq \exp\Big(-\kappa + \frac{\log c_n}{4 + \log c_n} \frac{K}{2} T(\rho_n, f) \Big) > \rho_n^5.$$

Thus,

$$f(B_n) \supseteq A(d_n^{-1} t_n, d_n t_n) + q \supseteq A(d_n^{-1} t_n + |c|, d_n t_n - |c|).$$

Set $s_n = \rho_n t_n \geq \rho_n$. Then

$$A(s_n, \rho_n^3 s_n) \subset A(d_n^{-1} t_n + |c|, d_n t_n - |c|).$$

This implies that the Fatou component U contains a sequence of annuli $D_n = A(s_n, \rho_n^3 s_n)$.

For a simple statement, we assume without loss of generality that $c = 0$. We can assume that $f(E_n) \cap \{z : |z| \leq 1\} \neq \emptyset$ with $E_n = A(\rho_n s_n, \rho_n^2 s_n)$. We can find two points $z_0' \in E_n$ and z_n' with $|z_n'| = \rho_n^{3/2} s_n$ such that

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$$|f(z_0')| = 1 \quad \text{and} \quad \log|f(z_n')| \geq KT(\rho_n^{3/2}s_n, f) > 2\log r_n.$$

Additionally,

$$d_{D_n}(z_0', z_n') \leq \frac{\sqrt{3}\pi}{9} + \frac{\pi^2}{3\log\rho_n} < \frac{9}{10}.$$

Therefore, in view of Lemma 3.3, $f(D_n) \supset A(|f(z_0')|, f(z_n')|) \supset A(1, r_n^2)$ and furthermore, $A(1, r_n^2) \subset U$ and so $A(1, R_n) \subset U$. This implies that $U \supset \{z : |z| > 1\}$. A contradiction is derived.

From Cases A and B, the Fatou set $F(f)$ contains no annuli mentioned in Theorem 1.7.

4.3. Proof of Theorem 1.8. Consider the meromorphic function with the following form:

$$f(z) = cz + d + \sum_{n=1}^{\infty} c_n\Big(\frac{1}{a_n - z} - \frac{1}{a_n}\Big),$$

where c, d, c_n, and a_n are real numbers with $a_n \to \infty (n \to \infty)$, $c \geq 0$, and $c_n > 0$ such that

$$\sum_{n=1}^{\infty} \frac{c_n}{a_n^2} < +\infty.$$

For such a function f, the real axis is completely invariant under f and so $J(f)$ is completely on the real axis, see [2].

Set $r_n = M2^n$ for a large $M > 0$. For a given real number $\lambda > \frac{1}{2}$, define $r_{n,k} = r_{n+1} - 1 + k/[r_n^\lambda]$, $1 \leq k \leq [r_n^\lambda]$ for each n, where $[x]$ is the maximal integer not greater than x. We prove that the function

$$g(z) = \sum_{n=1}^{\infty} \frac{1}{[r_n^\lambda]} \sum_{k=1}^{[r_n^\lambda]} \frac{2z}{r_{n,k}^2 - z^2} = \sum_{n=1}^{\infty} \frac{1}{[r_n^\lambda]} \sum_{k=1}^{[r_n^\lambda]} \Big(\frac{1}{r_{n,k} - z} - \frac{1}{r_{n,k} + z}\Big)$$

satisfies the requirement of Theorem 1.8.

Given an arbitrarily large real number r, we have $r_n \leq r < r_{n+1}$ for some n. Then,

$$\begin{aligned} n(r, f) &= \sum_{k=1}^{n-1}[r_k^\lambda] \leq r_n^\lambda \sum_{k=1}^{n-1}\Big(\frac{r_k}{r_n}\Big)^\lambda \\ &= r^\lambda \sum_{k=1}^{n-1}\Big(\frac{1}{2^{n-k}}\Big)^\lambda \\ &\leq \frac{r^\lambda}{2^\lambda - 1} \end{aligned}$$

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and

$$n(r,f) \geq \sum_{k=1}^{n-1}(r_k^\lambda - 1) = \frac{r_n^\lambda}{2^\lambda - 1} - n$$
$$= \frac{r_{n+1}^\lambda}{2^\lambda(2^\lambda - 1)} - \frac{\log(r_n/M)}{\log 2}$$
$$> \frac{r^\lambda}{2^\lambda(2^\lambda - 1)} - \frac{\log(r/M)}{\log 2}.$$

For $z \in A(\frac{4}{3}r_n, \frac{5}{3}r_n)$,

$$|g(z)| \leq \sum_{m=1}^{n-1}\frac{1}{[r_m^\lambda]}\sum_{k=1}^{[r_m^\lambda]}\frac{2|z|}{|z|^2 - r_{m,k}^2} + \sum_{m=n}^{\infty}\frac{1}{[r_m^\lambda]}\sum_{k=1}^{[r_m^\lambda]}\frac{2|z|}{r_{m,k}^2 - |z|^2} = I_1 + I_2(\text{say}).$$

We estimate

$$I_1 \leq \sum_{m=1}^{n-1}\frac{1}{[r_m^\lambda]}\sum_{k=1}^{[r_m^\lambda]}\frac{30r_n}{16r_n^2 - 9r_{m,k}^2}$$
$$\leq \frac{30nr_n}{16r_n^2 - 9r_n^2} = \frac{30n}{7r_n}.$$

Since $r_{m,k} \geq r_{m+1} - 1 = 2r_m - 1$, for $m \geq n$,

$$9r_{m,k}^2 - 25r_n^2 \geq 9(2r_m - 1)^2 - 25r_n^2 > 10r_m^2$$

so that

$$I_2 \leq \sum_{m=n}^{\infty}\frac{1}{[r_m^\lambda]}\sum_{k=1}^{[r_m^\lambda]}\frac{30r_n}{9r_{m,k}^2 - 25r_n^2}$$
$$< \sum_{m=n}^{\infty}\frac{30r_n}{10r_m^2} \leq 3r_n\sum_{m=n}^{\infty}r_m^{-2} = \frac{3}{r_n}\sum_{m=n}^{\infty}\left(\frac{r_n}{r_m}\right)^2 < \frac{3}{r_n}.$$

Therefore,

$$T(3r_n/2, g) = N(3r_n/2, g) + m(3r_n/2, g) < n(3r_n/2, g)\log(3r_n/2) + 2$$

and

$$T(3r_n/2, g) = N(3r_n/2, g) + m(3r_n/2, g) > n(r_n, g)\log(3/2) - 2,$$

and so

$$\lim_{n\to\infty}\frac{\log T(3r_n/2, g)}{\log(3r_n/2)} = \lambda.$$

This easily implies that g has order and lower order equal to λ.

Obviously, 0 is an attracting fixed point of g and we can choose a large M such that $B(0, M)$ is in its attracting basin. For all sufficiently large n, the annulus $A(4r_n/3, 5r_n/3)$

is mapped into $B(0, M)$. Therefore, the Fatou set $F(g)$ contains a sequence of annuli $A(4r_n/3, 5r_n/3)$ with $r_n \to \infty$ $(n \to \infty)$. Since $J(g)$ lies in the real axis and $r_1, r_2 \in J(g)$, g cannot take on the values r_1 and r_2 on the upper half plane and lower half plane.

We omit the proof of the case when $\lambda = \infty$.

5. Remarks

In the proofs of Theorems A and 1.4, the existence of so-called filling disks is a key point.

From the proof of [21, Theorem A], we observe that if $0 < \lambda(f) < \infty$, then there exist $R_0, \tau > 1$, and $N > 0$ such that for all $r > R_0$, the annulus $A(r/4, 3\tau T^{-1}(T^N(r)))$ contains a filling disk $D := \{z : |z - z_0| < (4\pi/\log r)|z_0|\}$ of f with index $m = c^*(T(R)/(\log R)^2)$, where c^* is a positive constant, $T(r) = T(r, f)$, and $\tau r < R < \tau T^{-1}(T^N(r))$.

From the proof of Theorem 1.4, we see that the annulus $A(r, 2r^2)$ for $r \geq R_0$ contains a filling disk $D := \{z : |z - z_0| < (4\pi/\log r)|z_0|\}$ with index $m = c^*(T(r^2)/(\log r)^4)$. However, for any two sequences $\{r_n\}$ and $\{R_n\}$ of positive numbers with $r_n < R_n < r_{n+1}$, there exists a transcendental meromorphic function which has no filling disks in annulus $A(r_n, R_n)$ with index $m_n \to \infty$ $(n \to \infty)$. For an example which has a sequence of large annuli in the Fatou set, see [34]. A transcendental entire function f with this property can be found in [10]. It is obvious that the Fatou set cannot contain a filling disk with large enough index m. This is an interesting contrast between the iterate theory and the value distribution of meromorphic functions.

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