NOTE ON ALMOST PRODUCT MANIFOLDS AND THEIR TANGENT BUNDLES

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Let M^n be an n-dimensional manifold of differentiability class C^∞ with an almost product structure ϕ^j_i. Let ϕ^j_i have eigenvalue +1 of multiplicity p and eigenvalue -1 of multiplicity q where $p+q=n$ and $p \geqq 1$, $q \geqq 1$. Let $T(M^n)$ be the tangent bundle of M^n. $T(M^n)$ is a 2n dimensional manifold of class C^∞. Let x^i be the local coordinates of a point P of M^n. The local coordinates of $T(M^n)$ can be expressed by 2n variables (x^i, y^i) where x^i are coordinates of the point P and y^i are components of a tangent vector at P with respect to the natural frame constituted by the vectors $\partial/\partial x^i$ at P. For convenience's sake we put

$$x^{i^*} = x^{n+i} = y^i .$$

We assume that all indices i, j, k, ... run through 1, 2, ..., n. So all indices $i^*, j^*, k^*, \ldots$ run through n+1, n+2, ..., 2n. The indeces A, B, C, ... are supposed to run through 1, 2, ..., 2n.

In § 1 we prove that if M^n has constant curvature and its metric connection is a ϕ-connection then M^n is locally flat. (Theorem 1). In § 2 we introduce an almost product structure ϕ^j_i in $T(M^n)$ and prove that it is integrable if and only if M^n

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is locally flat. Also it is proved that an infinitesimal transformation of M^n is affine if and only if its extension to $T(M^n)$ is almost decomposable. (Theorem 2). In § 3 we use the metric on $T(M^n)$ introduced by S. Sasaki [3] and prove that the metric connection is a ϕ-connection if and only if M^n is locally flat. (Theorem 3). Finally we suppose M^n is locally flat and show that an extension to $T(M^n)$ of a given infinitesimal transformation on M^n is affine if and only if the extension is almost decomposable. (Theorem 4). Theorems 1, 2 correspond to known theorems in a manifold with an almost complex structure. (Remarks to § 1, 2). Some tensor calculus are omitted in § 2, 3 and 4. Those are straight calculation though lengthy.

1. In an almost product manifold, an affine connection is called a ϕ-connection if the almost product structure $\phi_j^{\ i}$ is covariant constant with respect to that connection. (Yano [6], p. 255). It is known that we can introduce a positive definite Riemannian metric g_{ij}, which satisfies $g_{ij}\phi_h^{\ i}\phi_k^{\ j} = g_{hk}$, over M^n. $\{^{\ i}_{j\ k}\}$ are the Christoffel symbols constructed by g_{ij}. The Riemann-Christoffel curvature tensor is given by

$$R_{ijk}^{\ \ \ h} = \partial_i\{^{\ h}_{j\ k}\} - \partial_j\{^{\ h}_{i\ k}\} + \{^{\ h}_{i\ l}\}\{^{\ l}_{j\ k}\} - \{^{\ h}_{j\ l}\}\{^{\ l}_{i\ k}\},$$

where ∂_i denotes the partial differentiation with respect to x^i.

THEOREM 1. <u>If the almost product manifold</u> M^n <u>has constant curvature and its metric connection</u> $\{^{\ i}_{j\ k}\}$ <u>is a</u> ϕ-<u>connection, then</u> M^n <u>is locally flat.</u>

<u>Proof</u>. Let $\phi^{lh} = g^{lk}\phi_k^{\ h}$, $\overline{H}_{ji} = \phi^{lh}R_{hjil}$, $\overline{H} = \phi^{ji}\overline{H}_{ji}$; $R = g^{ij}R_{ij}$ be the curvature scalar. If $\{^{\ i}_{j\ k}\}$ is a ϕ-connection then (Hsu [2] proposition 5.3)

$$R = \overline{H}.$$

Now M^n has constant curvature K, then

$$R = n(n-1)K.$$

But

$$\bar{H} = \phi^{ji}\phi^{lh}R_{hjil}$$

$$= \phi^{ji}\phi^{lh}K(g_{hl}g_{ji} - g_{hi}g_{jl})$$

$$= K((\phi_i^{\ i})^2 - n) = K[(q-p)^2 - n].$$

So $\bar{H} = R$ implies $K = 0$.

Remark. This theorem corresponds to a known theorem in a Kähler manifold. (Yano [6] pp. 69-70).

2. Now we introduce an almost product structure in $T(M^n)$ as follows:

$$(1)\qquad \begin{aligned} \phi_j^{\ i} &= -\left\{{i \atop j\,l}\right\} y^l, \\ \phi_{j*}^{\ i} &= -\delta_j^i, \\ \phi_j^{\ i*} &= -\delta_j^i + \left\{{i \atop l\,t}\right\}\left\{{l \atop j\,m}\right\} y^t y^m, \\ \phi_{j*}^{\ i*} &= \left\{{i \atop j\,l}\right\} y^l. \end{aligned}$$

It is easy to show that

$$\phi_B^{\ A}\phi_C^{\ B} = \delta_C^A,$$

so $\phi_B^{\ A}$ is an almost product structure on $T(M^n)$.

The Nijenhuis tensor of the almost product structure (1) is:

$$N_{BC}^{\ \ A} = \phi_B^{\ D}(\partial_D\phi_C^{\ A} - \partial_G\phi_D^{\ A}) - \phi_C^{\ D}(\partial_D\phi_B^{\ A} - \partial_B\phi_D^{\ A}),$$

where ∂_D denotes the partial differentiation with respect to x^D.

After some tensor calculations we have the following result:

$$N_{jk}{}^{i} = (\{{}^{t}_{j\,m}\} R_{tkl}{}^{i} - \{{}^{t}_{k\,m}\} R_{tjl}{}^{i}) y^{m} y^{l}$$

$$N_{jk}{}^{i*} = -R_{jkm}{}^{i} y^{m} - \{{}^{l}_{j\,s}\}\{{}^{h}_{k\,m}\} y^{s} y^{m} R_{lht}{}^{i} y^{t}$$

$$- \{{}^{i}_{s\,t}\} y^{t}(\{{}^{l}_{j\,h}\} y^{h} R_{lkm}{}^{s} y^{m} - \{{}^{l}_{k\,h}\} y^{h} R_{ljm}{}^{s} y^{m}),$$

(2) $$N_{jk*}{}^{i} = R_{jkl}{}^{i} y^{l},$$

$$N_{j*k*}{}^{i} = 0,$$

$$N_{jk*}{}^{i*} = - \{{}^{t}_{j\,l}\} y^{l} R_{tkm}{}^{i} y^{m} - \{{}^{i}_{l\,t}\} y^{t} R_{jkm}{}^{l} y^{m},$$

$$N_{j*k*}{}^{i*} = -R_{jkt}{}^{i} y^{t},$$

where $R_{ijk}{}^{h}$ is the Riemann-Christoffel curvature tensor.

Let v^{i} be an infinitesimal transformation of M^{n} and V^{A} be the extension of v^{i}. V^{A} is an infinitesimal transformation of $T(M^{n})$ and defined by (S. Sasaki [3] § 2)

$$V^{i} = v^{i}, \qquad V^{i*} = y^{r} \partial_{r} v^{i}.$$

We call v^{i} an <u>almost contravariant decomposable</u> vector field of M^{n} if it satisfies

(3) $$\mathop{\mathcal{L}}_{v} \phi_{j}{}^{i} = 0$$

where $\mathop{\mathcal{L}}_{v}$ denotes the operator of Lie derivation with respect to v^{i}. If M^{n} is a locally product manifold which satisfies stronger conditions than the almost product manifold, then the v^{i} satisfying (3) turns out to be a contravariant decomposable vector field. (K. Yano [6] pp.222-223).

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Making use of (1) and the following formula for Lie derivation:

$$\underset{V}{\mathcal{L}}\phi_B{}^A = V^C \partial_C \phi_B{}^A - \phi_B{}^C \partial_C V^A + \phi_C{}^A \partial_B V^C ,$$

we have the following result:

$$
(4)\qquad
\begin{aligned}
&\underset{V}{\mathcal{L}}\phi_j{}^h = - y^r t_{jr}{}^h ,\\
&\underset{V}{\mathcal{L}}\phi_{j*}{}^h = 0 ,\\
&\underset{V}{\mathcal{L}}\phi_j{}^{h*} = (\{{}^{\ l}_{j\ n}\} t_{lm}{}^h + \{{}^{\ h}_{l\ m}\} t_{jn}{}^l) y^m y^n ,\\
&\underset{V}{\mathcal{L}}\phi_{j*}{}^{h*} = y^r t_{jr}{}^h
\end{aligned}
$$

where $t_{ji}{}^h$ is given by (K. Yano [6] p. 17)

$$t_{ji}{}^h = \nabla_j \nabla_i v^h + v^r R_{rji}{}^h = \underset{v}{\mathcal{L}}\{{}^{\ h}_{j\ i}\}$$

∇_j being the Riemannian covariant derivation in M^n.

From (2) and (4) we have the following theorem:

THEOREM 2. <u>In order that the almost product structure (1) of $T(M^n)$ is integrable, it is necessary and sufficient that the Riemannian manifold M^n is locally flat. In order that an infinitesimal transformation of a Riemannian manifold M^n is affine, it is necessary and sufficient that its extension in $T(M^n)$ with an almost structure (1) is almost decomposable.</u>

<u>Proof.</u> The almost product structure (1) is integrable if and only if $N_{AB}{}^C = 0$. (K. Yano [5] Theorem 1). Hence the first part of the theorem is the conclusion of (2). v^i is affine if and only if $\underset{v}{\mathcal{L}}\{{}^{\ h}_{i\ j}\} = t_{ij}{}^h = 0$. (K. Yano [6], p. 17). Hence the second part of this theorem is the conclusion of (4).

Remark. This theorem is an analogue of theorems in an almost complex manifold. (Hsu [1] Theorem 1.1, Tachibara and Okumura [4], Theorems 1, 2).

3. Let G_{AB} be the Riemannian metric defined by S. Sasaki [3] § 3 on $T(M^n)$, namely

$$
(5)\qquad
\begin{aligned}
G_{jk} &= g_{jk} + g_{st}\left\{ {s \atop j\ l} \right\}\left\{ {t \atop k\ m} \right\} y^l y^m ,\\
G_{jk^*} &= g_{kl}\left\{ {l \atop j\ r} \right\} y^r ,\\
G_{j^*k^*} &= g_{jk} .
\end{aligned}
$$

Then for $\phi_{AB} = \phi_A{}^C G_{CB}$ we have by (1) and (5):

$$
(6)\qquad
\begin{aligned}
\phi_{ij} &= -\partial_l g_{ij} y^l ,\\
\phi_{ij^*} &= \phi_{j^*i} = -g_{ij} ,\\
\phi_{i^*j^*} &= 0
\end{aligned}
$$

and

$$G_{AB}\,\phi_C{}^A \phi_D{}^B = G_{CD} .$$

The Christoffel symbols $\overline{\left\{ {A \atop B\ C} \right\}}$ of the metric G_{AB} of $T(M^n)$ are (S. Sasaki [3] § 7).

$$
(7)\qquad
\begin{aligned}
\overline{\left\{ {A \atop j^*\ k^*} \right\}} &= 0 ,\\
\overline{\left\{ {i \atop j\ k} \right\}} &= \left\{ {i \atop j k} \right\} + \frac{1}{2}\left(R_{khl}{}^i \left\{ {h \atop m\ j} \right\} + R_{jhl}{}^i \left\{ {h \atop m\ k} \right\}\right) y^l y^m ,\\
\overline{\left\{ {i \atop j^*\ k} \right\}} &= \frac{1}{2} R_{kjl}{}^i y^l ,\\
\overline{\left\{ {i^* \atop j\ k} \right\}} &= \frac{1}{2}\left(R_{jlk}{}^i + R_{klj}{}^i + 2\,\partial_l \left\{ {i \atop j\ k} \right\}\right) y^l + \frac{1}{2}\left\{ {i \atop p\ h} \right\}\left(R_{kml}{}^h \left\{ {l \atop n\ j} \right\}\right.\\
&\qquad \left. + R_{jml}{}^h \left\{ {l \atop n\ k} \right\}\right) y^n y^m y^p ,
\end{aligned}
$$

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$$\overline{\{{}_{j*}{}^{i*}{}_{k}\}} = \{{}_j{}^i{}_k\} - \frac{1}{2}\{{}_m{}^i{}_h\} R_{kjl}{}^h y^l y^m .$$

Let $\overline{\nabla}$ denote the covariant differentiation with respect to the metric connection $\overline{\{{}_B{}^A{}_C\}}$ on $T(M^n)$. We will prove the following theorem:

THEOREM 3. <u>The connection</u> $\overline{\{{}_B{}^A{}_C\}}$ <u>is a</u> ϕ-<u>connection, where</u> ϕ <u>is an almost product structure on</u> $T(M^n)$ <u>given by</u> (1), <u>if and only if</u> M^n <u>is locally flat.</u>

<u>Proof.</u> Suppose $\overline{\{{}_B{}^A{}_C\}}$ is a ϕ-connection, then $\overline{\nabla}_A \phi_C{}^B = 0$, also $\overline{\nabla}_A \phi_{CB} = 0$. Making use of (1), (6), (7) we have

$$\overline{\nabla}_k \phi_{j*}{}^i = \frac{1}{2}(-R_{khl}{}^i \{{}_m{}^h{}_j\} - R_{jhl}{}^i \{{}_m{}^h{}_k\} + R_{ksl}{}^i \{{}_j{}^s{}_m\})y^l y^m ,$$

$$\overline{\nabla}_k \phi_{i*j} = \frac{1}{2}(R_{likh}\{{}_j{}^h{}_m\} + R_{lijh}\{{}_m{}^h{}_k\} + R_{kils}\{{}_j{}^s{}_m\})y^l y^m .$$

So $\overline{\nabla}_k \phi_{j*}{}^i = 0$ yields

$$(-R_{likh}\{{}_m{}^h{}_j\} - R_{lijh}\{{}_m{}^h{}_k\} + R_{liks}\{{}_j{}^s{}_m\})y^l y^m = 0.$$

Combining with $\overline{\nabla}_k \phi_{i*j} = 0$ we have

$$(R_{liks} + R_{kils})\{{}_j{}^s{}_m\} y^l y^m = 0.$$

Substituting this relation to the right of the above $\overline{\nabla}_k \phi_{i*j}$ we have

(8) $$R_{lijh}\{{}_m{}^h{}_k\} y^l y^m = 0.$$

Computing $\overline{\nabla}_{k*} \phi_j{}^{i*}$ and making use of (1), (7) and (8) we have

$$\overline{\nabla}_{k*} \phi_j{}^{i*} = \frac{1}{2} R_{jks}{}^i y^s .$$

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$\bar{\nabla}_{k*} \phi_j^{i*} = 0$ implies $R_{jks}^{\ \ \ i} = 0$. So M^n is locally flat.

Conversely suppose M^n is locally flat, then $R_{jks}^{\ \ \ i} = 0$.
(7) turns out to be

$$\overline{\{{}_{j*}{}^{A}{}_{k*}\}} = 0, \quad \overline{\{{}_{j}{}^{i}{}_{k}\}} = \{{}_{j}{}^{i}{}_{k}\}, \quad \overline{\{{}_{j*}{}^{i}{}_{k}\}} = 0$$

$$\overline{\{{}_{j}{}^{i*}{}_{k}\}} = \partial_1 \{{}_{j}{}^{i}{}_{k}\} y^1, \quad \overline{\{{}_{j*}{}^{i*}{}_{k}\}} = \{{}_{j}{}^{i}{}_{k}\}.$$

Then from (6) and (5)

$$\bar{\nabla}_k \phi_{ij} = -\partial_k \partial_l g_{ij} y^l + (\partial_p [ik, j] + \partial_p [jk, i]) y^p = 0,$$

$$\bar{\nabla}_{k*} \phi_{ij} = -\partial_k g_{ij} + \{{}_{i}{}^{l}{}_{k}\} g_{lj} + \{{}_{j}{}^{l}{}_{k}\} g_{li} = 0,$$

$$\bar{\nabla}_A \phi_{i*j} = -\bar{\nabla}_A g_{ij} = -\bar{\nabla}_A G_{i*j*} = 0,$$

$$\bar{\nabla}_k \phi_{i*j*} = 0,$$

$$\bar{\nabla}_{k*} \phi_{i*j*} = 0.$$

Hence $\overline{\{{}_{B}{}^{A}{}_{C}\}}$ is a ϕ-connection. This completes the proof of the theorem.

Remark. It was known that an almost product manifold is integrable if and only if it is possible to introduce a symmetric affine connection with respect to which the structure tensor is covariantly constant. (Yano [6], p. 254). So by Theorem 2 we knew that we can introduce a symmetric affine ϕ-connection in $T(M^n)$ if and only if M^n is locally flat. Theorem 3 exhibits such a metric ϕ-connection.

4. Suppose M^n be locally flat and have an almost product structure ϕ. Let V^A be an extension of an infinitesimal transformation v^i of M^n. That is $V^i = v^i$, $V^{i*} = y^r \partial_r v^i$. Then

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$$\bar{t}_{BC}{}^{A} = \partial_B \partial_C V^A + V^D \partial_D \overline{\{{}_B{}^A{}_C\}} + \overline{\{{}_C{}^A{}_D\}} \partial_B V^C + \overline{\{{}_B{}^A{}_D\}} \partial_C V^D$$

$$- \overline{\{{}_B{}^D{}_C\}} \partial_D V^A .$$

Breaking down the indeces and making use of (7) and the fact that M^n is locally flat, we have

(9)
$$\bar{t}_{jk}{}^{i} = t_{jk}{}^{i},$$
$$\bar{t}_{jk}{}^{i*} = y^r \partial_r t_{jk}{}^{i}$$
$$\bar{t}_{j*k}{}^{i} = 0,$$
$$\bar{t}_{j*k*}{}^{i} = 0,$$
$$\bar{t}_{j*k}{}^{i*} = t_{jk}{}^{i},$$
$$\bar{t}_{j*k*}{}^{i*} = 0 .$$

THEOREM 4. <u>Suppose</u> M^n <u>is locally flat,</u> v^i <u>is a vector field on</u> M^n, V^A <u>is the extension of</u> v^i <u>on</u> $T(M^n)$. <u>Then</u> V^A <u>is an affine infinitesimal transformation on</u> $T(M_n)$ <u>if and only if</u> V^A <u>is almost decomposable with respect to the almost product structure</u> (1) <u>in</u> $T(M^n)$.

<u>Proof.</u> By (9) $\bar{t}_{BC}{}^{A} = 0$ if and only if $t_{jk}{}^{i} = 0$. So V^A is an affine infinitesimal transformation if and only if v^i is an affine infinitesimal transformation. Then by Theorem 2 it is necessary and sufficient that V^A is almost decomposable.

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