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ISOMORPHISM OF SOME SIMPLE 2-GRADED LIE ALGEBRAS

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Introduction. The grading is by integers modulo 2 and we refer to it as 2-grading. For the definition of 2-graded Lie algebras L and their properties we refer the reader to the papers [**1**; **2**; **3**]. All algebras considered here are *finite-dimensional* over a field F of *characteristic zero*.

We recall that if L is a 2-graded Lie algebra then $L = L_0 \oplus L_1$ and we have

$$[y, x] = -(-1)^{\alpha\beta}[x, y],$$
$$[x, [y, z]] = [[x, y], z] + (-1)^{\alpha\beta}[y, [x, z]]$$

for $x \in L_\alpha$, $y \in L_\beta$, $z \in L$.

Thus L_0 is an ordinary Lie algebra, L_1 is an L_0-module and in addition to that we have a bilinear map

$$L_1 \times L_1 \to L_0$$

sending (x, y) to $[x, y]$, which is symmetric and satisfies the identity

$$(1) \quad [x, y] \cdot z + [y, z] \cdot x + [z, x] \cdot y = 0.$$

Here $x, y, z \in L_1$ and for $a \in L_0$, $x \in L_1$ we write $a \cdot x$ instead of $[a, x]$. Moreover, the corresponding linear map $L_1 \otimes L_1 \to L_0$ is a homomorphism of L_0-modules. Conversely, if all these conditions are satisfied then $L = L_0 \oplus L_1$ is a 2-graded Lie algebra.

A 2-graded Lie algebra L is *simple* if its only homogeneous ideals are L and 0 and $L \neq 0$. The list of all simple 2-graded Lie algebras over an algebraically closed field F of characteristic 0 will soon be known (see [**4**]).

The object of this paper is to prove that the series III and IV of [**1**] coincide. That this may be so was suggested by I. Kaplansky.

We first describe the series III and IV and supply the details which were omitted in [**1**] and [**2**].

Series III. Let F be a field of characteristic 0 and V a finite-dimensional F-vector space. Let V^* be the dual of V. Then we have the canonical isomorphism

$$V \otimes V^* \to \mathrm{End}_F(V)$$

which we use to identify these two spaces. Recall that for $a \in V$, $f \in V^*$ we

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then have $(a \otimes f)(x) = f(x)a$ for all $x \in V$. Hence, $V \otimes V^*$ becomes an associative algebra with identity such that

$$(a \otimes f) \circ (b \otimes g) = f(b)a \otimes g$$

and the trace map is given by

$$\operatorname{tr}\,(a \otimes f) = f(a).$$

Because of our identification, $V \otimes V^*$ considered as a Lie algebra is nothing else but $\mathrm{gl}(V)$, the Lie algebra of endomorphisms of V. The elements of trace zero in $V \otimes V^*$ form a Lie-subalgebra, $\mathrm{sl}(V)$.

We denote by SV and ΛV the symmetric and exterior algebras of V, respectively. We shall only use the second symmetric and exterior powers S^2V and $\Lambda^2 V$. For $a, b \in V$ we write ab and $a \wedge b$ for their products in SV and ΛV. In fact, $ab \in S^2V$ and $a \wedge b \in \Lambda^2 V$.

We identify $S^2(V^*)$ with $(S^2V)^*$ by

$$(fg)(xy) = f(x)g(y) + f(y)g(x)$$

where $x, y \in V; f, g \in V^*$. Similarly, we identify $\Lambda^2(V^*)$ and $(\Lambda^2 V)^*$ by

$$(f \wedge g)(x \wedge y) = f(x)g(y) - f(y)g(x).$$

We construct now a 2-graded Lie algebra $P = P(V)$ by taking $P_0 = V \otimes V^*$ and $P_1 = S^2V \oplus \Lambda^2(V^*)$. Clearly, P_0 is an ordinary Lie algebra and since V is a P_0-module so are S^2V and $\Lambda^2(V^*)$. We define a symmetric bilinear map $P_1 \times P_1 \to P_0$ as follows:

(i) it is zero on $S^2V \times S^2V$ and on $\Lambda^2(V^*) \otimes \Lambda^2(V^*)$

(ii) $[ab, f \wedge g] = [f \wedge g, ab] = f(a)b \otimes g + f(b)a \otimes g - g(a)b \otimes f - g(b)a \otimes f$

for $a, b \in V$ and $f, g \in V^*$.

We claim that the corresponding linear map θ: $P_1 \otimes P_1 \to P_0$ is a homomorphism of P_0-modules. For this it suffices to check that

$$\theta((c \otimes h) \cdot ((ab) \otimes (f \wedge g))) = (c \otimes h) \cdot (f(a)b \otimes g + f(b)a \otimes g - g(a)b \otimes f - g(b)a \otimes f).$$

for all $a, b, c \in V$ and $f, g, h \in V^*$. This is so because

$$\begin{aligned}(c \otimes h) \cdot ((ab) \otimes (f \wedge g)) &= ((c \otimes h) \cdot (ab)) \otimes (f \wedge g) + (ab) \otimes ((c \otimes h) \cdot (f \wedge g)) \\ &= (h(a)bc + h(b)ac) \otimes (f \wedge g) + (ab) \otimes (-f(c)h \wedge g - g(c)f \wedge h)\end{aligned}$$

and

$$\begin{aligned}&(c \otimes h) \cdot (f(a)b \otimes g + f(b)a \otimes g - g(a)b \otimes f - g(b)a \otimes f) \\ &\quad = f(a)(h(b)c \otimes g - g(c)b \otimes h) + f(b)(h(a)c \otimes g - g(c)a \otimes h) \\ &\qquad - g(a)(h(b)c \otimes f - f(c)b \otimes h) - g(b)(h(a)c \otimes f - f(c)a \otimes h).\end{aligned}$$

We also claim that θ satisfies (1), i.e., that

(2) $\theta(u, v) \cdot w + \theta(v, w) \cdot u + \theta(w, u) \cdot v = 0$

for $u, v, w \in P_1$. It suffices to verify this in the following two cases:

Case 1. $u = ab, v = cd, w = f \wedge g$

Case 2. $u = ab, v = f \wedge g, w = h \wedge k$

where $a, b, c, d \in V$ and $f, g, h, k \in V^*$.

In Case 1, (2) becomes

$$[cd, f \wedge g] \cdot ab + [f \wedge g, ab] \cdot cd = 0.$$

This is true because

$$\begin{aligned}[f \wedge g, ab] \cdot cd &= (f(a)b \otimes g + f(b)a \otimes g - g(a)b \otimes f - g(b)a \otimes f) \cdot cd \\ &= f(a)(g(c)bd + g(d)bc) + f(b)(g(c)ad + g(d)ac) \\ &\qquad - g(a)(f(c)bd + f(d)bc) - g(b)(f(c)ad + f(d)ac) \\ &= ((f \wedge g)(a \wedge c))bd + ((f \wedge g)(a \wedge d))(bc) \\ &\qquad + ((f \wedge g)(b \wedge c))ad + ((f \wedge g)(b \wedge d))ac\end{aligned}$$

and the last term changes sign if we interchange (a, b) with (c, d).

In Case 2, (2) reads

$$[ab, f \wedge g] \cdot (h \wedge k) + [h \wedge k, ab] \cdot (f \wedge g) = 0.$$

This is true because

$$\begin{aligned}[ab, f \wedge g] \cdot (h \wedge k) &= (f(a)b \otimes g + f(b)a \otimes g - g(a)b \otimes f \\ &\qquad - g(b)a \otimes f) \cdot (h \wedge k) \\ &= -f(a)(h(b)g \wedge k + k(b)h \wedge g) - f(b)(h(a)g \wedge k + k(a)h \wedge g) \\ &\qquad + g(a)(h(b)f \wedge k + k(b)h \wedge f) + g(b)(h(a)f \wedge k + k(a)h \wedge f) \\ &= ((fh)(ab))k \wedge g + ((fk)(ab))g \wedge h \\ &\qquad + ((gh)(ab))f \wedge k + ((gk)(ab))h \wedge f\end{aligned}$$

and the last term changes sign if we interchange (f, g) with (h, k).

Hence, we have proved that $P = P_0 \oplus P_1$ is a 2-graded Lie algebra. Note that $\operatorname{tr}([ab, f \wedge g]) = 0$ and hence $[P_1, P_1] \subset \mathrm{sl}(V)$.

Let P' be the derived algebra $[P, P]$ of P. Then $P_0' = \mathrm{sl}(V)$ and $P_1' = P_1$. The 2-graded Lie algebra P' is simple and for $\dim V = 2, 3, 4, \ldots$ we obtain an infinite series which is the series III of [**1**].

Series IV. We use the notation from the previous section. We shall construct a 2-graded Lie algebra $Q = Q(V)$ such that $Q_0 = V \otimes V^*$ and $Q_1 = S^2(V^*) \oplus \Lambda^2 V$. The symmetric bilinear map $Q_1 \times Q_1 \to Q_0$ is defined by

(i) it is zero on $S^2(V^*) \times S^2(V^*)$ and on $(\Lambda^2 V) \times (\Lambda^2 V)$;

(ii) $[a \wedge b, fg] = [fg, a \wedge b] = -f(a)b \otimes g - g(a)b \otimes f + f(b)a \otimes g + g(b)a \otimes f$

for $a, b \in V$ and $f, g \in V^*$.

This time we omit the verifications of the facts that the corresponding linear map $Q_1 \otimes Q_1 \to Q_0$ is a homomorphism of Q_0-modules and that it satisfies the identity (1). These verifications are similar to ones in the preceeding section.

Hence, $Q = Q_0 \oplus Q_1$ is a 2-graded Lie algebra. Since tr $([ab, f \wedge g]) = 0$ we have again $[Q_1, Q_1] \subset \mathrm{sl}\ (V)$.

Let Q' be the derived algebra $[Q, Q]$ of Q. Then $Q_0' = \mathrm{sl}\ (V)$ and $Q_1' = Q_1$. The 2-graded Lie algebra Q' is simple and for dim $V = 2, 3, 4, \ldots$ we obtain an infinite series which is the series IV of [**1**].

Series III and IV coincide. Using the notation from the previous sections, we want to prove that if $P = P(V)$ and $Q = Q(V)$ then the 2-graded Lie algebras P' and Q' are isomorphic. Clearly, it suffices to prove that the 2-graded Lie algebras P and Q are isomorphic.

THEOREM. *The 2-graded Lie algebras $P = P(V)$ and $Q = Q(V)$ are isomorphic.*

Proof. Let $u: V \to V^*$ be any vector space isomorphism. Then its transpose ${}^t u: (V^*)^* = V \to V^*$ is also an isomorphism and we define $u^* = ({}^t u)^{-1}: V^* \to V$.

Using u and u^* we define vector space isomorphisms:

$$S^2 u: S^2 V \to S^2(V^*),$$
$$\Lambda^2(u^*): \Lambda^2(V^*) \to \Lambda^2 V,$$
$$\theta_0: P_0 = V \otimes V^* \to Q_0 = V \otimes V^*$$

where $S^2 u$ is the canonical extension of u, $\Lambda^2(u^*)$ the canonical extension of u^* and

$$\theta_0(a \otimes f) = -u^*(f) \otimes u(a)$$

for $a \in V, f \in V^*$.

We define a vector space isomorphism $\theta: P \to Q$ to be the direct sum of the isomorphisms θ_0, $S^2 u$ and $\Lambda^2(u^*)$.

We shall write $f(x) = \langle x, f\rangle = \langle f, x\rangle$ for $x \in V$ and $f \in V^*$. By the definition of u^* we have

$$u(x)(u^*(f)) = \langle u(x), u^*(f)\rangle = \langle u(x), {}^t u^{-1}(f)\rangle$$
$$= \langle u^{-1}(u(x)), f\rangle = \langle x, f\rangle = f(x).$$

We claim that θ is an isomorphism of 2-graded Lie algebras. We have to check that

(3) $\quad \theta([s, t]) = [\theta(s), \theta(t)]$

for $s, t \in P$. It clearly suffices to verify (3) in the following cases:

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(i) $s = a \otimes f, t = b \otimes g$;
(ii) $s = a \otimes f, t = bc$;
(iii) $s = a \otimes f, t = f \wedge h$;
(iv) $s = ab, t = f \wedge g$

where $a, b, c \in V$ and $f, g, h \in V^*$.
In Case (i) we have

$$\begin{aligned}\theta([a \otimes f, b \otimes g]) &= \theta(f(b)a \otimes g - g(a)b \otimes f)\\ &= -\langle b, f\rangle u^*(g) \otimes u(a) + \langle a, g\rangle u^*(f) \otimes u(b)\\ &= -\langle u(b), u^*(f)\rangle u^*(g) \otimes u(a) + \langle u(a), u^*(g)\rangle u^*(f) \otimes u(b)\\ &= [-u^*(f) \otimes u(a), -u^*(g) \otimes u(b)]\\ &= [\theta(a \otimes f), \theta(b \otimes g)].\end{aligned}$$

In Case (ii) we have

$$\begin{aligned}\theta([a \otimes f, bc]) &= \theta((a \otimes f) \cdot (bc))\\ &= \theta(f(b)ac + f(c)ab)\\ &= f(b)u(a)u(c) + f(c)u(a)u(b)\\ &= \langle u(b), u^*(f)\rangle u(a)u(c) + \langle u(c), u^*(f)\rangle u(a)u(b)\\ &= -(u^*(f) \otimes u(a)) \cdot (u(b)u(c))\\ &= [-u^*(f) \otimes u(a), u(b)u(c)]\\ &= [\theta(a \otimes f), \theta(bc)].\end{aligned}$$

In Case (iii) we have

$$\begin{aligned}\theta([a \otimes f, g \wedge h]) &= \theta((a \otimes f) \cdot (g \wedge h))\\ &= \theta(-g(a)f \wedge h - h(a)g \wedge f)\\ &= -g(a)u^*(f) \wedge u^*(h) - h(a)u^*(g) \wedge u^*(f)\\ &= -\langle u(a), u^*(g)\rangle u^*(f) \wedge u^*(h) - \langle u(a), u^*(h)\rangle u^*(g) \wedge u^*(f)\\ &= -(u^*(f) \otimes u(a)) \cdot (u^*(g) \wedge u^*(h))\\ &= [-u^*(f) \otimes u(a), u^*(g) \wedge u^*(h)]\\ &= [\theta(a \otimes f), \theta(g \wedge h)].\end{aligned}$$

Finally, in Case (iv) we have

$$\begin{aligned}\theta([ab, f \wedge g]) &= \theta(f(a)b \otimes g + f(b)a \otimes g - g(a)b \otimes f - g(b)a \otimes f)\\ &= -f(a)u^*(g) \otimes u(b) - f(b)u^*(g) \otimes u(a)\\ &\qquad + g(a)u^*(f) \otimes u(b) + g(b)u^*(f) \otimes u(a)\\ &= -\langle u(a), u^*(f)\rangle u^*(g) \otimes u(b) - \langle u(b), u^*(f)\rangle u^*(g) \otimes u(a)\\ &\qquad + \langle u(a), u^*(g)\rangle u^*(f) \otimes u(b) + \langle u(b), u^*(g)\rangle u^*(f) \otimes u(a)\\ &= [u(a)u(b), u^*(f) \wedge u^*(g)]\\ &= [\theta(ab), \theta(f \wedge g)].\end{aligned}$$

The theorem is proved.

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