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RUIN EXCURSIONS, THE G/G/∞ QUEUE, AND TAX PAYMENTS IN RENEWAL RISK MODELS

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RUIN EXCURSIONS, THE G/G/∞ QUEUE, AND TAX PAYMENTS IN RENEWAL RISK MODELS

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Abstract

In this paper we investigate the number and maximum severity of the ruin excursion of the insurance portfolio reserve process in the Cramér–Lundberg model with and without tax payments. We also provide a relation of the Cramér–Lundberg risk model with the G/G/∞ queue and use it to derive some explicit ruin probability formulae. Finally, the renewal risk model with tax is considered, and an asymptotic identity is derived that in some sense extends the tax identity of the Cramér– Lundberg risk model.

Keywords: Classical risk model; ruin probability; G/G/∞ queue; tax; renewal model

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1. Introduction

Consider the classical Cramér–Lundberg model in risk theory to describe the surplus process $\{R_t\}$ at time t of an insurance portfolio. Starting with an initial capital x, premium is collected according to a constant premium intensity (normalized to) 1. Claims occur according to a homogeneous Poisson process with intensity λ and are paid at the times of their occurrence. The claim sizes are independent and identically distributed random variables with distribution function $H(\cdot)$. Define $\phi_0(x) = \mathrm{P}(R_t \geq 0 \text{ for all } t \mid R_0 = x)$ as the probability of survival and correspondingly define the ruin probability as $\psi_0(x) = 1 - \phi_0(x)$. Furthermore, let $V_{\max}$ be the maximum workload in an M/G/1 queue with arrival rate λ and service time distribution $H(\cdot)$. Then the following relation between the Cramér–Lundberg risk model and the M/G/1 queueing model is classical:

$$\phi_0(x) = \exp\bigg(-\lambda \int_x^\infty \mathrm{P}(V_{\max} > y)\mathrm{d}y\bigg). \tag{1.1}$$

Let $G(\cdot)$ denote the distribution function of $V_{\max}$. One way to show (1.1) is to use the well-known relation

$$G(u) = \mathrm{P}(V_{\max} \leq u) = 1 - \frac{1}{\lambda}\frac{\mathrm{d}}{\mathrm{d}u}\ln \mathrm{P}(V \leq u), \tag{1.2}$$

where V is the stationary workload in the same M/G/1 queue as described above, and use the sample path duality result $\phi_0(x) = \mathrm{P}(V \leq x)$ (see, e.g. [5, Chapter III.2] for a recent survey). In [4] another more direct proof of (1.1) was given and subsequently used to establish a simple proof of the tax identity

$$\phi_\gamma(x) = (\phi_0(x))^{1/(1-\gamma)} = \exp\bigg(-\frac{\lambda}{1-\gamma}\int_x^\infty \mathrm{P}(V_{\max} > y)\mathrm{d}y\bigg), \tag{1.3}$$

Dedicated to Søren Asmussen, with much appreciation and deep admiration.

where $\phi_\gamma(x) = 1 - \psi_\gamma(x)$ is the survival probability in a Cramér–Lundberg model with tax rate $0 \le \gamma < 1$, i.e. whenever the risk process is in its running maximum (and, hence, in a profitable position), a constant proportion γ of the incoming premium is paid as tax ($\gamma = 0$ corresponds to the Cramér–Lundberg model without tax). For extensions of this identity in various directions, see [1], [2], [3], [6], and [9].

In this paper we will provide a relation between the Cramér–Lundberg risk model and the G/G/∞ queue, which will give rise to another view towards identity (1.1) and some explicit ruin probability formulae. Subsequently, we will consider the renewal risk model with tax, and establish an asymptotic identity that may be interpreted as an extension of the tax identity (1.3). We start with some refined results on the number and maximum severity of the ruin excursion in the Cramér–Lundberg model with and without tax.

2. Maximum severity of the ruin excursion

Consider the Cramér–Lundberg model with tax rate γ. Ruin can only occur during an 'interruption', i.e. a period in between running maxima. Denote the kth interruption period by P_k. Interruptions occur according to a Poisson process with intensity λ. The probability that no ruin occurs during an interruption that starts at surplus level z is given by $G(z) = 1 - \bar{G}(z)$ (cf. (1.2)). Let $R_{\min}$ be the lowest surplus value during the ruin excursion. Furthermore, let $A_k(x, d)$ be the probability that ruin occurs during the kth interruption P_k and $R_{\min} \le -d$, where $d \ge 0$. Then, for $k \in \mathbb{N}$,

$$A_k(x,d) = \int_{t=0}^{\infty} \lambda^k \frac{t^{k-1}}{(k-1)!} \mathrm{e}^{-\lambda t} \left(\int_{v=0}^{t} G(x+(1-\gamma)v) \frac{\mathrm{d}v}{t} \right)^{k-1} \bar{G}(x+(1-\gamma)t+d)\,\mathrm{d}t. \quad (2.1)$$

Here we have used the fact that the sum of k independent exponential arrival intervals is Erlang(k, λ) distributed, and given that their sum is t, the interruption epochs are uniformly distributed on $[0, t]$.

Proposition 2.1. *Let $A(x, d)$ be the probability that ruin occurs and the lowest surplus value of the ruin excursion is smaller than $-d \le 0$. Then*

$$A(x,d) = \int_x^\infty \frac{\phi'_\gamma(w+d)}{\phi_\gamma(w+d)} \frac{\phi_\gamma(x)}{\phi_\gamma(w)}\,\mathrm{d}w.$$

Proof. We have

$$\begin{aligned} A(x,d) &= \sum_{k=1}^{\infty} A_k(x,d) \\ &= \int_{t=0}^{\infty} \lambda \mathrm{e}^{-\lambda t} \bar{G}(x+(1-\gamma)t+d) \exp\left(\lambda \int_{v=0}^{t} G(x+(1-\gamma)v)\,\mathrm{d}v \right) \mathrm{d}t \\ &= \int_{t=0}^{\infty} \lambda \bar{G}(x+(1-\gamma)t+d) \exp\left(-\lambda \int_{v=0}^{t} \bar{G}(x+(1-\gamma)v)\,\mathrm{d}v \right) \mathrm{d}t. \end{aligned} \quad (2.2)$$

Now the result follows from (1.2) and (1.3).

Remark 2.1. Clearly, $d = 0$ gives $A(x, 0) = 1 - \phi_\gamma(x) = \psi_\gamma(x)$, so that in this case we indeed recover the usual ruin probability.

Remark 2.2. An alternative way to establish (2.2) is to use the joint distribution of the maximum surplus before ruin, $R_{\max} = \sup_{t\geq 0} R_t \, \mathbf{1}_{\{R_u \geq 0 \text{ for all } u\in[0,t]\}}$, and the maximum deficit of the ruin excursion, $R_{\min}$. Concretely,

$$\begin{aligned}
&\mathrm{P}(R_{\max} \in [y, y+\mathrm{d}y]; R_{\min} \leq -d) \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}y}\bigg[1 - \exp\bigg(-\frac{\lambda}{1-\gamma}\int_{v=x}^{y} \mathrm{P}(V_{\max} > v)\,\mathrm{d}v\bigg)\bigg]\mathrm{P}(V_{\max} > y+d \mid V_{\max} > y) \\
&\quad = \frac{\lambda}{1-\gamma}\mathrm{P}(V_{\max} > y+d)\exp\bigg(-\frac{\lambda}{1-\gamma}\int_{v=x}^{y} \mathrm{P}(V_{\max} > v)\,\mathrm{d}v\bigg),
\end{aligned}$$

which also yields (2.2) upon integration over $y \geq x$. Note in addition that the time spent in the running maximum until ruin is given by $(R_{\max} - x)/(1-\gamma)$.

Proposition 2.2. *The generating function* $\Phi(z, x, d) := \sum_{k=1}^{\infty} z^k A_k(x, d)$ *is given by*

$$\Phi(z, x, d) = z\int_{x}^{\infty} \frac{\phi'_\gamma(w+d)}{\phi_\gamma(w+d)}\bigg(\frac{\phi_\gamma(x)}{\phi_\gamma(w)}\bigg)^{z} \mathrm{e}^{-\lambda(1-z)(w-x)/(1-\gamma)}\,\mathrm{d}w. \tag{2.3}$$

Proof. From (2.1), it follows that

$$\begin{aligned}
\Phi(z, x, d) &= z\int_{t=0}^{\infty} \lambda \mathrm{e}^{-\lambda t}\exp\bigg(z\lambda\int_{v=0}^{t} G(x+(1-\gamma)v)\,\mathrm{d}v\bigg)\bar{G}(x+(1-\gamma)t+d)\,\mathrm{d}t \\
&= z\int_{t=0}^{\infty} \lambda\bar{G}(x+(1-\gamma)t+d)\mathrm{e}^{-\lambda(1-z)t} \\
&\qquad\qquad \times \exp\bigg(-\lambda z\int_{v=0}^{t} \bar{G}(x+(1-\gamma)v)\,\mathrm{d}v\bigg)\,\mathrm{d}t,
\end{aligned}$$

so that the assertion again follows from (1.2) and (1.3).

Denote by K the number of the interruption that leads to ruin (K is a defective random variable on the positive integers). Then starting at (2.3) with $d = 0$, some elementary calculations lead to the following result.

Corollary 2.1. *We have*

$$\begin{aligned}
&\mathrm{E}[K \mid \text{ruin occurs with } R_0 = x] \\
&\quad = \frac{1}{\psi_\gamma(x)}\frac{\partial\Phi(z, x, 0)}{\partial z}\bigg|_{z=1} \\
&\quad = \ln\phi_\gamma(x)\bigg(1 - \frac{1}{\psi_\gamma(x)}\bigg) - \frac{\lambda}{1-\gamma}\bigg(x - \frac{\phi_\gamma(x)}{\psi_\gamma(x)}\int_{x}^{\infty} \frac{w\phi'_\gamma(w)}{\phi_\gamma^2(w)}\,\mathrm{d}w\bigg).
\end{aligned}$$

On the other hand, we may rewrite (2.1) as

$$\begin{aligned}
A_k(x, d) &= \int_{t=0}^{\infty} \frac{\lambda}{(k-1)!}\mathrm{e}^{-\lambda t}\bigg(\lambda\int_{v=0}^{t} G(x+(1-\gamma)v)\,\mathrm{d}v\bigg)^{k-1}\bar{G}(x+(1-\gamma)t+d)\,\mathrm{d}t \\
&= \int_{t=0}^{\infty} \frac{\mathrm{e}^{-\lambda t}}{(k-1)!}\bigg(\lambda t - \int_{v=0}^{t} \frac{\phi'_0(x+(1-\gamma)v)}{\phi_0(x+(1-\gamma)v)}\,\mathrm{d}v\bigg)^{k-1}\frac{\phi'_0(x+(1-\gamma)t+d)}{\phi_0(x+(1-\gamma)t+d)}\,\mathrm{d}t \\
&= \int_{t=0}^{\infty} \frac{\mathrm{e}^{-\lambda t}}{(k-1)!}\bigg(\lambda t - \ln\frac{\phi_\gamma(x+(1-\gamma)t)}{\phi_\gamma(x)}\bigg)^{k-1}\frac{\phi'_0(x+(1-\gamma)t+d)}{\phi_0(x+(1-\gamma)t+d)}\,\mathrm{d}t.
\end{aligned}$$

Integrating over d and some elementary algebra then gives the following expressions.

Corollary 2.2. *The expected maximum severity of the ruin excursion, with ruin occurring at the kth interruption, is given by*

$$\begin{aligned}&\mathrm{E}[|R_{\min}|\,\mathbf{1}_{\{ruin\ at\ P_k\}}\mid R_0=x]\\&\quad=-\frac{1}{(1-\gamma)^k}\int_x^\infty\frac{\mathrm{e}^{-\lambda(w-x)/(1-\gamma)}}{(k-1)!}\bigg(\lambda(w-x)-\ln\frac{\phi_0(w)}{\phi_0(x)}\bigg)^{k-1}\ln\phi_0(w)\,\mathrm{d}w.\end{aligned}$$

Furthermore, the expected maximum severity of the ruin excursion, given that ruin occurs, is given by

$$\mathrm{E}[|R_{\min}|\mid ruin\ occurs\ with\ R_0=x]=-\frac{\phi_\gamma(x)}{\psi_\gamma(x)}\int_x^\infty\frac{\ln\phi_\gamma(w)}{\phi_\gamma(w)}\,\mathrm{d}w.$$

Remark 2.3. From the above formulae, it is straightforward to write down the probability that the ruin excursion stays above surplus level $-d<0$, given that ruin occurs, as

$$\frac{A(x,0)-A(x,d)}{\psi_\gamma(x)}=\frac{1}{\psi_\gamma(x)}\int_x^\infty\bigg(\frac{\phi'_\gamma(w)}{\phi_\gamma(w)}-\frac{\phi'_\gamma(w+d)}{\phi_\gamma(w+d)}\bigg)\frac{\phi_\gamma(x)}{\phi_\gamma(w)}\,\mathrm{d}w.$$

For the case without tax ($\gamma=0$), this formula can be compared with the following related classical formula for the distribution of the maximum severity M of ruin, where M is defined as the smallest value of the risk process between the time of ruin and the time of recovery to surplus level 0 (instead of the time that the running maximum is reached again):

$$\mathrm{P}(M\le d\mid R_0=x\text{ and ruin occurs})=\frac{\phi_0(x+d)-\phi_0(x)}{\phi_0(d)(1-\phi_0(x))}$$

(see [7]).

3. Relation with the G/G/∞ queue

Consider the following situation. We have a sequence of pairs of random variables (X_1,Y_1), (X_2,Y_2), (X_3,Y_3), ..., for which we want to calculate

$$\phi(x)=\mathrm{P}\bigg(Y_i\le x+\sum_{j=1}^{i}X_j\text{ for all }i=1,2,\dots\bigg).$$

As a first interpretation, the function $\phi(x)$ is the survival probability in the risk model, if the X_is represent the increase of the surplus during periods in which the surplus process is in its running maximum (in the absence of tax payments, the X_is equivalently represent the lengths of the periods during which the surplus process is in its running maximum) and the Y_is represent the maximal decreases of the surplus process in periods during which the surplus process is not in a profitable situation (i.e. the Y_is correspond to identically distributed copies of the random variable $V_{\max}$).

A second interpretation of the function $\phi(x)$ is as the steady-state probability that at an arrival instant in a G/G/∞ queue the residual service times of all the customers present in the system are less than x. Here, the X_is represent the interarrival times of the customers and the Y_is represent the service times of the customers.

Let us first assume that the random vectors (X_i,Y_i), $i=1,2,\dots$, are independent and identically distributed. Furthermore, assume that, for each $i=1,2,\dots$, the random variables X_i and Y_i are independent.

Remark 3.1. The above assumptions are satisfied in the Cramér–Lundberg risk model, where the claim arrival process is a Poisson process. However, when the claim arrival process is a general renewal process, the random variables Y_i and X_{i+1} are dependent. In the related G/G/∞ queueing model this will mean that the service time of a customer depends on the previous interarrival time.

Let us denote by $F(\cdot)$ the common distribution function of the random variables X_i (with corresponding probability density function $f(\cdot)$). Furthermore, we denote by $G(\cdot)$ the common distribution function of the random variables Y_i.

Conditioning on the value of X_1 we obtain

$$\phi(x) = \int_{x_1=0}^{\infty} \phi(x+x_1)G(x+x_1)f(x_1)\,\mathrm{d}x_1. \tag{3.1}$$

Iteration of this equation yields

$$\begin{aligned}\phi(x) &= \int_{x_1=0}^{\infty}\int_{x_2=0}^{\infty} \phi(x+x_1+x_2)G(x+x_1+x_2)G(x+x_1)f(x_2)f(x_1)\,\mathrm{d}x_2\,\mathrm{d}x_1 \\ &= \cdots \\ &= \lim_{M\to\infty}\int_{x_1=0}^{\infty}\cdots\int_{x_M=0}^{\infty}\phi\Big(x+\sum_{j=1}^{M}x_j\Big)\prod_{i=1}^{M}\Big\{G\Big(x+\sum_{j=1}^{i}x_j\Big)f(x_i)\Big\}\,\mathrm{d}x_M\cdots\mathrm{d}x_1.\end{aligned}$$

Example 3.1. (*The X_is are deterministic.*) If the X_is are deterministic, say $X_i = w$, we have

$$\phi(x) = \phi(x+w)G(x+w) = \prod_{i=1}^{\infty} G(x+wi).$$

Example 3.2. (*The Y_is are deterministic.*) If the Y_is are deterministic, say $Y_i = v$, we have

$$\phi(x) = \begin{cases} 1 & \text{for } x \geq v, \\ 1 - F(v-x) & \text{for } x < v. \end{cases}$$

Example 3.3. (*The X_is are exponential with parameter λ.*) This is the case of the Cramér–Lundberg risk model. For an M/G/∞ queue, it is well known (see, e.g. [8, Section 3.2]) that the steady-state distribution of the number of customers is Poisson distributed and that the residual service times of the customers are all independent and identically distributed according to the excess lifetime distribution

$$G_\mathrm{e}(x) := \frac{1}{\mathrm{E}[Y]}\int_0^x \bar{G}(y)\,\mathrm{d}y.$$

Hence, we find that

$$\phi(x) = \sum_{n=0}^{\infty}\frac{(\lambda \mathrm{E}[Y])^n}{n!}\mathrm{e}^{-\lambda\mathrm{E}[Y]}[G_\mathrm{e}(x)]^n = \mathrm{e}^{-\lambda\mathrm{E}[Y](1-G_\mathrm{e}(x))} = \exp\Big(-\lambda\int_x^{\infty}\bar{G}(y)\,\mathrm{d}y\Big), \tag{3.2}$$

which can be interpreted as yet another approach to establish (1.1). Of course, (3.2) can also be obtained from (3.1), which in this case takes the form

$$\phi(x) = \lambda\int_0^{\infty}\phi(x+x_1)G(x+x_1)\mathrm{e}^{-\lambda x_1}\,\mathrm{d}x_1.$$

Introducing $T(x) := \mathrm{e}^{-\lambda x}\phi(x)$ yields

$$T(x) = \lambda \int_x^\infty T(u)G(u)\,\mathrm{d}u,$$

which gives $T'(x) = -\lambda G(x)T(x)$. It follows that $T(x) = C\exp(-\lambda\int_0^x G(y)\,\mathrm{d}y)$, so that $\phi(x) = C\exp(\lambda\int_0^x \bar{G}(y)\,\mathrm{d}y)$ with C some constant to be determined. Letting $x\to\infty$, we find that $C = \exp(-\lambda\int_0^\infty \bar{G}(y)\,\mathrm{d}y)$, and, hence, $\phi(x) = \exp(-\lambda\int_x^\infty \bar{G}(y)\,\mathrm{d}y)$.

Example 3.4. (*The Y_is are exponential with parameter ν.*) For a G/M/∞ queue, it is well known (see, e.g. [8, Theorem 2, p. 166]) that the steady-state probability for an arriving customer to find n customers in the system is given by

$$p_n = \sum_{r=n}^{\infty}(-1)^{r-n}\binom{r}{n}B_r,$$

where B_r is given by

$$B_r = \prod_{i=1}^{r}\frac{\tilde{F}(i\nu)}{1-\tilde{F}(i\nu)}$$

and $\tilde{F}(s)$ is the Laplace–Stieltjes transform of the interarrival time distribution. Exploiting the lack-of-memory property of the exponential distribution, we therefore have

$$\begin{aligned}
\phi(x) &= \sum_{n=0}^{\infty} p_n(1-\mathrm{e}^{-\nu x})^n \\
&= \sum_{n=0}^{\infty}\sum_{r=n}^{\infty}(-1)^{r-n}\binom{r}{n}\prod_{i=1}^{r}\frac{\tilde{F}(i\nu)}{1-\tilde{F}(i\nu)}(1-\mathrm{e}^{-\nu x})^n \\
&= \sum_{r=0}^{\infty}\prod_{i=1}^{r}\frac{\tilde{F}(i\nu)}{1-\tilde{F}(i\nu)}\sum_{n=0}^{r}\binom{r}{n}(-1)^{r-n}(1-\mathrm{e}^{-\nu x})^n \\
&= \sum_{r=0}^{\infty}\bigg(\prod_{i=1}^{r}\frac{\tilde{F}(i\nu)}{1-\tilde{F}(i\nu)}\bigg)(-\mathrm{e}^{-\nu x})^r.
\end{aligned}$$

In the special case that the interarrival times are exponential as well (with parameter λ), we have

$$\frac{\tilde{F}(i\nu)}{1-\tilde{F}(i\nu)} = \frac{\lambda}{i\nu}$$

and, correspondingly,

$$\phi(x) = \sum_{r=0}^{\infty}\prod_{i=1}^{r}\frac{\lambda}{i\nu}(-\mathrm{e}^{-\nu x})^r = \sum_{r=0}^{\infty}\frac{1}{r!}\bigg(-\frac{\lambda}{\nu}\mathrm{e}^{-\nu x}\bigg)^r = \mathrm{e}^{-\lambda\mathrm{e}^{-\nu x}/\nu} = \exp\bigg(-\lambda\int_x^\infty \mathrm{e}^{-\nu y}\,\mathrm{d}y\bigg), \tag{3.3}$$

as before.

If, on the other hand, the interarrival times are Erlang$(2,\lambda)$ distributed, we have

$$\frac{\tilde{F}(i\nu)}{1-\tilde{F}(i\nu)} = \frac{\lambda^2}{(i\nu)^2+2\lambda i\nu}$$

and, consequently,

$$\phi(x) = \sum_{r=0}^{\infty} \prod_{i=1}^{r} \frac{\lambda^2}{(i\nu)^2 + 2\lambda i\nu} (-\mathrm{e}^{-\nu x})^r = \sum_{r=0}^{\infty} \left(\frac{\lambda}{\nu}\right)^{2r} \frac{1}{r!} \prod_{i=1}^{r} \frac{1}{i + 2\lambda/\nu} (-\mathrm{e}^{-\nu x})^r.$$

Introducing $\alpha = 2\lambda/\nu$ and using

$$\prod_{i=1}^{r} \frac{1}{i+\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+r+1)}$$

gives

$$\phi(x) = \Gamma(\alpha+1) \sum_{r=0}^{\infty} \frac{[-(\lambda/\nu)^2 \mathrm{e}^{-\nu x}]^r}{r!\,\Gamma(\alpha+r+1)} = \frac{\Gamma(\alpha+1)}{(\lambda \mathrm{e}^{-\nu x/2}/\nu)^{\alpha}} J_{\alpha}(\alpha \mathrm{e}^{-\nu x/2}), \tag{3.4}$$

where $J_\alpha(\cdot)$ is the Bessel function of the first kind, defined by

$$J_\alpha(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r+\alpha+1)} \left(\frac{x}{2}\right)^{2r+\alpha}.$$

Formula (3.4) can also be obtained via (3.1): substituting $f(x) = \lambda^2 x \mathrm{e}^{-\lambda x}$ and $G(x) = 1 - \mathrm{e}^{-\nu x}$ into (3.1), and differentiating twice yields

$$(\mathrm{e}^{-\lambda x}\phi(x))'' = \lambda^2 \phi(x)(1 - \mathrm{e}^{-\nu x}),$$

or, equivalently,

$$\phi''(x) - 2\lambda\phi'(x) + \lambda^2 \mathrm{e}^{-\nu x}\phi(x) = 0.$$

This ordinary differential equation has the solution

$$\phi(x) = \left(\frac{\nu \mathrm{e}^{\nu x/2}}{\lambda}\right)^{\alpha} [C_1 \Gamma(1+\alpha) J_\alpha(\alpha \mathrm{e}^{-x\nu/2}) + C_2 \Gamma(1-\alpha) J_{-\alpha}(\alpha \mathrm{e}^{-x\nu/2})],$$

where C_1 and C_2 are constants, and $\alpha = 2\lambda/\nu$ again. The boundary condition

$$\lim_{x\to\infty} \phi(x) = 1$$

then gives $C_2 = 0$ and $C_1 = 1$, and, hence, (3.4).

It is interesting to examine the asymptotic behavior of $\phi(x_\lambda)$, with $x_\lambda := \kappa + (1/\nu)\log\lambda$ as $\lambda \to \infty$. It is easily verified that

$$\begin{aligned}
\lim_{\lambda\to\infty} \phi(x_\lambda) &= \lim_{\lambda\to\infty} \sum_{r=0}^{\infty} \left(\frac{\lambda}{\nu}\right)^{2r} \frac{1}{r!} \prod_{i=1}^{r} \frac{1}{i + 2\lambda/\nu} \left(\frac{-\mathrm{e}^{-\kappa\nu}}{\lambda}\right)^r \\
&= \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{-\mathrm{e}^{-\kappa\nu}}{2\nu}\right)^r \\
&= \mathrm{e}^{-\mathrm{e}^{-\kappa\nu}/2\nu}.
\end{aligned}$$

Note that this limit is the same as the value of $\phi(x_\lambda)$ in the case of exponential interarrival times with parameter $\lambda/2$ (cf. (3.3)).

4. An asymptotic result for renewal risk models with tax

Assume that potential 'catastrophes' occur according to a delayed renewal process with initial delay T_0 and interrenewal periods $T_1, T_2, \ldots$. At time $S_n := T_0 + \cdots + T_n$, an actual catastrophe occurs if V_n exceeds $f(S_n)$, with $f(\cdot)$ some increasing function, and $V_0, V_1, V_2, \ldots$ a sequence of independent and identically distributed random variables. The random variables T_{n+1} and V_n may be dependent. Let the 0–1 variable $I_n := \mathbf{1}_{\{V_n > f(S_n)\}}$ indicate whether or not an actual catastrophe occurs at time S_n, and define

$$p(t) := \mathrm{P}(V_n > f(t)).$$

We are interested in the probability of the event E_τ that no actual catastrophe occurs during the time interval $[0, \tau]$, i.e.

$$E_\tau = \bigcup_{n=-1}^{\infty} \{S_n \le \tau < S_{n+1};\ I_0 = \cdots = I_n = 0\},$$

with the notational convention that $S_{-1} := 0$.

Now consider the surplus process in the Sparre Andersen risk model where claims of generic size Y occur according to a renewal process with generic interrenewal time X, and a marginal tax rate γ applies whenever the free surplus is at a running maximum. Let Q be a single-server queue with generic interarrival time X and generic service time Y. Let $V_{\max}$ and T be a pair of random variables whose joint distribution is that of the maximum workload during a busy period of Q and the subsequent idle period. Furthermore, suppose that we take the joint distribution of T_{n+1} and V_n to be that of T and $V_{\max}$, and let $f(t) = x + (1-\gamma)t$. Then the probability of the event E_τ with $\tau = (v-x)/(1-\gamma)$ equals the probability $\phi_\gamma(x, v)$ that the surplus process reaches level v, starting from level x, before ruin occurs. In particular, the survival probability in the renewal model with tax is $\phi_\gamma(x) = \mathrm{P}(E_\infty)$, with $E_\infty = \{V_n \le x + (1-\gamma)S_n \text{ for all } n = 0, 1, 2, \ldots\}$.

Remark 4.1. Following Section 3, the probability of the event E_∞ may also be interpreted as the probability that no customer with a remaining service time exceeding x is present in a G/G/∞ system where the joint distribution of the interarrival time and subsequent service time is that of $(1-\gamma)T_{n+1}$ and V_n, given that the past interarrival time is T_0.

In order to characterize the probability of interest, i.e. $\mathrm{P}(E_\tau)$, we will consider a scenario where the interrenewal periods are relatively short (compared to the time interval $[0, \tau]$), i.e. the number of potential catastrophes is relatively large, whereas the probability that an actual catastrophe occurs is relatively small, such that the value of the ratio $p(t)/\mathrm{E}[T]$ is moderate. More specifically, we assume an asymptotic regime where time is accelerated by a factor s, i.e. with interrenewal periods $T^{(s)} := T/s$, while the function $f^{(s)}(\cdot)$ is simultaneously boosted in such a manner that the ratio $p^{(s)}(t)/\mathrm{E}[T^{(s)}] = p(t)/\mathrm{E}[T]$, i.e. $p^{(s)}(t) = p(t)/s$. For each fixed value of s, denote the resulting event E_τ by $E_\tau^{(s)}$.

The next theorem states the main result of this section.

Theorem 4.1. *Under the abovementioned assumptions,*

$$\mathrm{P}(E_\tau^{(s)}) \to \exp\biggl(-\lambda \int_{t=0}^{\tau} p(t)\,\mathrm{d}t\biggr) \tag{4.1}$$

as $s \to \infty$, with $\lambda := 1/\mathrm{E}[T]$.

Remark 4.2. Theorem 4.1 suggests that the expression on the right-hand side should provide a reasonable approximation for $\mathrm{P}(E_\tau^{(s)})$ in the above-described asymptotic regime where the interrenewal periods are relatively short compared to the time interval $[0, \tau]$. Note that (4.1) has a similar form as the earlier result (1.1) for the Cramér–Lundberg risk process.

In order to prove Theorem 4.1, we will establish lower and upper bounds for the unscaled process. Lemmas 4.2 and 4.4 below will show that these two bounds, while crude, coincide in the asymptotic regime under consideration.

For ease of notation, we henceforth drop the subscript τ from the notation $E_\tau^{(s)}$, and simply write $E^{(s)}$ or just E. Note that

$$\lim_{K\to\infty} \frac{\tau}{K} \sum_{k=1}^{K} p\left(k\frac{\tau}{K}\right) = \lim_{K\to\infty} \frac{\tau}{K} \sum_{k=1}^{K} p\left((k-1)\frac{\tau}{K}\right) = \int_{t=0}^{\tau} p(t)\,\mathrm{d}t. \tag{4.2}$$

Let us now focus on the lower bound. Let $K \geq 1$ and $N \geq 1$ be integers, and let $t_0 = 0 \leq t_1 \leq \cdots \leq t_K = \tau$. For any $k = 1, \ldots, K$, define the events

$$D_k := \{S_{kN} > t_k\}, \qquad F_k := \{V_{(k-1)N} \leq f(t_{k-1}), \ldots, V_{kN-1} \leq f(t_{k-1})\},$$

and

$$E^{\text{lower}} := \bigcap_{k=1}^{K} D_k \cap \bigcap_{k=1}^{K} F_k.$$

Lemma 4.1. *The event E^{lower} implies the event E.*

Proof. Suppose that the event E^{lower} occurs, i.e. all the events D_k and F_k occur. Let i be such that $(k-1)N \leq i \leq kN-1$ for some $k = 1, \ldots, K$. The event D_k gives $S_i \geq S_{(k-1)N} > t_{k-1}$, while the event F_k implies that $V_i \leq f(t_{k-1})$. Since the function $f(\cdot)$ is increasing, it follows that $V_i \leq f(S_i)$. Hence, $I_i = 0$ for all $i = 0, \ldots, KN-1$. The event D_K implies that there exists an $n \leq KN - 1$ with $S_n \leq \tau < S_{n+1}$. Thus, the event E occurs.

Lemma 4.2. *We have*

$$\lim_{s\to\infty} \mathrm{P}(E^{(s)}) \geq \exp\left(-\lambda \int_{t=0}^{\tau} p(t)\,\mathrm{d}t\right). \tag{4.3}$$

Proof. Lemma 4.1 yields

$$\begin{aligned}
\mathrm{P}(E) &\geq \mathrm{P}(E^{\text{lower}}) \\
&= \mathrm{P}\left(\bigcap_{k=1}^{K} D_k \cap \bigcap_{k=1}^{K} F_k\right) \\
&\geq \mathrm{P}\left(\bigcap_{k=1}^{K} F_k\right) - \mathrm{P}\left(\overline{\bigcap_{k=1}^{K} D_k}\right) \\
&\geq \prod_{k=1}^{K} \mathrm{P}(F_k) - \sum_{k=1}^{K} \mathrm{P}(\bar{D}_k) \\
&= \prod_{k=1}^{K} \mathrm{P}\Big(V_{(k-1)N} \leq f(t_{k-1}), \ldots, V_{kN-1} \leq f(t_{k-1})\Big) - \sum_{k=1}^{K} \mathrm{P}(S_{kN} \leq t_k) \\
&= \prod_{k=1}^{K} (\mathrm{P}(V \leq f(t_{k-1})))^N - \sum_{k=1}^{K} \mathrm{P}(S_{kN} \leq t_k).
\end{aligned}$$

We now take $N = \lceil N(s) \rceil$, with $N(s) = (1+\varepsilon)\tau s / K\mathrm{E}[T]$, and $t_k = k\tau/K$, $k = 1, \ldots, K$. Then

$$\mathrm{P}(S_{kN} \le t_k) = \mathrm{P}\biggl(\frac{T_0}{s} + \frac{T_1}{s} + \cdots + \frac{T_{k\lceil N(s)\rceil}}{s} \le \frac{k\tau}{K}\biggr)$$
$$= \mathrm{P}\biggl(T_0 + T_1 + \cdots + T_{k\lceil N(s)\rceil} \le \frac{kN(s)\mathrm{E}[T]}{1+\varepsilon}\biggr),$$

which, by the law of large numbers, tends to 0 as $s \to \infty$. Also,

$$\lim_{s\to\infty} \prod_{k=1}^{K} (\mathrm{P}(V < f(t_{k-1})))^{N(s)} = \lim_{s\to\infty} \prod_{k=1}^{K} \mathrm{e}^{-N(s)p^{(s)}(t_{k-1})}$$
$$= \exp\biggl(-\sum_{k=1}^{K} \lim_{s\to\infty} N(s) p^{(s)}(t_{k-1})\biggr)$$
$$= \exp\biggl(-\sum_{k=1}^{K} \frac{\tau p(t_{k-1})}{K\mathrm{E}[T]}\biggr)$$
$$= \exp\biggl(-\frac{\tau}{K\mathrm{E}[T]} \sum_{k=1}^{K} p(t_{k-1})\biggr).$$

We deduce that

$$\lim_{s\to\infty} \mathrm{P}(E^{(s)}) \ge \exp\biggl(-\frac{\tau}{K\mathrm{E}[T]} \sum_{k=1}^{K} p(t_{k-1})\biggr)$$

for any $K \ge 1$. Letting $K \to \infty$ and applying (4.2), we obtain the lower bound (4.3).

Next, we establish an upper bound that asymptotically matches the lower bound. Let $K \ge 1$ and $N \ge 1$ be integers, and let $t_0 = 0 \le t_1 \le \cdots \le t_K = \tau$. For any $k = 1, \ldots, K$, define the events

$$G_k := \{V_{(k-1)N} \le f(t_k), \ldots, V_{kN-1} \le f(t_k)\}$$

and

$$E^{\text{upper}} := \bigcup_{k=1}^{K} D_k \cup \bigcap_{k=1}^{K} G_k.$$

Lemma 4.3. *The event E implies the event E^{upper}.*

Proof. Suppose that the event E occurs, i.e. there exists an $n(\tau)$ with $S_{n(\tau)} \le \tau < S_{n(\tau)+1}$ and $I_0 = \cdots = I_{n(\tau)} = 0$. Also, assume that all the events $\bar{D}_k$ occur, i.e. $S_{kN} \le t_k$ for all $k = 1, \ldots, K$, because otherwise there is nothing to prove. This in particular implies that $n(\tau) \ge KN - 1$, and, hence, $I_0 = \cdots = I_{KN-1} = 0$, i.e. $V_i \le f(S_i)$ for all $i = 0, \ldots, KN-1$. Let i be such that $(k-1)N \le i \le kN - 1$ for some $k = 1, \ldots, K$, so that $S_i \le S_{kN}$. Since the function $f(\cdot)$ is increasing, it follows that $V_i \le f(t_k)$, and, thus, all the events G_k occur, and, hence, the event E^{upper} occurs.

Lemma 4.4. *We have*

$$\lim_{s\to\infty} \mathrm{P}(E^{(s)}) \le \exp\biggl(-\lambda \int_{t=0}^{\tau} p(t)\,\mathrm{d}t\biggr). \tag{4.4}$$

Proof. Lemma 4.3 yields

$$\begin{aligned}
\mathrm{P}(E) &\le \mathrm{P}(E^{\mathrm{upper}}) \\
&= \mathrm{P}\bigg(\bigcup_{k=1}^{K} D_k \cup \bigcap_{k=1}^{K} G_k\bigg) \\
&\le \mathrm{P}\bigg(\bigcap_{k=1}^{K} G_k\bigg) + \mathrm{P}\bigg(\bigcup_{k=1}^{K} D_k\bigg) \\
&\le \prod_{k=1}^{K} \mathrm{P}(G_k) + \sum_{k=1}^{K} \mathrm{P}(D_k) \\
&= \prod_{k=1}^{K} (\mathrm{P}(V \le f(t_k)))^N + \sum_{k=1}^{K} \mathrm{P}(S_{kN} > t_k).
\end{aligned} \tag{4.5}$$

We now take $N = \lceil N(s) \rceil$, with $N(s) = (1-\varepsilon)\tau s / K\mathrm{E}[T]$, and $t_k = k\tau/K$, $k = 1, \ldots, K$, and proceed to evaluate the upper bound (4.5) in the asymptotic regime of interest. Note that

$$\begin{aligned}
\mathrm{P}(S_{kN} > t_k) &= \mathrm{P}\bigg(\frac{T_0}{s} + \frac{T_1}{s} + \cdots + \frac{T_{k\lceil N(s) \rceil}}{s} > \frac{k\tau}{K}\bigg) \\
&= \mathrm{P}\bigg(T_0 + T_1 + \cdots + T_{k\lceil N(s) \rceil} > \frac{kN(s)\mathrm{E}[T]}{1-\varepsilon}\bigg),
\end{aligned}$$

which tends to 0 as $s \to \infty$ because of the law of large numbers. Also,

$$\begin{aligned}
\lim_{s\to\infty} \prod_{k=1}^{K} (\mathrm{P}(V \le f(t_k)))^{N(s)} &= \lim_{s\to\infty} \prod_{k=1}^{K} \mathrm{e}^{-N(s)p^{(s)}(t_k)} \\
&= \exp\bigg(-\sum_{k=1}^{K} \lim_{s\to\infty} N(s)p^{(s)}(t_k)\bigg) \\
&= \exp\bigg(-\sum_{k=1}^{K} \frac{\tau p(t_k)}{K\mathrm{E}[T]}\bigg) \\
&= \exp\bigg(-\frac{\tau}{K\mathrm{E}[T]} \sum_{k=1}^{K} p(t_k)\bigg).
\end{aligned}$$

We conclude that

$$\lim_{s\to\infty} \mathrm{P}(E^{(s)}) \le \exp\bigg(-\frac{\tau}{K\mathrm{E}[T]} \sum_{k=1}^{K} p(t_k)\bigg)$$

for any $K \ge 1$. Letting $K \to \infty$ and invoking (4.2), we obtain the upper bound (4.4).

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