# A COVERING THEOREM FOR TYPICALLY REAL FUNCTIONS

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(Received 8 June, 1968)

Let $T$ denote the class of functions

$$f(z) = z + a_2 z^2 + \dots$$

that are analytic in $U = \{|z| < 1\}$, and satisfy the condition

$$\operatorname{Im} f(z) . \operatorname{Im} z \geqq 0 \qquad (z \in U).$$

Thus $T$ denotes the class of typically real functions introduced by W. Rogosinski [**5**].

One of the most striking results in the theory of functions

$$g(z) = z + b_2 z^2 + \dots$$

that are analytic and univalent in $U$ is the Koebe–Bieberbach covering theorem which states that $\{|w| < \frac{1}{4}\} \subset g(U)$. In this note we point out that the same result holds for functions in the class $T$, a fact which seems to have been overlooked previously. We also determine the largest subdomain of $U$ in which every $f(z)$ in $T$ is univalent, extending previous results in [**1**] and [**2**].

Our basic tool is the following theorem of M. P. Remizova.

THEOREM 1. [**3**] *If* $f(z) \in T$ *and* $z = re^{i\theta} \in U$, *then*

$$|f(z)| \geqq \begin{cases} \dfrac{r}{|1+z|^2} & \text{if} \quad \operatorname{Re}\left(z + \dfrac{1}{z}\right) \geqq 2, \\ \dfrac{r}{|1-z|^2} & \text{if} \quad \operatorname{Re}\left(z + \dfrac{1}{z}\right) \leqq -2, \\ \dfrac{r(1-r^2)|\sin\theta|}{|1-z^2|^2} & \text{if} \quad \left|\operatorname{Re}\left(z + \dfrac{1}{z}\right)\right| \leqq 2. \end{cases} \tag{1}$$

We shall also require the following

LEMMA. *Let* $C_1$ *denote the arc of* $|z+i| = \sqrt{2}$ *on which* $\operatorname{Im} z \geqq 0$. *For* $z$ *on* $C_1$,

$$\left|\operatorname{Re}\left(z + \frac{1}{z}\right)\right| \leqq 2. \tag{2}$$

*Proof.* Let $s = \frac{1}{2}(r + 1/r)$, so that $s > 1$ for $0 < r < 1$. If $z \in C_1$, $2r\sin\theta = 1 - r^2$, and so $\cos^2\theta = 2 - s^2$. Now $z$ satisfies (2) if and only if $|\cos\theta| \leqq 1/s$, and this is certainly true for $z$ on $C_1$, since $2 - s^2 \leqq s^{-2}$ for all $s \geqq 1$.

† Supported in part by NSF GP 6891.

THEOREM 2. *Let $w = f(z)$ belong to $T$. Then $\{|w| < \frac{1}{4}\} \subset f(U)$.*

*Proof.* We assume first that $f(z)$ is continuous in $|z| \leqq 1$. Let $z$ be a point of $C_1$ not equal to $\pm 1$. By the Lemma and Theorem 1,

$$|f(z)| \geqq \frac{r(1-r^2)\sin\theta}{|1-z^2|^2} = \frac{1}{4},$$

since $2r\sin\theta = 1-r^2$ and $|1-z^2|^2 = 1+r^4-2r^2\cos 2\theta = 2(1-r^2)^2$ for $z = re^{i\theta}$ on $C_1$. On the other hand, it follows from (1) that, for real values of $z$,

$$|f(z)| \geqq \frac{|z|}{(1+|z|)^2}.$$

Thus, $|f(\pm 1)| \geqq \frac{1}{4}$, and hence $|f(z)| \geqq \frac{1}{4}$ for $z$ on $C_1$. If $C_2$ denotes the arc of $|z-i| = \sqrt{2}$ on which $\operatorname{Im} z \leqq 0$, then $C_1$ and $C_2$ intersect at 1 and $-1$. Since $f(\bar{z}) = \overline{f(z)}$ for each $f(z)$ in $T$, $|f(z)| \geqq \frac{1}{4}$ on $C_2$. If we set $C = C_1 \cup C_2$, then we have shown that $|f(z)| \geqq \frac{1}{4}$ on $C$; hence $\{|w| < \frac{1}{4}\} \subset f(U)$ by Rouché's Theorem, since $f(0) = 0$.

If $f(z)$ is not continuous in $|z| \leqq 1$, then we apply the above argument to the function

$$g_R(z) = \frac{1}{R} f(Rz) \qquad (0 < R < 1).$$

Since $\lim_{R \to 1} g_R(z) = f(z)$ for $z$ in $U$, the result follows.

Let $L$ denote the inside of the curve $C$ of the previous theorem; i.e.,

$$L = \{|z+i| < \sqrt{2}\} \cap \{|z-i| < \sqrt{2}\}.$$

In [3], Remizova has shown that the largest subdomain $U'$ of $U$ in which every $f(z)$ in $T$ is univalent is contained in $L$. Using a result of L. Čakalov [1], we point out that $U' = L$.

THEOREM 3. *If $f(z) \in T$, then $f(z)$ is univalent in $L$. Furthermore, if $z_0 \in U - L$, there is a function in $T$ whose derivative vanishes at $z_0$.*

*Proof.* M. S. Robertson [4] has shown that, if $f(z) \in T$, then

$$f(z) = \int_{-1}^{1} \frac{z}{1-2tz+z^2}\, d\alpha(t) = \frac{1}{2}\int_{-1}^{1} \frac{d\alpha(t)}{w-t},$$

where $\alpha(t)$ is increasing on $[-1, 1]$, $\alpha(1) - \alpha(-1) = 1$, and $w = \frac{1}{2}(z + 1/z)$. In [1], Čakalov proved that $\displaystyle\int_{-1}^{1} \frac{d\alpha(t)}{w-t}$ is univalent in $\{|w| > 1\}$. The curve $|w| = 1$ is the image of $C$ under the transformation $w = \frac{1}{2}(z + 1/z)$, and, consequently, $f(z)$ is univalent in $L$.

Finally, for any $\lambda$ $(0 < \lambda < 1)$, the function defined by

$$f_\lambda(z) = \frac{\lambda z}{(1-z)^2} + \frac{(1-\lambda)z}{(1+z)^2}$$

is in $T$, since $T$ is a convex set containing $z/(1\pm z)^2$. Now $f'_\lambda(z)=0$ when

$$\frac{1+z}{1-z}=\left(-\frac{1-\lambda}{\lambda}\right)^{1/4},$$

and as $z$ varies round $C$, $\{(1+z)/(1-z)\}^4$ attains all real negative values; hence, if $z_0$ is a point of $U$ which lies on $C$, $\lambda$ can be chosen so that $f'_\lambda(z_0)=0$. If $z_0$ is any point of $U-L$, then, by considering the function $R^{-1}f_\lambda(Rz)$ $(0<R<1)$, we obtain a function in $T$ whose derivative vanishes at $z_0$.

*Note.* In [2] it was shown that any function in $T$ maps $\{|z|<\sqrt{(2)}-1\}\subset L$ univalently onto a domain starlike with respect to $w=0$.

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https://doi.org/10.1017/S0017089500000719 Published online by Cambridge University Press