A LOWER BOUND ON THE HOMOLOGICAL BIDIMENSION OF A NON-UNITAL C*-ALGEBRA

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(Received 15 January, 1997)

1. Introduction. Let A be a C*-algebra. For each Banach A-bimodule X, the second continuous Hochschild cohomology group $\mathcal{H}^2(A, X)$ of A with coefficients in X is defined (see [**6**]); there is a natural correspondence between the elements of this group and equivalence classes of singular, admissible extensions of A by X. Specifically this means that $\mathcal{H}^2(A, X) \neq \{0\}$ for some X if and only if there exists a Banach algebra B with Jacobson radical R such that $R^2 = \{0\}$, R is complemented as a Banach space, and $B/R \cong A$, but B has no strong Wedderburn decomposition; i.e., there is no closed subalgebra C of B such that $B \cong C \oplus R$. In turn this is equivalent to $\operatorname{db} A \geqslant 2$, where $\operatorname{db} A$ is the *homological bidimension* of A; i.e., the homological dimension of $A^{\#}$, the unitization of A, as an A-bimodule [**6**, III.5.15]. This paper is concerned with the following basic question, which was posed in [**7**].

Is $\operatorname{db} A \geqslant 2$ for each infinite-dimensional C*-algebra A?

A positive answer to this question has been obtained in each of the following cases.

(i) A is commutative [**8**];

(ii) A is separable and has a closed ideal of finite codimension that cannot be complemented as a subalgebra, or A is separable and non-unital [**1**];

(iii) A is a CCR-algebra [**11**].

Actually in each case the stronger result is established that $\operatorname{dg} A \geqslant 2$ in the case where A is an infinite-dimensional member of the specified class; here $\operatorname{dg} A$ is the *global homological dimension* of A [**6**, III.5.7], and it is known that $\operatorname{dg} A \geqslant 2$ if and only if there exist Banach left A-modules Y and Z such that $\mathcal{H}^2(A, \mathcal{B}(Y, Z)) \neq \{0\}$, where $\mathcal{B}(Y, Z)$ denotes the A-bimodule of continuous linear mappings from Y into Z.

Let A be a C*-algebra, and suppose that A admits a non-unital, closed ideal I of finite codimension. We show in §3 of this paper that $\mathcal{H}^2(A, I\hat{\otimes}I) \neq \{0\}$ in this case, and so in particular $\operatorname{db} A \geqslant 2$. In fact, we give an explicit formula for a cocycle μ of A with values in $I\hat{\otimes}I$ and show that μ does not cobound. As a corollary, we obtain the fact that $\operatorname{db} A \geqslant 2$ for each infinite-dimensional type I C*-algebra.

Finally we shall demonstrate that our methods may also be used to establish that $\operatorname{dg} A \geqslant 2$ in certain cases.

2. Preliminaries. Let A be a C*-algebra. If A is unital, we write 1_A for the identity of A. We write A_+ (respectively, A_{sa}) for the positive (respectively, self-adjoint) elements of A, and

The author was supported by HCM grant No. ERB CHBI-CT94-1366.

https://doi.org/10.1017/S0017089500032778 Published online by Cambridge University Press

we write $A^{\#}$ for the unitization of A in the sense of [**12**, 1.1.3], so that $A^{\#}= A$ if A is unital, and $A^{\#}= A \oplus \mathbb{C}$ otherwise. By an *approximate unit* for A we shall mean an increasing net of elements of A_+ of norm at most one that is an approximate identity for A in the usual sense. We denote by $\Lambda(A)$ the set of elements a in A_+ such that $\|a\| < 1$. By [**12**, 1.4.2], $\Lambda(A)$ is an approximate unit for A in the partial ordering on A_{sa}; as in [**12**], we shall refer to it as the *canonical approximate unit* for A. We denote by $S(A)$ the set of states on A.

Let E be a Banach space. We denote by E^* the continuous dual of E. Also, we write $E \otimes F$ (respectively, $E\hat{\otimes}F$) for the algebraic (respectiely, projective) tensor product of Banach spaces E and F. We write $\mathcal{B}(E, F)$ for the Banach space of bounded linear maps from E into F. The (projective) tensor product of two operators $S \in \mathcal{B}(E_1, F_1)$ and $T \in \mathcal{B}(E_2, F_2)$ is denoted by $S \otimes T \in \mathcal{B}(E_1\hat{\otimes}E_2, F_1\hat{\otimes}F_2)$.

Let A be a Banach algebra. Then A^* is a Banach A-bimodule for the operations

$$\langle b, \varphi \cdot a\rangle = \langle ab, \varphi\rangle \text{ and } \langle b, a \cdot \varphi\rangle = \langle ba, \varphi\rangle \ (\varphi \in A^*, a, b \in A).$$

Also, the Banach space $A\hat{\otimes}A$ is a Banach A-bimodule for the operations

$$a \cdot (b \otimes c) = ab \otimes c \text{ and } (a \otimes b) \cdot c = a \otimes bc \ (a, b, c \in A).$$

We shall use the fact that $((\varphi \cdot a) \otimes \psi)(u) = (\varphi \otimes \psi)(a \cdot u)$ and $(\varphi \otimes (a \cdot \psi))(u) = (\varphi \otimes \psi)(u \cdot a)$ for $\varphi, \psi \in A^*, a \in A$ and $u \in A\hat{\otimes}A$; these formulae are immediate from the definitions.

Let A be a Banach algebra, and let X be a Banach A-bimodule. We denote by $\mathcal{Z}^2(A, X)$ the Banach space of continuous bilinear maps $\mu : A \times A \to X$ that satisfy the cocycle identity

$$a \cdot \mu(b, c) - \mu(ab, c) + \mu(a, bc) - \mu(a, b) \cdot c = 0 \ (a, b, c \in A);$$

the elements in $\mathcal{Z}^2(A, X)$ are the 2-cocycles of A with coefficients in X. For $T \in \mathcal{B}(A, X)$ we define

$$(\delta^1 T)(a, b) = a \cdot T(b) - T(ab) + T(a) \cdot b \ (a, b \in A);$$

the map $T \mapsto \delta^1 T$ is a continuous linear map from $\mathcal{B}(A, X)$ into $\mathcal{Z}^2(A, X)$ whose range is denoted by $\mathcal{N}^2(A, X)$; the elements in $\mathcal{N}^2(A, X)$ are the 2-coboundaries of A with coefficients in X. The quotient group $\mathcal{Z}^2(A, X)/\mathcal{N}^2(A, X)$, denoted by $\mathcal{H}^2(A, X)$, is the *second continuous Hochschild cohomology group of A with coefficients in X*. For more information on (Banach) Hochschild cohomology see [**6**] and [**10**]; on the question of the role of second cohomology groups and the splittings of extensions of a Banach algebra, we refer to [**2**].

3. Nontrivial cocycles for nonunital C*-algebras. We start this section with a general lemma, which is inspired by [**6**, V.2.14].

LEMMA 1. *Let A be a* C**-algebra, let E be a Banach space, and let $\Theta : A \to E\hat{\otimes}E$ be a linear map. Let $\Lambda(A)$ be the canonical approximate unit for A, and let $\mathcal{U}(0)$ denote the system of neighbourhoods of* 0 *in $(E^*, \sigma(E^*, E))$. Suppose that, for each $\varepsilon > 0$, there exists a non-empty subset Y of the unit ball of E^* such that, for all $\varphi_1, \dots, \varphi_n \in Y, a \in \Lambda(A)$ and $U \in \mathcal{U}(0)$, there exist $b \in \Lambda(A)$ and $\varphi \in U \cap Y$ with $b \geqslant a$ and*

$$|(\varphi_i \otimes \varphi)(\Theta(b)) - 1| < \varepsilon, \text{ for } 1 \leqslant i \leqslant n. \tag{3.1}$$

Then Θ is unbounded.

https://doi.org/10.1017/S0017089500032778 Published online by Cambridge University Press

Proof. Assume that Θ is bounded. Let $\varepsilon > 0$, and let Y be the corresponding subset of the unit ball of E^*. First we inductively construct, for every positive integer n, elements φ_n, ψ_n of Y and a_n, b_n of $\Lambda(A)$ such that the following hold for each n.

$$|(\varphi_i \otimes \psi_n)(\Theta(b_n - a_n)) - 1| < 2\varepsilon \text{ for } i \leqslant n; \tag{3.2}$$

$$|(\varphi_i \otimes \psi_j)(\Theta(b_n - a_n))| < \varepsilon \text{ for } i \leqslant n, j < n; \tag{3.3}$$

$$|(\varphi_i \otimes \psi_n)(\Theta(b_j - a_j))| < \varepsilon \text{ for } i \leqslant n, j < n; \tag{3.4}$$

$$|(\varphi_n \otimes \psi_i)(\Theta(b_j - a_j))| < \varepsilon \text{ for } i, j < n; \tag{3.5}$$

$$a_n \leqslant b_n, \text{ and } b_{n-1} \leqslant a_n \text{ if } n \geqslant 2. \tag{3.6}$$

If $n = 1$, then it is immediate from the condition in the lemma that $\varphi_1, \psi_1 \in Y$ and $a_1, b_1 \in \Lambda(A)$ exist to satisfy (3.2) to (3.6).

Now assume that $n \geqslant 2$, and that φ_i, ψ_i, a_i and b_i have already been constructed for $1 \leqslant i \leqslant n-1$. It follows from the condition in the lemma that there exists $\varphi_n \in Y$ such that (3.5) is true; and we may then inductively choose elements $\rho_1, \rho_2, \ldots$ of Y and an increasing sequence $x_0, x_1, \ldots$ of elements of $\Lambda(A)$ such that $x_0 = b_{n-1}$ and

$$|(\varphi_i \otimes \rho_r)(\Theta(b_j - a_j))| < \varepsilon, \quad |(\varphi_i \otimes \rho_r)(\Theta(x_r - x_{r-1})) - 1| < 2\varepsilon$$

for $i \leqslant n, j < n$ and $r \geqslant 1$. Since $(x_r)_r$ is an increasing sequence bounded above, we have that $x_r - x_{r-1} \to 0$ weakly, and so there exists $r \geqslant 1$ such that

$$|(\varphi_i \otimes \psi_j)(\Theta(x_r - x_{r-1}))| < \varepsilon$$

for $i \leqslant n$ and $j < n$. We choose $\psi_n = \rho_r, a_n = x_{r-1}$ and $b_n = x_r$. Then (3.2) to (3.6) are satisfied, and the induction continues.

We now fix a positive integer n and consider the element $a = \sum_{i=1}^{n}(b_i - a_i)$. Certainly we have $\|a\| \leqslant 1$. The conditions (3.2) to (3.6) imply that

$$|(\varphi_i \otimes \psi_j)(\Theta(a)) - 1| < 2n\varepsilon \text{ for } 1 \leqslant i \leqslant j \leqslant n$$

and

$$|(\varphi_i \otimes \psi_j)(\Theta(a))| < n\varepsilon \text{ for } 1 \leqslant j < i \leqslant n.$$

We may thus deduce from [**6**, II.2.48] (see also [**9**, Lemma 3.1]) that

$$\|\Theta\| \geqslant \|\Theta(a)\| \geqslant \frac{1}{2\pi} \log n - 2n^2\varepsilon,$$

which cannot be true because ε and n were chosen arbitrarily. We have arrived at a contradiction, and the result follows.

We can now state the main result of this paper.

THEOREM 1. *Let A be a* C*-*algebra and let I be a non-unital closed ideal of A of finite codimension. Then* $\mathcal{H}^2(A, I\hat{\otimes}I) \neq 0$. *In particular*, $\operatorname{db} A \geqslant 2$.

Proof. Since $\mathcal{H}^2(A, I\hat{\otimes}I)$ and $\mathcal{H}^2(A^{\#}, I\hat{\otimes}I)$ are isomorphic, we may suppose that A is unital and that $A/I \neq 0$. We shall regard $A\hat{\otimes}A$ and $I\hat{\otimes}I$ as Banach A-bimodules in the usual way, so that $a \cdot (b \otimes c) = ab \otimes c$ and $(a \otimes b) \cdot c = a \otimes bc$ for $a, b, c \in A$. Likewise, we shall regard $(A/I)\hat{\otimes}(A/I)$ as a Banach A/I-bimodule. Let $\{e^k_{ij} : k = 1, \ldots, N; i, j = 1, \ldots, n_k\}$ be a *-matricial basis (in the sense of [5, p. 113]) for the finite-dimensional C*-algebra A/I, so that $\sum_{k,i} e^k_{ii} = 1_{A/I}$ and

$$e^k_{ij}e^r_{st} = \delta_{kr}\delta_{js}e^k_{it}, \left(e^k_{ij}\right)^* = e^k_{ji}$$

for $k, r = 1, \ldots, N$, $i, j = 1, \ldots, n_k$ and $s, t = 1, \ldots, n_r$. We set

$$\Delta = \sum_{i,k} e^k_{i1} \otimes e^k_{1i},$$

and we denote by $\pi : (A/I)\hat{\otimes}(A/I) \to A/I, a \otimes b \mapsto ab$, the product map for A/I. Then Δ is a diagonal for A/I; i.e., we have $\pi(\Delta) = 1$ and

$$x \cdot \Delta = \Delta \cdot x \text{ for } x \in A/I. \tag{3.7}$$

Let $\kappa : A \to A/I$ be the quotient map, and let $\rho : A/I \to A$ be a linear map such that $\kappa \circ \rho = \mathrm{id}_{A/I}$; note that ρ is continuous because A/I is finite-dimensional. We consider the continuous linear map

$$T : A \to A\hat{\otimes}A : a \mapsto a \cdot (\rho \otimes \rho)(\Delta) - (\rho \otimes \rho)(\kappa(a) \cdot \Delta).$$

Set $\mu = \delta^1 T$, and let $a, b \in A$. A simple calculation shows that

$$\mu(a, b) = a \cdot (\rho \otimes \rho)(\Delta) \cdot b - a \cdot (\rho \otimes \rho)(\kappa(b) \cdot \Delta) + (\rho \otimes \rho)(\kappa(ab) \cdot \Delta) - (\rho \otimes \rho)(\kappa(a) \cdot \Delta) \cdot b.$$

It follows that

$$\begin{aligned}(\kappa \otimes \mathrm{id}_A)(\mu(a, b)) &= \kappa(a) \cdot \left(\mathrm{id}_{A/I} \otimes \rho\right)(\Delta) \cdot b - \kappa(a) \cdot \left(\mathrm{id}_{A/I} \otimes \rho\right)(\kappa(b) \cdot \Delta) \\ &\quad + \left(\mathrm{id}_{A/I} \otimes \rho\right)(\kappa(ab) \cdot \Delta) - \left(\mathrm{id}_{A/I} \otimes \rho\right)(\kappa(a) \cdot \Delta) \cdot b \\ &= \left(\kappa(a) \cdot \left(\mathrm{id}_{A/I} \otimes \rho\right)(\Delta) - \left(\mathrm{id}_{A/I} \otimes \rho\right)(\kappa(a) \cdot \Delta)\right) \cdot b \\ &\quad + \left(\mathrm{id}_{A/I} \otimes \rho\right)(\kappa(ab) \cdot \Delta) - \kappa(a) \cdot \left(\mathrm{id}_{A/I} \otimes \rho\right)(\kappa(b) \cdot \Delta) \\ &= 0 \cdot b + 0 = 0.\end{aligned}$$

Thus $\mu(a, b) \in \ker(\kappa \otimes \mathrm{id}_A)$. But also $\mu(a, b) \in A \otimes A$, the algebraic tensor product of A with itself, and so $\mu(a, b) \in \ker(\kappa) \otimes A = I \otimes A$ by a standard piece of linear algebra. Analogously we can show (using (3.7)) that $\mu(a, b) \in A \otimes I$. Hence $\mu(a, b)$ lies in $(I \otimes A) \cap (A \otimes I) = I \otimes I$, and so we have shown that $\mu \in \mathcal{Z}^2(A, I\hat{\otimes}I)$.

We claim that μ defines a non-trivial element of the group $\mathcal{H}^2(A, I\hat{\otimes}I)$. To see this, let us assume that this is not the case. Then there exists a continuous linear map $\tilde{T} : A \to I\hat{\otimes}I$ such that

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$$\mu(a, b) = a \cdot \tilde{T}(b) - \tilde{T}(ab) + \tilde{T}(a) \cdot b \quad (a, b \in A). \tag{3.8}$$

Recall that each $\varphi \in S(I)$ has a unique extension to a state on A, that we shall again denote by φ. Let (u_λ) be an approximate unit for I, and let $\varphi, \psi \in S(I)$. From (3.8) and the definition of μ, we see that

$$u_\lambda \cdot (\rho \otimes \rho)(\Delta) \cdot u_\mu = u_\lambda \cdot \tilde{T}(u_\mu) - \tilde{T}(u_\lambda u_\mu) + \tilde{T}(u_\lambda) \cdot u_\mu$$

and therefore

$$\begin{aligned}((\varphi \cdot u_\lambda) \otimes (u_\mu \cdot \psi))((\rho \otimes \rho)(\Delta)) = ((\varphi \cdot u_\lambda) \otimes \psi)\big(\tilde{T}(u_\mu)\big) - (\varphi \otimes \psi)\big(\tilde{T}(u_\lambda u_\mu)\big) \\ + \big(\varphi \otimes (u_\mu \cdot \psi)\big)\big(\tilde{T}(u_\lambda)\big)\end{aligned}$$

for all λ, μ. Taking limits, first over λ (for fixed μ) and then over μ, and using the fact that the nets $(\varphi \cdot u_\lambda)$ (respectively, $(u_\lambda \cdot \psi)$) are norm-convergent to φ (respectively, to ψ), we obtain that

$$(\varphi \otimes \psi)((\rho \otimes \rho)(\Delta)) = \lim_\lambda (\varphi \otimes \psi)\big(\tilde{T}(u_\lambda)\big). \tag{3.9}$$

Since I is non-unital, it admits a net (φ_α) of states that is weak-* convergent to 0 [**4**, 2.12.13]. We may suppose that (φ_α) converges in the weak-* topology on A^* to a state φ on A. Then φ vanishes on I, and $\varphi \circ \rho$ is a state on A/I. Hence there exists a unitary element u in A/I and $1 \leqslant l \leqslant N$ such that $\varphi\big(\rho(ue^l_{11})\rho(ue^l_{11})^*\big) = \varphi\big(\rho(ue^l_{11}u^*)\big) > 0$; we may suppose that $\varphi_\alpha\big(\rho(ue^l_{11})\rho(ue^l_{11})^*\big) \geqslant \delta > 0$, for some δ independent of α. We consider the positive functionals

$$\psi_\alpha : I \to \mathbb{C} : a \mapsto \Big(\varphi_\alpha\Big(\rho(ue^l_{11})\rho(ue^l_{11})^*\Big)\Big)^{-1} \varphi_\alpha\Big(\rho(ue^l_{11})a\rho(ue^l_{11})^*\Big).$$

Let (u_λ) be an approximate identity for I which is quasi-central (see [**12**, 3.12.14]), so that $\|u_\lambda a - a u_\lambda\| \to 0$, for each $a \in A$. Then it is easily checked that $\lim_\lambda \psi_\alpha(u_\lambda) = 1$, and so each ψ_α is a state on I. Clearly we have

$$\psi_\alpha \to 0 \text{ (in } \sigma(I^*, I)) \text{ and } \psi_\alpha\big(\rho(e^l_{11})\big) \to 1. \tag{3.10}$$

Now let Θ be the restriction of $\tilde{T}$ to I, and let $\Lambda(I)$ be the canonical approximate unit for I. We wish to apply Lemma 1 to Θ, and so we choose $\varepsilon > 0$. Let Y be the set of φ in $S(I)$ such that $|\varphi\big(\rho(e^l_{11})\big) - 1| < \varepsilon$. Choose $\varphi_1, \ldots, \varphi_n \in Y, a \in \Lambda(I)$ and a neighbourhood U of 0 in $(I^*, \sigma(I^*, I))$. From (3.10) we see that we can find α_0 such that $\psi_\alpha \in Y \cap U$ for each $\alpha \geqslant \alpha_0$; it also follows that the ψ_α, considered as functionals on A, are weak-* convergent to the state $\psi \circ \kappa$, where ψ is the pure state on A/I that satisfies $\psi(e^l_{11}) = 1$ and $\psi(e^k_{ij}) = 0$ whenever $(k, i, j) \neq (l, 1, 1)$. Let $\varphi \in S(I)$. Then

$$\begin{aligned}\lim_\alpha (\varphi \otimes \psi_\alpha)((\rho \otimes \rho)(\Delta)) &= \big(\varphi \otimes \psi \circ \kappa\big)((\rho \otimes \rho)(\Delta)) \\ &= (\varphi \otimes \psi)((\rho \otimes \mathrm{id})(\Delta)) \\ &= \sum_{k,i} \varphi\big(\rho(e^k_{i1})\big)\psi(e^k_{1i}) = \varphi\big(\rho(e^l_{11})\big).\end{aligned}$$

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Hence, by (3.9) and the definition of Y, we may choose $\alpha_1 \geqslant \alpha_0$ and then $b \in \Lambda(I)$ such that $b \geqslant a$ and

$$|(\varphi_i \otimes \psi_{\alpha_1})(\Theta(b)) - 1| < \varepsilon \quad (1 \leqslant i \leqslant n).$$

Choose $\varphi = \psi_{\alpha_1}$. Then $\varphi \in Y \cap U$, and the condition (3.1) of Lemma 1 is satisfied. Consequently Θ is unbounded. This is a contradiction.

REMARK. In the case where A is non-unital and $I = A$, the cocycle that we have constructed is the map $\mu : (a, b) \mapsto a \otimes b, A \times A \to A \hat{\otimes} A$. This cocycle has been considered before in [**9**, Theorem 3.2], where it was shown that μ defines a non-trivial element of $\mathcal{H}^2(A, A\hat{\otimes}A)$ in the case where A is a non-unital, amenable Banach function algebra. It would be interesting to know whether the condition that A be amenable is needed here.

We shall need the fact that each infinite-dimensional type I C*-algebra contains a closed non-unital ideal of finite codimension. A proof of this is (implicitly) contained in [**1**, §7]; we shall give another, shorter, proof here for the sake of completeness.

Recall that a closed ideal I of a C*-algebra A is *essential* ([**12**, 3.12.7]) if each non-zero closed ideal of A has non-zero intersection with I, or, equivalently, when the *annihilator* $I^\perp = \{a \in A : aI = 0\}$ is zero. If I is unital, then $I^\perp = (1_{A^\#} - 1_I)A$ and $I \oplus I^\perp = A$; hence if I is unital and essential, then $I = A$.

COROLLARY 1. *Let A be an infinite-dimensional type I C*-algebra. Then* $\mathrm{db}\, A \geqslant 2$.

Proof. Let $\mathcal{C}$ be the set of non-unital closed ideals of A. An application of Zorn's lemma shows that the set $\mathcal{C} \cup \{0\}$ has a maximal element, I say. Set $B = A/I$. By [**12**, 6.2.11], B contains a closed, essential ideal J which has continuous trace. The maximality of I implies that J is unital. Thus $J = B$ and B has continuous trace. Hence by [**12**, 6.1.11] the primitive ideal space $\mathrm{Prim}(B)$ of B is a compact Hausdorff space. Let $P \in \mathrm{Prim}(B)$, and let $F = \mathrm{Prim}(B)\backslash\{P\}$. Then F is homeomorphic to $\mathrm{Prim}(P)$. But P is unital, and so (see [**4**, 3.1.8]) $\mathrm{Prim}(P)$ is compact. Hence F is a compact, and therefore closed, subset of $\mathrm{Prim}(B)$. We have shown that $\mathrm{Prim}(B)$ is a discrete compact space, which must therefore be finite. Thus $B = A/I$ is finite-dimensional. The result now follows from Theorem 1.

Let I be a closed ideal of a C*-algebra A, and let X be a Banach A/I-bimodule. Then the quotient map $\kappa : A \to A/I$ induces a Banach A-bimodule structure on X. We have a canonical map

$$\mathcal{H}^2(A/I, X) \to \mathcal{H}^2(A, X), \mu + \mathcal{N}^2(A/I, X) \mapsto \mu \circ (\kappa \otimes \kappa) + \mathcal{N}^2(A, X).$$

We claim that this is an embedding. Indeed, let $\mu \in \mathcal{Z}^2(A/I, X)$, and suppose that $\mu \circ (\kappa \otimes \kappa) \in \mathcal{N}^2(A, X)$. Then there exists a continuous linear map $T : A \to X$ such that

$$\mu(a + I, b + I) = (a + I) \cdot T(b) - T(ab) + T(a) \cdot (b + I) \quad (a, b \in A).$$

It follows that T vanishes on I^2, the set of all products of two elements of I. But $I^2 = I$ by Cohen's factorization theorem. Thus $I \subseteq \ker T$ and T induces a continuous linear map $\tilde{T} : A/I \to X$. Clearly we have $\mu = \delta^1 \tilde{T}$. Hence $\mu \in \mathcal{N}^2(A/I, X)$, and our claim follows.

In particular, we see that $\mathrm{db}\, A \geqslant 2$ if $\mathrm{db}\, A/I \geqslant 2$. This fact together with an easy adaptation of the proof of [**1**, Theorem 4] implies that the following theorem analogous to [**1**, Theorem 4] is true.

THEOREM 2. *Suppose that* $\operatorname{db} A \geqslant 2$ *for each infinite-dimensional, unital, simple* C^**-algebra* A. *Then* $\operatorname{db} A \geqslant 2$, *for each infinite-dimensional* C^**-algebra* A.

Although primarily designed to give lower bounds on $\operatorname{db} A$, our methods may also be used to establish that $\operatorname{dg} A \geqslant 2$ in certain cases. We shall demonstrate this in our next theorem, where we give another application of Lemma 1.

For Banach left A-modules Y and Z, the Banach space $\mathcal{B}(Y, Z)$ will always be endowed with the A-bimodule structure given by

$$(a \cdot T)(y) = a \cdot T(y) \text{ and } (T \cdot b)(y) = T(b \cdot y) \quad (y \in Y) \tag{3.11}$$

for $a, b \in A$ and $T \in \mathcal{B}(Y, Z)$. From the general theory it is known that $\operatorname{dg} A \geqslant 2$ if and only if $\mathcal{H}^2(A, \mathcal{B}(Y, Z)) \neq \{0\}$, for some Y and Z. (See [**6**, III.5.15].)

THEOREM 3. *Let* A *be a* C^**-algebra and let* I *be a closed ideal of* A *of finite codimension. Suppose that* I *admits a sequence of states that is weak-* convergent to* 0. *Then we have* $\mathcal{H}^2(A, \mathcal{B}(I^{**}, I \hat{\otimes} (I \cdot I^{**}))) \neq \{0\}$. *In particular*, $\operatorname{dg} A \geqslant 2$.

REMARKS. (i) Here $I \cdot I^{**}$ denotes the set of all products ab, where $a \in I$ and $b \in I^{**}$. By the Cohen factorization theorem, $I \cdot I^{**}$ is a closed subspace of I^{**}.

(ii) We shall regard I^{**} and $I \hat{\otimes} (I \cdot I^{**})$ as left Banach A-modules in the usual way, so that $\mathcal{B}(I^{**}, I \hat{\otimes} (I \cdot I^{**}))$ carries the A-bimodule structure defined in (3.11).

(iii) Clearly the condition on I in the theorem is satisfied in the case where I is non-unital and separable. Hence our theorem contains the main result in [**1**].

Proof. Suppose that I satisfies the condition in the theorem. Let $\mu : A \times A \to I \hat{\otimes} I$ be the cocycle constructed in Theorem 1. For $a, b \in A$ and $c \in I^{**}$ we set

$$\nu(a, b)(c) = \mu(a, b) \cdot c.$$

This defines a map $\nu : A \times A \to \mathcal{B}(I^{**}, I \hat{\otimes} (I \cdot I^{**}))$, and it is easily checked that ν is 2-cocycle. We claim that ν defines a non-trivial element of the group $\mathcal{H}^2(A, \mathcal{B}(I^{**}, I \hat{\otimes} (I \cdot I^{**})))$. To see this, let us assume that this is not the case. Then there is a continuous linear map $T : A \to \mathcal{B}(I^{**}, I \hat{\otimes} (I \cdot I^{**}))$ such that

$$a \cdot T(b)(c) - T(ab)(c) + T(a)(bc) = \mu(a, b) \cdot c \tag{3.12}$$

for all $a, b \in A$ and $c \in I^{**}$. We set $E = I \cdot I^{**}$, and for $a \in I$ we set

$$\Theta(a) = (\iota \otimes \mathrm{id}_E)(T(a)(1_{I^{**}})),$$

where $\iota : I \hookrightarrow E$ is the inclusion map. Then $\Theta : I \to E \hat{\otimes} E$ is a continuous linear map. We wish to apply Lemma 1 to Θ. Let $\varepsilon > 0$ be given.

As in Theorem 1, let $\rho : A/I \to A$ be a linear map that is a right inverse for the quotient map $\kappa : A \to A/I$. The condition on I implies that the construction in the proof of Theorem 1 yields a *-matrical basis $\{e_{ij}^k\}$ of A/I and a sequence (ψ_m) of states on I that is weak-* convergent to 0 such that

$$\lim_m (\psi \otimes \psi_m)((\rho \otimes \rho)(\Delta)) = \psi(\rho(e_{11}^1)) \quad (\psi \in S(I)), \tag{3.13}$$

https://doi.org/10.1017/S0017089500032778 Published online by Cambridge University Press

where $\Delta = \sum_{k,i} e^k_{i1} \otimes e^k_{1i}$, and

$$\lim_m \psi_m(\rho(e^1_{11})) = 1. \tag{3.14}$$

Let Y_0 be the set of all $\phi \in S(I)$ such that $|\phi(\rho(e^1_{11})) - 1| < \varepsilon$. Let $\alpha : I^* \hookrightarrow I^{***}$ be the canonical embedding, and let $\beta : I^{***} \to E^*$ be the map that restricts an element ψ in I^{***} to E. We choose Y to be the image of Y_0 under the mapping $\beta\circ\alpha : I^* \to E^*$. Choose $\varphi_1, \dots, \varphi_n \in Y$, $a \in \Lambda(I)$ and a neighbourhood U of 0 in $(E^*, \sigma(E^*, E))$. By the definition of Y, there are $\phi_1, \dots, \phi_n \in Y_0$ such that $\varphi_i = (\beta\circ\alpha)(\phi_i)$, $(i = 1, \dots, n)$. For each m we set $\tau_m = (\beta\circ\alpha)(\psi_m)$. Clearly each τ_m is an extension of ψ_m to E and it is easily checked that $\tau_m \to 0$ in the weak-* topology on E^*. Hence, by (3.14) there exists m_0 such that $\tau_m \in Y \cap U$, for all $m \geqslant m_0$. Also it follows from (3.13) and the definition of Y that there exist $\delta \in (0, \varepsilon)$, $c \in \Lambda(I)$ and $m_1 \geqslant m_0$ such that

$$|((\phi_i \cdot c) \otimes \psi_m)((\rho \otimes \rho)(\Delta)) - 1| \leqslant \varepsilon - \delta \quad (1 \leqslant i \leqslant n), \tag{3.15}$$

for all $m \geqslant m_1$, and

$$\|\phi_i \cdot c - \phi_i\| \|\Theta\| < \delta \quad (1 \leqslant i \leqslant n). \tag{3.16}$$

Now let $u_0 = 0 \leqslant u_1 \leqslant u_2 \dots$ be an increasing sequence in $\Lambda(I)$ such that $u_1 \geqslant a$ and

$$\lim_m \|cu_m - c\| = 0 \text{ and } \lim_m \psi_m(u_m) = 1.$$

By (3.12) and the definition of μ we have that

$$c \cdot \Theta(u_m) - \Theta(cu_m) + T(c)(u_m) = \mu(c, u_m) = c \cdot (\rho \otimes \rho)(\Delta) \cdot u_m$$

for all m. Hence by (3.15)

$$\begin{aligned} |((\varphi_i \cdot c) \otimes \tau_m)(\Theta(u_m)) - 1| &\leqslant |((\phi_i \cdot c) \otimes ((1_{I^{**}} - u_m) \cdot \psi_m))((\rho \otimes \rho)(\Delta))| \\ &\quad + |(\varphi_i \otimes \tau_m)(\Theta(cu_m))| + |(\phi_i \otimes \tau_m)(T(c)(u_m))| + \varepsilon - \delta \end{aligned} \tag{3.17}$$

for all $i \in \{1, \dots, n\}$ and $m \geqslant m_1$. We have that

$$|(\varphi_i \otimes \tau_m)(\Theta(cu_m))| \leqslant |(\varphi_i \otimes \tau_m)(\Theta(c))| + \|\Theta\| \|cu_m - c\|$$

for all m. But $\Theta(c) \in E\hat{\otimes}E$ and consequently

$$\lim_m |(\varphi_i \otimes \tau_m)(\Theta(cu_m))| = 0 \quad (1 \leqslant i \leqslant n). \tag{3.18}$$

Also we have $\|(1_{I^{**}} - u_m) \cdot \psi_m\| \leqslant (1 - \psi_m(u_m))^{\frac{1}{2}} \to 0$ as $m \to \infty$ and therefore

$$\lim_m |((\phi_i \cdot c) \otimes ((1_{I^{**}} - u_m) \cdot \psi_m))((\rho \otimes \rho)(\Delta))| = 0 \tag{3.19}$$

for all $i \in \{1, \dots, n\}$.

For $\lambda = (\lambda_j) \in c_0(\mathbb{N})$ we set $\sigma(\lambda) = \sum_{j=1}^{\infty} \lambda_j(u_j - u_{j-1})$. It is straightforward to check that σ is a well-defined continuous linear map from $c_0(\mathbb{N})$ into I. Therefore σ^{**} maps $l^\infty(\mathbb{N})$ into I^{**}. Fix $\varphi \in S(I)$. For each positive integer m we define

$$\mathcal{X}_m : l^\infty(\mathbb{N}) \to \mathbb{C} : \lambda \mapsto (\varphi \otimes \tau_m)(T(c)(\sigma^{**}(\lambda))).$$

Then $(\mathcal{X}_m)$ is a sequence of continuous functionals on $l^\infty(\mathbb{N})$ that is weak-* convergent to 0. By Phillips's lemma, we have that $\lim_m \sum_{j=1}^{\infty} |\mathcal{X}_m(e_j)| = 0$, where e_j is the sequence which has 1 in the j-th position and 0 elsewhere. (For a proof see [**3**, p.83]; note that this lemma has been used, in similar situations, in [**6**, V.2.15] and also [**1**].) But

$$|(\varphi \otimes \tau_m)(T(c)(u_m))| \leqslant \sum_{j=1}^{m} |(\varphi \otimes \tau_m)(T(c)(u_j - u_{j-1})| = \sum_{j=1}^{m} |\mathcal{X}_m(e_j)| \leqslant \sum_{j=1}^{\infty} |\mathcal{X}_m(e_j)|$$

for all m, and so we conclude that $\lim_m |(\varphi \otimes \tau_m)(T(c)(u_m))| = 0$. This together with (3.16), (3.17), (3.18) and (3.19) shows that there exists $m_2 \geqslant m_1$ such that

$$|(\varphi_i \otimes \tau_{m_2})(\Theta(u_{m_2})) - 1| \leqslant \varepsilon \text{ for } 1 \leqslant i \leqslant n.$$

We now choose $\varphi = \tau_{m_2}$ and $b = u_{m_2}$. Then the condition (3.1) in Lemma 1 is satisfied, and so by this lemma Θ is unbounded. This is a contradiction.

Acknowledgements. The author would like to thank Professor H.G. Dales, Dr. Z. Lykova and Dr. O. Aristov for stimulating and helpful discussions on the subject of this note.

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