CORRESPONDENCE.

ON THE GENERAL EXPRESSION FOR THE FORCE OF MORTALITY.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—The general expression for the value of the force of mortality given by you, but not demonstrated, in a foot-note to Mr. Sheppard's communication in the last number of the *Journal* (*J.I.A.*, xxxii, 295), appears to me to be of sufficient importance to call for a formal proof, and I therefore venture to communicate the following.

Let the value of l_x be expressed, by means of Lagrange's interpolation formula, in terms of $l_{-n} \ldots \ldots l_0 \ldots \ldots l_{+n}$; we shall have

$$l_x = \Sigma_{k=-n}^{k=+n}\left[l_k \frac{\overline{x-(-n)} \ldots\ldots\ldots \overline{x-(+n)}, \text{ omitting } x-k}{\overline{k-(-n)} \ldots\ldots\ldots \overline{k-(+n)}, \text{ omitting } k-k}\right]$$

The general term, whether k be $+$, 0 or $-$, may be put into the form

$$l_k \frac{\dfrac{\overline{x+n} \ldots\ldots \overline{x-n}}{x-k}}{\underline{|n+k}\ \underline{|n-k}} (-1)^{n-k}$$

whence

$$\frac{l_0-l_x}{x} = \frac{l_0}{x} - \Sigma_{k=-n}^{k=+n}\left[l_k \frac{\dfrac{\overline{x+n} \ldots\ldots \overline{x-n},}{x\ \overline{x-k}}}{\underline{|n+k}\ \underline{|n-k}} (-1)^{n-k}\right]$$

$$= \frac{l_0}{x} + \Sigma_{k=-n}^{k=+n}\left[l_k \frac{\dfrac{\overline{x+n} \ldots\ldots \overline{x-n},}{x\ \overline{x-k}}}{\underline{|n+k}\ \underline{|n-k}} (-1)^{n-k+1}\right] \quad \ldots\ldots \quad (a)$$

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Put $x=0$; then $\frac{l_0 - l_x}{x} = -\frac{d}{d_x} l_0 = l_0 \times \mu_0$, and for all values of k, except zero, the general term in the expression (a) becomes

$$l_k \frac{\frac{\underline{|n} \times \underline{|n} \times (-1)^n}{-k}}{\underline{|n+k}\ \underline{|n-k}} (-1)^{n-k+1} = \frac{l_k}{k} \cdot \frac{(\underline{|n})^2}{\underline{|n+k}\ \underline{|n-k}} (-1)^{2n-k}$$

The term involving l_0 becomes, by adding the two coefficients which appear in (a),

$$\frac{l_0}{\underline{|n}\ \underline{|n}} \times Lt_{x=0} \left[\frac{\overline{x+n} \ldots\ldots \overline{x+1} \cdot \overline{x-1} \ldots\ldots \overline{x-n} (-1)^{n+1} + (\underline{|n})^2}{x} \right]$$

which takes the form $\frac{0}{0}$, but it may be shown that when $x=0$ the term vanishes (*). We therefore have

$$\mu_0 = \Sigma \frac{l_k}{k l_0} \cdot \frac{(\underline{|n})^2}{\underline{|n+k}\ \underline{|n-k}} (-1)^{2n-k}, \text{ omitting } l_0.$$

Now we have, since any even power of (-1) is equal to unity

$$\frac{(-1)^{2n-k}}{k} = -\frac{(-1)^{2n-(-k)}}{-k}$$

i.e., the coefficient of l_{+k} is the same as that of l_{-k} with the sign changed; hence the terms may be arranged in pairs. It is also evident that the signs are alternately + and —, and we shall have finally

$$\mu_0 = \frac{l_{-1} - l_{+1}}{l_0} \frac{(\underline{|n})^2}{\underline{|n+1}\ \underline{|n-1}} - \frac{l_{-2} - l_{+2}}{2 l_0} \frac{(\underline{|n})^2}{\underline{|n+2}\ \underline{|n-2}} + \ \ldots\ldots \qquad (\beta)$$

(*) $\overline{x+n} \ldots\ldots \overline{x+1}\ \overline{x-1} \ldots\ldots \overline{x-n}\ (-1)^{n+1}$

$= \overline{x^2 - n^2}\ \overline{x^2 - \overline{n-1}^2} \ldots\ldots \overline{x^2 - 1^2}\ (-1)^{n+1}$

$= \overline{n^2 - x^2}\ \overline{\overline{n-1}^2 - x^2} \ldots\ldots \overline{1^2 - x^2} \times (-1)$

$= -[(\underline{|n})^2 +$ terms involving x^2, &c.].

Thus, when $x=0$ the fraction of which we require the limiting value is in the form $\frac{\text{Terms involving } x^2, \text{\&c.}}{x}$, and the limiting value is therefore zero.

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In Mr. Sheppard's formulæ the numerical coefficient of $\dfrac{l_{-k}-l_{+k}}{l_0}$ is given in the form

$$\frac{\dfrac{n!}{k!\,(n-k)!}}{(k+n)\ldots\ldots(n+1)}(k-1)! = \frac{\dfrac{n!}{k!\,(n-k)!}}{\dfrac{(n+k)!}{n!}}(k-1)! = \frac{(n!)^2}{(n+k)!\,(n-k)!}\,\frac{1}{k}$$

as given above.

The formula having been obtained by the use of $2n+1$ values of l will be correct to $(2n)$th differences.

I am, Sir,

Your obedient servant,

GEORGE J. LIDSTONE.

Bartholomew Lane, E.C.,
February 1896.