HOMOLOGY OF DELETED PRODUCTS OF CONTRACTIBLE 2-DIMENSIONAL POLYHEDRA. I

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1. Introduction. If X is a space and $k > 1$, the *kth deleted product space* X_k^* of X is the topological product $X \times X \times \ldots \times X$ of k copies of X minus the set of all points of the form $(x, x, \ldots, x)$, where $x \in X$. In **(4)**, the author shows that the homology groups of X_k^*, where X is a tree, produce as much information about trees as counting the orders of vertices.

The space X_2^* is called the *deleted product space* and is denoted by X^*. The homology groups of the deleted product of a finitely triangulable, 1-dimensional space have been described by Copeland **(1)** and the author **(5)**. Cones are the simplest examples of contractible spaces, and in **(2)**, Copeland describes the homology groups of the deleted product of a 2-dimensional cone. The present paper extends the results of Copeland **(2)**. In particular, we compute the homology groups of the deleted product of an arbitrary, finite, contractible, 2-dimensional polyhedron. Knowing these groups is a step towards distinguishing between the contractible spaces by means of algebraic invariants.

The homology groups used throughout this paper will be the reduced homology groups with integral coefficients, and the customary "tilde" over the H has been omitted. If σ is a simplex of a polyhedron X, we let $\mathrm{St}(\sigma, X)$ denote the open star of σ in X, and if $v_1, v_2, \ldots, v_n$ are the vertices of a simplex σ, we denote σ by $\langle v_1, v_2, \ldots, v_n\rangle$. If X is a space and p is a point not in X, then a *cone* over X is the join $\hat{X} = X * p$ of X with p. Note that if v is a vertex of a locally finite polyhedron X, then $\partial(\mathrm{St}(v, X)) * v = \overline{\mathrm{St}(v, X)}$. We also let Z denote the group of integers.

If X is a finite polyhedron, let $P(X^*) = \bigcup\{\sigma \times \tau \mid \sigma$ and τ are simplexes of X and $\sigma \cap \tau = \emptyset\}$. Hu **(3)** has proved that X^* and $P(X^*)$ are homotopically equivalent. If X is a 2-simplex and X^1 denotes the 1-skeleton of X, then $P(X^*) = P((X^1)^*)$, and the author **(5)** has shown that $P((X^1)^*)$ is a simple closed curve.

If X is a finite, contractible, 2-dimensional polyhedron and A is a 2-simplex, then a homeomorph of X can be constructed out of A by appending n-simplexes ($n = 1, 2$). The construction may be factored

$$A = X_1 \to X_2 \to \ldots \to X_p = X$$

so that X_i is obtained from X_{i-1} by

Received August 23, 1966.

https://doi.org/10.4153/CJM-1968-039-3 Published online by Cambridge University Press

(a) adding a 1-simplex which meets X_{i-1} in just one of its vertices,
(b) adding a 2-simplex which meets X_{i-1} in just one of its vertices,
(c) adding a 2-simplex which meets X_{i-1} in just one of its 1-faces,
or
(d) adding a 2-simplex which meets X_{i-1} in exactly two of its 1-faces.

In constructing the homeomorph of X from A, we may choose the order in which we add simplexes so that if τ is a 2-simplex such that $X_i = X_{i-1} \cup \tau$ and $X_{i-1} \cap \tau = s_1 \cup s_2$, where s_1 and s_2 are 1-simplexes of X_{i-1} and τ, $s_1 \cap s_2 = \{u_3\}$, and u_i is the vertex of s_i different from u_3, then there is a sequence $r_1, r_2, \ldots, r_n$ of 1-simplexes in $\partial(\mathrm{St}(u_3, X_{i-1}))$ such that u_1 is a vertex of r_1, u_2 is a vertex of r_n, $r_j \cap r_{j+1}$ is a vertex, and $r_j \cap r_k = \emptyset$ if $|j - k| > 1$; and if, in addition, S is a simple closed curve in $\partial(\mathrm{St}(u_3, X_{i-1}))$ such that u_1 and u_2 are not in S, then the sequence $r_1, r_2, \ldots, r_n$ can be chosen so that $r_j \cap S = \emptyset$ for each j.

Since X_1^* is connected, in order to show that X^* is connected, it is sufficient to assume that X_{i-1}^* is connected and show that $H_0(X_i^*)$ is isomorphic to $H_0(X_{i-1}^*)$. Also, since $H_1(X_1^*)$ is isomorphic to Z and $H_2(X_1^*) = 0$, in order to show that $H_j(X^*)$, $j = 1, 2$, is a free abelian group, it is sufficient to assume that $H_j(X_{i-1}^*)$ is free abelian and show that $H_j(X_i^*)$ is free abelian.

In a forthcoming paper, we describe the finite, contractible, 2-dimensional polyhedra whose deleted products have the homotopy type of the 2-sphere, and we examine the relation between the number of isotopy classes of embeddings of one of these polyhedra in an arbitrary, finite, contractible 2-dimensional polyhedron X (see **6**) and the 2-dimensional Betti number of the deleted product of X.

2. Some preliminary results.

THEOREM 1. *Let X be a polyhedron such that X is the union of two subpolyhedra A and B, where $A \cap B$ is an n-dimensional polyhedron. If there is an $(n + 1)$-cycle z such that $[z] \in H_{n+1}(X)$ but $[z]$ is not an element of the direct sum $H_{n+1}(A) + H_{n+1}(B)$, then there is an n-cycle c in $A \cap B$ which bounds in A and in B.*

Proof. Since $[z]$ is in $H_{n+1}(X)$ but not in $H_{n+1}(A) + H_{n+1}(B)$, $z = z_1 + z_2$, where z_1 is a non-trivial $(n + 1)$-chain in A and z_2 is a non-trivial $(n + 1)$-chain in B. Now $0 = \partial z = \partial(z_1 + z_2)$. Therefore $\partial z_1 = -\partial z_2$, and hence ∂z_1 is an n-cycle in $A \cap B$. Also z_1 is an $(n + 1)$-chain in A whose boundary is ∂z_1 and $-z_2$ is an $(n + 1)$-chain in B whose boundary is ∂z_1.

Let X be a finite, contractible, 2-dimensional polyhedron. If S_1 and S_2 are simple closed curves in the 1-skeleton of X, the simplexes of X may be oriented so that if $r_{i1}, r_{i2}, \ldots, r_{in_i}$ are the 1-simplexes of S_i, then $r_{i1} + r_{i2} + \ldots + r_{in_i}$ is a 1-cycle for each $i = 1, 2$. In this paper, we assume that this has been done whenever we want to talk about two simple closed curves in the 1-skeleton of

X. We also let z_{S_i} denote the 1-cycle $r_{i1} + r_{i2} + \ldots + r_{in_i}$ associated with S_i. Moreover, if z is a 1-cycle in X, then

$$z = \sum_{i=1}^{p} z_i,$$

where each z_i is a 1-cycle and the union of the 1-simplexes with non-zero coefficients in z_i is a simple closed curve. Thus if each z_i bounds, then z bounds. Therefore in order to show that z bounds, we may assume that there is a simple closed curve S associated with z such that if $r_1, r_2, \ldots, r_n$ are the 1-simplexes of S, then

$$z = a \sum_{i=1}^{n} r_i.$$

If X is a polyhedron and p is a point, then $X \times \{p\}$ is just a copy of X. Thus, for convenience, we do not make any distinction between chains in $X \times \{p\}$ and chains in X. The meaning will be clear from the context.

THEOREM 2. *If X is a finite, contractible, 2-dimensional polyhedron and v is a vertex of X, then each 1-cycle in $\{v\} \times \partial(\mathrm{St}(v, X))$ is homologous in $P(X^*)$ to a 1-cycle in $\partial(\mathrm{St}(v, X)) \times \{v\}$.*

Proof. Let z be a 1-cycle in $\{v\} \times \partial(\mathrm{St}(v, X))$. It is sufficient to prove the theorem under the assumption that the union of the 1-simplexes with non-zero coefficients in z is a simple closed curve S. Furthermore, without loss of generality, we may assume that S is the union of three 1-simplexes, r_1, r_2, r_3, of X and that

$$z = \sum_{i=1}^{3} r_i.$$

Let v_1 denote the vertex $r_1 \cap r_2$, v_2 the vertex $r_2 \cap r_3$, and v_3 the vertex $r_3 \cap r_1$. Let r_{12} denote the 1-simplex with vertices v and v_1, r_{23} the 1-simplex with vertices v and v_2, and r_{31} the 1-simplex with vertices v and v_3. For each $i = 1, 2, 3$, let σ_i denote the 2-simplex which has r_i as a face and v as a vertex. Then

$$\begin{gathered}(r_{12} \times r_2) \cup (r_{23} \times r_1) \cup (r_{31} \times r_2) \cup (\sigma_1 \times \{v_2\}) \cup (\sigma_2 \times \{v_3\}) \\ \cup (\sigma_3 \times \{v_1\}) \cup (r_3 \times r_{12}) \cup (r_1 \times r_{23}) \cup (r_2 \times r_{31}) \cup (\{v_2\} \times \sigma_1) \\ \cup (\{v_3\} \times \sigma_2) \cup (\{v_1\} \times \sigma_3) \subset P(X^*),\end{gathered}$$

and it is clear that there is a 2-chain associated with this subset of $P(X^*)$ whose boundary is $z - z'$, where z' is a 1-cycle in $\partial(\mathrm{St}(v, X)) \times \{v\}$.

Definition 1. If X is a finite, contractible, 2-dimensional polyhedron and v is a vertex of X, then X is *pronged* at v provided $\partial(\mathrm{St}(v, X))$ contains a simple closed curve, and if $\partial(\mathrm{St}(v, X))$ is a simple closed curve S, then there is a simple closed curve S' in the 1-skeleton of $X - \mathrm{St}(v, X)$, a 2-chain

$$c = \sum_{j=1}^{n} a_j \sigma_j$$

($a_j \neq 0$ for each $j = 1, 2, \ldots, n$) in $X - \mathrm{St}(v, X)$, and either a 1-simplex $r \in X - \mathrm{St}(v, X)$ such that $\partial c = z_S - z_{S'}$, $r \cap S' = \emptyset$, and $r \cap (\sigma_1 \cup \sigma_2 \cup \ldots \cup \sigma_n)$ is a vertex, or a 2-simplex $\tau \in X - \mathrm{St}(v, X)$ and a 1-face μ of τ such that if L denotes the line segment in τ from the barycentre of τ to the barycentre of μ, then $\partial c = z_S - z_{S'}$, $L \cap S' = \emptyset$, and $L \cap (\sigma_1 \cup \sigma_2 \cup \ldots \cup \sigma_n)$ is a vertex. If s is a 1-simplex of X, then X is *pronged* at s provided the first barycentric subdivision of X is pronged at the barycentre of s.

THEOREM 3. *Let X be a finite, contractible, 2-dimensional polyhedron and v a vertex of X such that $\partial(\mathrm{St}(v, X))$ contains a simple closed curve. If X is pronged at v, then each 1-cycle in $\partial(\mathrm{St}(v, X)) \times \{v\}$ bounds in $P(X^*)$. If there is a 1-cycle in $\partial(\mathrm{St}(v, X)) \times \{v\}$ which bounds in $P(X^*)$, then X is pronged at v.*

Proof. Suppose X is pronged at v, and let z be a 1-cycle in $\partial(\mathrm{St}(v, X)) \times \{v\}$. Let S be the simple closed curve in $\partial(\mathrm{St}(v, X))$ associated with z. If $S \neq \partial(\mathrm{St}(v, X))$, let $u \in \partial(\mathrm{St}(v, X)) - S$. Since we can subdivide X so that u is a vertex of the subdivision, we may assume that u is a vertex of X. Then $(S \times \langle u, v\rangle) \cup ((S * v) \times \{u\}) \subset P(X^*)$, and it is clear that there is a 2-chain associated with this subset of $P(X^*)$ whose boundary is z. If $S = \partial(\mathrm{St}(v, X))$ and there is a simple closed curve S', a 2-chain c, and a 1-simplex r satisfying Definition 1, let u be the vertex of r which is not in $\sigma_1 \cup \sigma_2 \cup \ldots \cup \sigma_n$, and let $r_1, r_2, \ldots, r_q$ be a sequence of 1-simplexes such that v is a vertex of r_1, $r \cap (\sigma_1 \cup \sigma_2 \cup \ldots \cup \sigma_n)$ is a vertex of r_q, $r_1 \cap S' = \emptyset$,

$$\bigcup_{i=2}^{q} r_i \subset \bigcup_{j=1}^{n} \sigma_j - S',$$

$r_i \cap r_{i+1}$ is a vertex for each i, and $r_i \cap r_k = \emptyset$ if $|i - k| > 1$. (We may assume that X is subdivided so that this is possible.) Then

$$\left(\bigcup_{j=1}^{n} \sigma_j \times \{v\}\right) \cup \left(S' \times \left(\bigcup_{i=1}^{q} r_i \cup r\right)\right) \cup \left(\left(\bigcup_{j=1}^{n} \sigma_j \cup (S * v)\right) \times \{u\}\right) \subset P(X^*),$$

and it is clear that there is a 2-chain associated with this subset of $P(X^*)$ whose boundary is z. If $S = \partial(\mathrm{St}(v, X))$ and there is a simple closed curve S', a 2-chain c, and a 2-simplex τ with 1-face μ satisfying Definition 1, we do essentially the same thing in order to obtain a 2-chain in $P(X^*)$ whose boundary is z.

Suppose there is a non-trivial 1-cycle z in $\partial(\mathrm{St}(v, X)) \times \{v\}$ which bounds in $P(X^*)$. We may assume that

$$z = a \sum_{i=1}^{n} r_i,$$

where $S = r_1 \cup r_2 \cup \ldots \cup r_n$ is a simple closed curve. If there is a 1-simplex s with vertex v such that $S \times s \subset P(X^*)$, then $\partial(\mathrm{St}(v, X))$ contains a simple

https://doi.org/10.4153/CJM-1968-039-3 Published online by Cambridge University Press

closed curve but is not a simple closed curve. Suppose there is no such 1-simplex. Then $S = \partial(\mathrm{St}(v, X))$. Now $P((S * v)^*)$ is homeomorphic to a cylinder with one end $\{v\} \times S$ and the other $S \times \{v\}$. Hence z cannot bound in $P((S * v)^*)$, and thus there is a 1-simplex s_1 in S such that s_1 is a face of a 2-simplex σ_1 which is not in $S * v$. Let S_1 be the simple closed curve consisting of all the 1-simplexes in S and all the 1-faces of σ_1 except those in $S \cap \sigma_1$. If $S \cap \sigma_1$ consists of two 1-simplexes and the common vertex of these two 1-simplexes is a vertex of a 1-simplex r not in $(S * v) \cup \sigma_1$, let c be the elementary 2-chain which assigns to σ_1 either $\pm a$ depending upon the orientations. Then S_1, c, and r satisfy Definition 1. If $S \cap \sigma_1 = s_1$ and there is a 2-simplex τ such that $\tau \neq \sigma_1$, τ is not in $S * v$, and s_1 is a face of τ, let c be the elementary 2-chain which assigns to σ_1 either $\pm a$ depending upon the orientations. Then S_1, c, τ and the 1-face s_1 of τ satisfy Definition 1. If neither r nor τ exist, then since z bounds in $P(X^*)$, there is a 1-simplex s_2 in S_1 such that s_2 is a face of a 2-simplex σ_2, where $\sigma_2 \neq \sigma_1$ and σ_2 is not in $S * v$. Let S_2 be the simple closed curve consisting of all the 1-simplexes in S_1 and all the 1-faces of σ_2 except those in $S_1 \cap \sigma_2$. Now we repeat the above argument and continue this process. Since X is finite and contractible and z bounds in $P(X^*)$, after a finite number of steps we shall obtain either an S', c, and r satisfying Definition 1 or an S', c, and 2-simplex τ with 1-face μ satisfying Definition 1.

THEOREM 4. *Let X be a finite, contractible, 2-dimensional polyhedron, and let v be a vertex of X such that X is not pronged at v. If z is a 1-cycle in $P(X^*)$ which does not bound and there is an integer p such that pz is homologous to a non-trivial 1-cycle in $\partial(\mathrm{St}(v, X)) \times \{v\}$, then z is homologous to a 1-cycle in*

$$\partial(\mathrm{St}(v, X)) \times \{v\}.$$

Proof. Suppose z' is a non-trivial 1-cycle in $\partial(\mathrm{St}(v, X)) \times \{v\}$ such that pz is homologous to z'. Since X is not pronged at v,

$$z' = a \sum_{i=1}^{m} r_i,$$

where

$$S = \bigcup_{i=1}^{m} r_i = \partial(\mathrm{St}(v, X))$$

is a simple closed curve. Since $P((\overline{\mathrm{St}(v, X)})^*)$ is just a cyclinder with one end $S \times \{v\}$ and the other $\{v\} \times S$, it is sufficient to prove the theorem by working in $P(X^*) - (\mathrm{St}(v, X))^*$ rather than in $P(X^*)$. Since X is not pronged at v, each r_i in S is a face of at most one 2-simplex in $X - \mathrm{St}(v, X)$. If there is a 2-simplex σ_1 not in $\mathrm{St}(v, X)$ such that some 1-simplex in S is a face of σ_1, let S_1 be the simple closed curve consisting of all the 1-simplexes in S and all the 1-faces of σ_1 except those in $S \cap \sigma_1$. If there is a 2-simplex σ_2 such that $\sigma_2 \neq \sigma_1$, σ_2 is not in $\mathrm{St}(v, X)$, but some 1-simplex in S_1 is a face of σ_2, let S_2 be the simple closed curve consisting of all the 1-simplexes in S_1 and the 1-faces of σ_2 except those in $S_1 \cap \sigma_2$. Continue this process until we run out of 2-simplexes. Let

$\sigma_1, \sigma_2, \ldots, \sigma_n$ denote the collection of 2-simplexes obtained in this manner, and let S_n be the last simple closed curve that we obtain. Let z_n be the 1-cycle associated with S_n which has coefficient $\pm a$ (depending upon orientations) on each 1-simplex in S_n. If the proper sign has been chosen for each coefficient of z_n, then z' is homologous to z_n. Since X is not pronged at v, any 1-cycle in $P(X^*) - (\mathrm{St}(v, X))^*$ which is homologous in $P(X^*) - (\mathrm{St}(v, X))^*$ to z' must lie in $S' \times \{u\}$, where

$$S' \subset \bigcup_{i=1}^{n} \sigma_i$$

and u is "inside" S'. Therefore, since X is contractible, each coefficient of any such 1-cycle must be $\pm a$, and hence z is homologous to a 1-cycle in

$$\partial(\mathrm{St}(v, X)) \times \{v\}.$$

3. Addition of a 1-simplex at one vertex.

THEOREM 5. *If A is a finite, contractible, 2-dimensional polyhedron, B is a 1-simplex, $A \cap B = \{v\}$, where v is a vertex of A and B such that A is not pronged at v, A^* is connected, $H_1(A^*)$ is the free abelian group on α generators, $H_1(\partial(\mathrm{St}(v, A)))$ is the free abelian group on β generators, $H_0(\partial(\mathrm{St}(v, A)))$ is the free abelian group on γ generators, and $X = A \cup B$, then $H_k(X^*)$ is isomorphic to $H_k(A^*)$ for $k = 0$ and $k \geqslant 3$, $H_2(X^*)$ is isomorphic to the direct sum $H_2(A^*) + H_1(\partial(\mathrm{St}(v, A)))$, and $H_1(X^*)$ is the free abelian group on $\alpha - \beta + 2\gamma$ generators.* (Note that β is either 0 or 1 since A is not pronged at v.)

Proof. Let b denote the other vertex of B. Then

$$\begin{aligned} P(X^*) = P(A^*) &\cup ([A - \mathrm{St}(v, A)] \times B) \cup (\overline{\mathrm{St}(v, A)} \times \{b\}) \\ &\cup (B \times [A - \mathrm{St}(v, A)]) \cup (\{b\} \times \overline{\mathrm{St}(v, A)}). \end{aligned}$$

Now $P(A^*) \cap ([A - \mathrm{St}(v, A)] \times B) = [A - \mathrm{St}(v, A)] \times \{v\}$, and hence $H_k(P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B))$ is isomorphic to $H_k(P(A^*))$ for all k. Also

$$\begin{aligned} [P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B)] &\cap (\overline{\mathrm{St}(v, A)} \times \{b\}) \\ &= [\partial(\mathrm{St}(v, A)) \times \{b\}. \end{aligned}$$

Since A^* is connected and $\dim (\overline{\mathrm{St}(v, A)} \times \{b\}) \leqslant 2$,

$$H_k(P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup (\overline{\mathrm{St}(v, A)} \times \{b\}))$$

is isomorphic to $H_k(P(A^*))$ for $k = 0$ and $k \geqslant 3$. Suppose z is a 1-cycle in $\partial(\mathrm{St}(v, A)) \times \{b\}$. Then z bounds in $\overline{\mathrm{St}(v, A)} \times \{b\}$, z does not bound in $[A - \mathrm{St}(v, A)] \times B$, but z is homologous in $[A - \mathrm{St}(v, A)] \times B$ to a 1-cycle z' in $\partial(\mathrm{St}(v, A)) \times \{v\}$. Since A is not pronged at v, no 1-cycle in $\partial(\mathrm{St}(v, A)) \times \{v\}$ bounds in $P(A^*)$ by Theorem 3. Therefore, by Theorem 1,

$$H_2(P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup (\overline{\mathrm{St}(v, A)} \times \{b\}))$$

is isomorphic to $H_2(P(A^*))$, and, by Theorem 4,

$$H_1(P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup (\overline{\mathrm{St}(v, A)} \times \{b\}))$$

is the free abelian group on $\alpha - \beta + \gamma$ generators. Now

$$\begin{aligned}[P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup (\overline{\mathrm{St}(v, A)} \times \{b\})] \\ \cap (B \times [A - \mathrm{St}(v, A)]) = \{v\} \times [A - \mathrm{St}(v, A)],\end{aligned}$$

and hence

$$H_k(P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup (\overline{\mathrm{St}(v, A)} \times \{b\}) \cup (B \times [A - \mathrm{St}(v, A)]))$$

is isomorphic to $H_k(P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup (\overline{\mathrm{St}(v, A)} \times \{b\}))$ for all k. Also

$$\begin{aligned}[P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup (\overline{\mathrm{St}(v, A)} \times \{b\}) \cup (B \times [A - \mathrm{St}(v, A)])] \\ \cap (\{b\} \times \overline{\mathrm{St}(v, A)}) = \{b\}[\times \partial(\mathrm{St}(v, A)).\end{aligned}$$

Again, since A^* is connected and $\dim(\{b\} \times \overline{\mathrm{St}(v, A)}) \leqslant 2$, $H_k(P(X^*))$ is isomorphic to $H_k(P(A^*))$ for $k = 0$ and $k \geqslant 3$. Let z be a 1-cycle in $\{b\} \times \partial(\mathrm{St}(v, A))$. Then z bounds in $\{b\} \times \overline{\mathrm{St}(v, A)}$, and z is homologous in $B \times [A - \mathrm{St}(v, A)]$ to a 1-cycle z_1 in $\{v\} \times \partial(\mathrm{St}(v, A))$. By Theorem 2, z_1 is homologous in $P(A^*)$ to a 1-cycle z_2 in $\partial(\mathrm{St}(v, A)) \times \{v\}$. Now z_2 is homologous in $[A - \mathrm{St}(v, A)] \times B$ to a 1-cycle z_3 in $\partial(\mathrm{St}(v, A)) \times \{b\}$, and z_3 bounds in $\overline{\mathrm{St}(v, A)} \times \{b\}$. Therefore $H_2(P(X^*))$ is isomorphic to the direct sum $H_2(P(A^*)) + H_1(\partial(\mathrm{St}(v, A)))$, and $H_1(P(X^*))$ is the free abelian group on $\alpha - \beta + 2\gamma$ generators.

THEOREM 6. *If A is a finite, contractible, 2-dimensional polyhedron, B is a 1-simplex, $A \cap B = \{v\}$, where v is a vertex of A and B such that A is pronged at v, A^* is connected, and $X = A \cup B$, then $H_k(X^*)$ is isomorphic to $H_k(A^*)$ for $k = 0$ and $k \geqslant 3$, $H_2(X^*)$ is isomorphic to the direct sum*

$$H_2(A^*) + H_1(\partial(\mathrm{St}(v, A))) + H_1(\partial(\mathrm{St}(v, A))),$$

and $H_1(X^*)$ is isomorphic to the direct sum

$$H_1(A^*) + H_0(\partial(\mathrm{St}(v, A))) + H_0(\partial(\mathrm{St}(v, A))).$$

Proof. Again let b denote the other vertex of B, and express $P(X^*)$ as the union of sets in the same way that it was expressed in the previous proof. The previous proof applies to show that

$$H_k(P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B))$$

is isomorphic to $H_k(P(A^*))$ for all k and

$$H_k(P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup (\overline{\mathrm{St}(v, A)} \times \{b\}))$$

is isomorphic to $H_k(P(A^*))$ for $k = 0$ and $k \geqslant 3$. Let z be a 1-cycle in $\partial(\mathrm{St}(v, A)) \times \{b\}$. Then z bounds in $\overline{\mathrm{St}(v, A)} \times \{b\}$, z is homologous in

$[A - \mathrm{St}(v, A)] \times B$ to a 1-cycle z' in $\partial(\mathrm{St}(v, A)) \times \{v\}$, and, by Theorem 3, z' bounds in $P(A^*)$. Therefore

$$H_2(P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup \overline{\mathrm{St}(v, A)} \times \{b\}))$$

is isomorphic to the direct sum $H_2(P(A^*)) + H_1(\partial(\mathrm{St}(v, A)))$, and

$$H_1(P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup \overline{\mathrm{St}(v, A)} \times \{b\}))$$

is isomorphic to the direct sum $H_1(P(A^*)) + H_0(\partial(\mathrm{St}(v, A)))$. Again the previous proof applies to show that

$$H_k(P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup \overline{\mathrm{St}(v, A)} \times \{b\}) \cup (B \times [A - \mathrm{St}(v, A)]))$$

is isomorphic to $H_k(P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup \overline{\mathrm{St}(v, A)} \times \{b\}))$ for all k and $H_k(P(X^*))$ is isomorphic to $H_k(P(A^*))$ for $k = 0$ and $k \geqslant 3$. Let z be a 1-cycle in $\{b\} \times \partial(\mathrm{St}(v, A))$. Then z bounds in $\{b\} \times \overline{\mathrm{St}(v, A)}$, z is homologous in $B \times [A - \mathrm{St}(v, A)]$ to a 1-cycle z' in $\{v\} \times \partial(\mathrm{St}(v, A))$ and, by Theorem 3, z' bounds in $P(A^*)$. Therefore $H_2(P(X^*))$ is isomorphic to the direct sum

$$H_2(P(A^*)) + H_1(\partial(\mathrm{St}(v, A))) + H_1(\partial(\mathrm{St}(v, A))),$$

and $H_1(P(X^*))$ is isomorphic to the direct sum

$$H_1(P(A^*)) + H_0(\partial(\mathrm{St}(v, A))) + H_0(\partial(\mathrm{St}(v, A))).$$

4. Addition of a 2-simplex at one vertex.

THEOREM 7. *If A is a finite, contractible, 2-dimensional polyhedron, B is a 2-simplex, $A \cap B = \{v\}$, where v is a vertex of A and B such that A is not pronged at v, A^* is connected, $H_1(A^*)$ is the free abelian group on α generators, $H_1(\partial(\mathrm{St}(v, A)))$ is the free abelian group on β generators, $H_0(\partial(\mathrm{St}(v, A)))$ is the free abelian group on γ generators, and $X = A \cup B$, then $H_k(X^*)$ is isomorphic to $H_k(A^*)$ for $k = 0$ and $k \geqslant 3$, $H_2(X^*)$ is isomorphic to the direct sum $H_2(A^*) + H_1(\partial(\mathrm{St}(v, A)))$, and $H_1(X^*)$ is the free abelian group on $\alpha - \beta + 2\gamma + 2$ generators.*

Proof. Let s denote the 1-face of B which does not have v as a vertex. Then

$$P(X^*) = P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup \overline{\mathrm{St}(v, A)} \times s) \cup (B \times [A - \mathrm{St}(v, A)]) \cup (s \times \overline{\mathrm{St}(v, A)}) \cup P(B^*).$$

The proof of Theorem 5 applies to show that

$$H_k(P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B))$$

is isomorphic to $H_k(P(A^*))$ for all k. Now

$$[P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B)] \cap \overline{\mathrm{St}(v, A)} \times s) = \partial(\mathrm{St}(v, A)) \times s.$$

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Therefore $H_0(P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup (\overline{\mathrm{St}(v, A)} \times s))$ is isomorphic to $H_0(P(A^*))$. Since $H_3(\overline{\mathrm{St}(v, A)} \times s) = 0$ and $\partial(\mathrm{St}(v, A)) \times s$ does not contain a 2-cycle,

$$H_k(P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup (\overline{\mathrm{St}(v, A)} \times s))$$

is isomorphic to $H_k(P(A^*))$ for all $k \geqslant 3$. Just as in the proof of Theorem 5, if z is a 1-cycle in $\partial(\mathrm{St}(v, A)) \times s$, then z bounds in $\overline{\mathrm{St}(v, A)} \times s$ but not in $P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B)$. Therefore, by Theorem 1,

$$H_2(P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup (\overline{\mathrm{St}(v, A)} \times s))$$

is isomorphic to $H_2(P(A^*))$, and, by Theorem 4,

$$H_1(P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup (\overline{\mathrm{St}(v, A)} \times s))$$

is the free abelian group on $\alpha - \beta + \gamma$ generators. Again the proof of Theorem 5 applies to show that

$$H_k(P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup (\overline{\mathrm{St}(v, A)} \times s) \cup (B \times [A - \mathrm{St}(v, A)]))$$

is isomorphic to $H_k(P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup (\overline{\mathrm{St}\,(v, A)} \times s))$ for all k. Also

$$[P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup (\overline{\mathrm{St}(v, A)} \times s) \cup (B \times [A - \mathrm{St}(v, A)])] \\ \cap (s \times \overline{\mathrm{St}(v, A)}) = s \times \partial(\mathrm{St}(v, A)).$$

For the same reason as when we added $\overline{\mathrm{St}(v, A)} \times s$,

$$H_k(P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup (\overline{\mathrm{St}(v, A)} \times s) \\ \cup (B \times [A - \mathrm{St}(v, A)]) \cup (s \times \overline{\mathrm{St}(v, A)}))$$

is isomorphic to $H_k(P(A^*))$ for $k = 0$ and $k \geqslant 3$. Again, just as in the proof of Theorem 5, each 1-cycle in $s \times \partial(\mathrm{St}(v, A))$ bounds in $s \times \overline{\mathrm{St}(v, A)}$ and in $P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup (\overline{\mathrm{St}(v, A)} \times s) \cup (B \times [A - \mathrm{St}(v, A)])$. Therefore

$$H_2(P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup (\overline{\mathrm{St}(v, A)} \times s) \\ \cup (B \times [A - \mathrm{St}(v, A)]) \cup (s \times \overline{\mathrm{St}(v, A)}))$$

is isomorphic to the direct sum $H_2(P(A^*)) + H_1(\partial(\mathrm{St}(v, A)))$, and

$$H_1(P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup (\overline{\mathrm{St}(v, A)} \times s) \\ \cup (B \times [A - \mathrm{St}(v, A)]) \cup (s \times \overline{\mathrm{St}(v, A)}))$$

is the free abelian group on $\alpha - \beta + 2\gamma$ generators. Now

$$[P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup (\overline{\mathrm{St}(v, A)} \times s) \cup (B \times [A - \mathrm{St}(v, A)]) \\ \cup (s \times \overline{\mathrm{St}(v, A)})] \cap P(B^*) = (\{v\} \times s) \cup (s \times \{v\}).$$

Therefore, since $P(B^*)$ is a simple closed curve, $H_k(P(X^*))$ is isomorphic to $H_k(P(A^*))$ for $k = 0$ and $k \geqslant 3$, $H_2(P(X^*))$ is isomorphic to the direct sum $H_2(P(A^*)) + H_1(\partial(\mathrm{St}(v, A)))$, and $H_1(P(X^*))$ is the free abelian group on $\alpha - \beta + 2\gamma + 2$ generators.

THEOREM 8. *If A is a finite, contractible, 2-dimensional polyhedron, B is a 2-simplex, $A \cap B = \{v\}$, where v is a vertex of A and B such that A is pronged at v, A^* is connected, and $X = A \cup B$, then $H_k(X^*)$ is isomorphic to $H_k(A^*)$ for $k = 0$ and $k \geqslant 3$, $H_2(X^*)$ is isomorphic to the direct sum*

$$H_2(A^*) + H_1(\partial(\mathrm{St}(v, A))) + H_1(\partial(\mathrm{St}(v, A))),$$

and $H_1(X^*)$ is isomorphic to the direct sum

$$H_1(A^*) + H_0(\partial(\mathrm{St}(v, A))) + H_0(\partial(\mathrm{St}(v, A))) + Z + Z.$$

Proof. Again let s denote the 1-face of B which does not have v as a vertex and express $P(X^*)$ as the union of sets in the same way that it was expressed in the preceding proof. The preceding proof applies to show that

$$H_k(P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B))$$

is isomorphic to $H_k(P(A^*))$ for all k and

$$H_k(P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup \overline{(\mathrm{St}(v, A)} \times s))$$

is isomorphic to $H_k(P(A^*))$ for $k = 0$ and $k \geqslant 3$. By Theorem 3, each 1-cycle in $\partial(\mathrm{St}(v, A)) \times s$ bounds in $\overline{\mathrm{St}(v, A)} \times s$ and in $P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B)$. Therefore

$$H_2(P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup \overline{(\mathrm{St}(v, A)} \times s))$$

is isomorphic to the direct sum $H_2(P(A^*)) + H_1(\partial(\mathrm{St}(v, A)))$, and

$$H_1(P(A^*) \cup ([A - \mathrm{St}(v, A)] \times B) \cup \overline{(\mathrm{St}(v, A)} \times s))$$

is isomorphic to the direct sum $H_1(P(A^*)) + H_0(\partial(\mathrm{St}(v, A)))$. Now the remainder of the proof of Theorem 7 applies to complete the proof of this theorem.

5. Addition of a 2-simplex at one 1-simplex.

THEOREM 9. *If A is a finite, contractible, 2-dimensional polyhedron, B is a 2-simplex, $A \cap B = s$, where s is a 1-simplex of A and B such that A is not pronged at s, A^* is connected, $H_1(A^*)$ is the free abelian group on α generators, $H_1(\partial(\mathrm{St}(v, A)))$ is the free abelian group on β generators, and $X = A \cup B$, then $H_k(X^*)$ is isomorphic to $H_k(A^*)$ for $k = 0$ and $k \geqslant 3$, $H_2(X^*)$ is isomorphic to the direct sum $H_2(A^*) + H_1(\partial(\mathrm{St}(s, A)))$, and $H_1(X^*)$ is the free abelian group on $\alpha - \beta$ generators.*

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Proof. Let b denote the vertex of B which is not a vertex of s, let u_1 and u_2 denote the vertices of s, and for each $i = 1, 2$, let s_i denote the 1-face of B not in A which has u_i as a vertex. Then

$$\begin{aligned} P(X^*) = P(A^*) &\cup \left(\left[A - \bigcup_{i=1}^{2} \operatorname{St}(u_i, A)\right] \times B\right) \\ &\cup ([\overline{\operatorname{St}(u_2, A)} - \operatorname{St}(u_1, A)] \times s_1) \cup ([\overline{\operatorname{St}(u_1, A)} - \operatorname{St}(u_2, A)] \times s_2) \\ &\cup (\overline{\operatorname{St}(s, A)} \times \{b\}) \cup \left(B \times \left[A - \bigcup_{i=1}^{2} \operatorname{St}(u_i, A)\right]\right) \\ &\cup (s_1 \times [\overline{\operatorname{St}(u_2, A)} - \operatorname{St}(u_1, A)]) \cup (s_2 \times [\overline{\operatorname{St}(u_1, A)} - \operatorname{St}(u_2, A)]) \\ &\cup (\{b\} \times \overline{\operatorname{St}(s, A)}). \end{aligned}$$

Now

$$P(A^*) \cap \left(\left[A - \bigcup_{i=1}^{2} \operatorname{St}(u_i, A)\right] \times B\right) = \left[A - \bigcup_{i=1}^{2} \operatorname{St}(u_i, A)\right] \times s,$$

and hence

$$H_k\left(P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{2} \operatorname{St}(u_i, A)\right] \times B\right)\right)$$

is isomorphic to $H_k(P(A^*))$ for all k. Also

$$\begin{aligned} &\left[P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{2} \operatorname{St}(u_i, A)\right] \times B\right)\right] \cap ([\overline{\operatorname{St}(u_2, A)} - \operatorname{St}(u_1, A)] \times s_1) \\ &\qquad = ([\overline{\operatorname{St}(u_2, A)} - \operatorname{St}(u_1, A)] \times \{u_1\}) \cup ([\partial(\operatorname{St}(u_2, A)) - \operatorname{St}(u_1, A)] \times s_1), \end{aligned}$$

and hence

$$H_k\left(P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{2} \operatorname{St}(u_i, A)\right] \times B\right) \cup ([\overline{\operatorname{St}(u_2, A)} - \operatorname{St}(u_1, A)] \times s_1)\right)$$

is isomorphic to $H_k(P(A^*))$ for all k. Similarly

$$\begin{aligned} &\left[P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{2} \operatorname{St}(u_i, A)\right] \times B\right) \cup ([\overline{\operatorname{St}(u_2, A)} - \operatorname{St}(u_1, A)] \times s_1)\right] \\ &\qquad\qquad \cap ([\overline{\operatorname{St}(u_1, A)} - \operatorname{St}(u_2, A)] \times s_2) \\ &\qquad = ([\overline{\operatorname{St}(u_1, A)} - \operatorname{St}(u_2, A)] \times \{u_2\}) \cup ([\partial(\operatorname{St}(u_1, A)) - \operatorname{St}(u_2, A)] \times s_2). \end{aligned}$$

Therefore

$$\begin{aligned} H_k\Big(P(A^*) &\cup \left(\left[A - \bigcup_{i=1}^{2} \operatorname{St}(u_i, A)\right] \times B\right) \\ &\cup ([\overline{\operatorname{St}(u_2, A)} - \operatorname{St}(u_1, A)] \times s_1) \cup ([\overline{\operatorname{St}(u_1, A)} - \operatorname{St}(u_2, A)] \times s_2)\Big) \end{aligned}$$

is isomorphic to $H_k(P(A^*))$ for all k. Now

$$\left[P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times B\right) \cup ([\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)] \times s_1) \cup ([\overline{\mathrm{St}(u_1, A)} - \mathrm{St}(u_2, A)] \times s_2)\right] \cap (\overline{\mathrm{St}(s, A)} \times \{b\}) = \partial(\mathrm{St}(s, A)) \times \{b\}.$$

Since A^* is connected and $\dim(\mathrm{St}(s, A) \times \{b\}) \leqslant 2$,

$$H_k\left(P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times B\right) \cup ([\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)] \times s_1) \cup ([\overline{\mathrm{St}(u_1, A)} - \mathrm{St}(u_2, A)] \times s_2) \cup (\overline{\mathrm{St}(s, A)} \times \{b\})\right)$$

is isomorphic to $H_k(P(A^*))$ for $k = 0$ and $k \geqslant 3$. Suppose z is a 1-cycle in $\partial(\mathrm{St}(s, A)) \times \{b\}$. Clearly Theorem 3 applies, and hence z bounds in $\overline{\mathrm{St}(s, A)} \times \{b\}$ but not in

$$P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times B\right) \cup ([\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)] \times s_1) \cup ([\overline{\mathrm{St}(u_1, A)} - \mathrm{St}(u_2, A)] \times s_2).$$

Therefore, by Theorem 1,

$$H_2\left(P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times B\right) \cup ([\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)] \times s_1) \cup ([\overline{\mathrm{St}(u_1, A)} - \mathrm{St}(u_2, A)] \times s_2) \cup (\overline{\mathrm{St}(s, A)} \times \{b\})\right)$$

is isomorphic to $H_2(P(A^*))$. Clearly Theorem 4 applies, and hence

$$H_1\left(P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times B\right) \cup ([\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)] \times s_1) \cup ([\overline{\mathrm{St}(u_1, A)} - \mathrm{St}(u_2, A)] \times s_2) \cup (\overline{\mathrm{St}(s, A)} \times \{b\})\right)$$

is the free abelian group on $\alpha - \beta$ generators because $\partial(\mathrm{St}(s, A))$ is connected. Now

$$\left[P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times B\right) \cup ([\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)] \times s_1) \cup ([\overline{\mathrm{St}(u_1, A)} - \mathrm{St}(u_2, A)] \times s_2) \cup (\overline{\mathrm{St}(s, A)} \times \{b\})\right] \cap \left(B \times \left[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right]\right) = s \times \left[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right],$$

and therefore

$$H_k\left(P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times B\right) \cup ([\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)] \times s_1)\right.$$
$$\cup ([\overline{\mathrm{St}(u_1, A)} - \mathrm{St}(u_2, A)] \times s_2) \cup (\overline{\mathrm{St}(s, A)} \times \{b\})$$
$$\left.\cup \left(B \times \left[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right]\right)\right)$$

is isomorphic to

$$H_k\left(P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times B\right) \cup ([\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)] \times s_1)\right.$$
$$\left.\cup ([\overline{\mathrm{St}(u_1, A)} - \mathrm{St}(u_2, A)] \times s_2) \cup (\overline{\mathrm{St}(s, A)} \times \{b\})\right)$$

for all k. Also

$$\left[P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times B\right) \cup ([\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)] \times s_1)\right.$$
$$\cup ([\overline{\mathrm{St}(u_1, A)} - \mathrm{St}(u_2, A)] \times s_2) \cup (\overline{\mathrm{St}(s, A)} \times \{b\})$$
$$\left.\cup \left(B \times \left[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right]\right)\right] \cap (s_1 \times [\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)])$$
$$= (\{u_1\} \times [\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)]) \cup (s_1 \times [\partial(\mathrm{St}(u_2, A)) - \mathrm{St}(u_1, A)]),$$

and hence

$$H_k\left(P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times B\right) \cup ([\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)] \times s_1)\right.$$
$$\cup ([\overline{\mathrm{St}(u_1, A)} - \mathrm{St}(u_2, A)] \times s_2) \cup (\overline{\mathrm{St}(s, A)} \times \{b\})$$
$$\left.\cup \left(B \times \left[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right]\right) \cup (s_1 \times [\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)])\right)$$

is isomorphic to

$$H_k\left(P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times B\right) \cup ([\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)] \times s_1)\right.$$
$$\left.\cup ([\overline{\mathrm{St}(u_1, A)} - \mathrm{St}(u_2, A)] \times s_2) \cup (\overline{\mathrm{St}(s, A)} \times \{b\})\right)$$

for all k. Similarly

$$\left[P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times B\right) \cup ([\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)] \times s_1)\right.$$
$$\cup ([\overline{\mathrm{St}(u_1, A)} - \mathrm{St}(u_2, A)] \times s_2) \cup (\overline{\mathrm{St}(s, A)} \times \{b\})$$
$$\left.\cup \left(B \times \left[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right]\right) \cup (s_1 \times [\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)])\right]$$
$$\cap (s_2 \times [\overline{\mathrm{St}(u_1, A)} - \mathrm{St}(u_2, A)])$$
$$= (\{u_2\} \times [\overline{\mathrm{St}(u_1, A)} - \mathrm{St}(u_2, A)]) \cup (s_2 \times [\partial(\mathrm{St}(u_1, A)) - \mathrm{St}(u_2, A)]),$$

and hence

$$H_k\Big(P(A^*) \cup \Big(\Big[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\Big] \times B\Big) \cup ([\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)] \times s_1)$$
$$\cup ([\overline{\mathrm{St}(u_1, A)} - \mathrm{St}(u_2, A)] \times s_2) \cup (\overline{\mathrm{St}(s, A)} \times \{b\})$$
$$\cup \Big(B \times \Big[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\Big]\Big) \cup (s_1 \times [\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)])$$
$$\cup (s_2 \times [\overline{\mathrm{St}(u_1, A)} - \mathrm{St}(u_2, A)])\Big)$$

is isomorphic to

$$H_k\Big(P(A^*) \cup \Big(\Big[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\Big] \times B\Big) \cup ([\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)] \times s_1)$$
$$\cup ([\overline{\mathrm{St}(u_1, A)} - \mathrm{St}(u_2, A)] \times s_2) \cup (\overline{\mathrm{St}(s, A)} \times \{b\})\Big)$$

for all k. Now

$$\Big[P(A^*) \cup \Big(\Big[A - \bigcup_{i=1}^{2} \mathrm{St}(u_1, A)\Big] \times B\Big) \cup ([\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)] \times s_1)$$
$$\cup ([\overline{\mathrm{St}(u_1, A)} - \mathrm{St}(u_2, A)] \times s_2) \cup (\overline{\mathrm{St}(s, A)} \times \{b\})$$
$$\cup \Big(B \times \Big[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\Big]\Big) \cup (s_1 \times [\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)])$$
$$\cup (s_2 \times [\overline{\mathrm{St}(u_1, A)} - \mathrm{St}(u_2, A)])\Big] \cap (\{b\} \times \overline{\mathrm{St}(s, A)})$$
$$= \{b\} \times \partial(\mathrm{St}(s, A)).$$

Again, since A^* is connected and $\dim(\{b\} \times \overline{\mathrm{St}(s, A)}) \leqslant 2$, $H_k(P(X^*))$ is isomorphic to $H_k(P(A^*))$ for $k = 0$ and $k \geqslant 3$. Suppose z is a 1-cycle in $\{b\} \times \partial(\mathrm{St}(s, A))$. Then z bounds in $\{b\} \times \overline{\mathrm{St}(s, A)}$ and in

$$P(A^*) \cup \Big(\Big[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\Big] \times B\Big) \cup ([\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)] \times s_1)$$
$$\cup ([\overline{\mathrm{St}(u_1, A)} - \mathrm{St}(u_2, A)] \times s_2) \cup (\overline{\mathrm{St}(s, A)} \times \{b\})$$
$$\cup \Big(B \times \Big[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\Big]\Big) \cup (s_1 \times [\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)])$$
$$\cup (s_2 \times [\overline{\mathrm{St}(u_1, A)} - \mathrm{St}(u_2, A)]).$$

Therefore $H_2(P(X^*))$ is isomorphic to the direct sum

$$H_2(P(A^*)) + H_1(\partial(\mathrm{St}(s, A))),$$

and $H_1(P(X^*))$ is the free abelian group on $\alpha - \beta$ generators.

THEOREM 10. *If A is a finite, contractible, 2-dimensional polyhedron, B is a 2-simplex, $A \cap B = s$, where s is a 1-simplex of A and B such that A is pronged at s, A^* is connected, and $X = A \cup B$, then $H_k(X^*)$ is isomorphic to $H_k(A^*)$ for $k \neq 2$ and $H_2(X^*)$ is isomorphic to the direct sum*

$$H_2(A^*) + H_1(\partial(\mathrm{St}(s, A))) + H_1(\partial(\mathrm{St}(s, A))).$$

Proof. Again let b denote the vertex of B which is not a vertex of s, let u_1 and u_2 denote the vertices of s, and for each $i = 1, 2$, let s_i denote the 1-face of B not in A which has u_i as a vertex. Express $P(X^*)$ as the union of sets in the same way that it was expressed in the preceding proof. The preceding proof applies to show that

$$H_k\left(P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times B\right) \cup ([\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)] \times s_1) \cup ([\overline{\mathrm{St}(u_1, A)} - \mathrm{St}(u_2, A)] \times s_2)\right)$$

is isomorphic to $H_k(P(A^*))$ for all k and

$$H_k\left(P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times B\right) \cup ([\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)] \times s_1) \cup ([\overline{\mathrm{St}(u_1, A)} - \mathrm{St}(u_2, A)] \times s_2) \cup (\overline{\mathrm{St}(s, A)} \times \{b\})\right)$$

is isomorphic to $H_k(P(A^*))$ for $k = 0$ and $k \geqslant 3$. But now, since A is pronged at s, each 1-cycle in $\partial(\mathrm{St}(s, A)) \times \{b\}$ bounds in $\overline{\mathrm{St}(s, A)} \times \{b\}$ and in

$$P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times B\right) \cup ([\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)] \times s_1) \cup ([\overline{\mathrm{St}(u_1, A)} - \mathrm{St}(u_2, A)] \times s_2).$$

Therefore

$$H_2\left(P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times B\right) \cup ([\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)] \times s_1) \cup ([\overline{\mathrm{St}(u_1, A)} - \mathrm{St}(u_2, A)] \times s_2) \cup (\overline{\mathrm{St}(s, A)} \times \{b\})\right)$$

is isomorphic to the direct sum $H_2(P(A^*)) + H_1(\partial(\mathrm{St}(s, A)))$, and

$$H_1\left(P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times B\right) \cup ([\overline{\mathrm{St}(u_2, A)} - \mathrm{St}(u_1, A)] \times s_1) \cup ([\overline{\mathrm{St}(u_1, A)} - \mathrm{St}(u_2, A)] \times s_2) \cup (\overline{\mathrm{St}(s, A)} \times \{b\})\right)$$

is isomorphic to $H_1(P(A^*))$. Again the preceding proof applies to show that $H_k(P(X^*))$ is isomorphic to $H_k(P(A^*))$ for $k \neq 2$ and $H_2(P(X^*))$ is isomorphic to the direct sum

$$H_2(P(A^*)) + H_1(\partial(\mathrm{St}(s, A))) + H_1(\partial(\mathrm{St}(s, A))).$$

6. Addition of a 2-simplex at two 1-simplexes. Throughout this section, we assume that

(1) A is a finite, contractible, 2-dimensional polyhedron;
(2) B is a 2-simplex;
(3) $A \cap B = s_1 \cup s_2$, where s_1 and s_2 are 1-simplexes of A and B;
(4) $s_1 \cap s_2 = \{u_3\}$, where u_3 is a vertex;
(5) if u_i is the vertex of s_i different from u_3, then there is a sequence $r_1, r_2, \ldots, r_n$ of 1-simplexes in $\partial(\mathrm{St}(u_3, A))$ such that u_1 is a vertex of r_1, u_2 is a vertex of r_n, $r_j \cap r_{j+1}$ is a vertex, and $r_j \cap r_k = \emptyset$ if $|j - k| > 1$;
(6) if S is a simple closed curve in $\partial(\mathrm{St}(u_3, A))$ and u_1 and u_2 are not in S, then there is a sequence of 1-simplexes satisfying (5) such that $r_j \cap S = \emptyset$ for each j;
(7) A^* is connected;
(8) $H_2(A^*)$ is the free abelian group on α generators;
(9) $H_1(A^*)$ is the free abelian group on β generators;
(10) $H_0(\partial(\mathrm{St}(u_3, A)) - \bigcup_{i=1}^2 \mathrm{St}(u_i, A))$ is the free abelian group on γ generators; and
(11) $X = A \cup B$.

Let s denote the 1-face of B which is not in A. Then

$$\begin{aligned} P(X^*) = P(A^*) &\cup \left(\left[A - \bigcup_{i=1}^{3} \mathrm{St}(u_i, A)\right] \times B\right) \\ &\cup \left(\left[\overline{\mathrm{St}(u_3, A)} - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times s\right) \cup \left(B \times \left[A - \bigcup_{i=1}^{3} \mathrm{St}(u_i, A)\right]\right) \\ &\cup \left(s \times \left[\overline{\mathrm{St}(u_3, A)} - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right]\right). \end{aligned}$$

THEOREM 11. *If z is a 2-cycle in*

$$\begin{aligned} &\left(\left[\overline{\mathrm{St}(u_3, A)} - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times \{u_1\}\right) \\ &\cup \left(\left[\overline{\mathrm{St}(u_3, A)} - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times \{u_2\}\right) \\ &\cup \left(\left[\partial(\mathrm{St}(u_3, A)) - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times s\right), \end{aligned}$$

then z bounds in

$$P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{3} \mathrm{St}(u_i, A)\right] \times B\right).$$

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Proof. If there is a 2-cycle z in

$$\left(\left[\overline{\mathrm{St}(u_3, A)} - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times \{u_1\}\right)$$
$$\cup \left(\left[\overline{\mathrm{St}(u_3, A)} - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times \{u_2\}\right)$$
$$\cup \left(\left[\partial(\mathrm{St}(u_3, A)) - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times s\right),$$

then there is a simple closed curve S in

$$\left[\partial(\mathrm{St}(u_3, A)) - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times s.$$

Let $D = S * u_3$. It is sufficient to prove the theorem under the assumption that z is a 2-chain which assigns to each 2-cell in

$$(D \times \{u_1\}) \cup (S \times s) \cup (D \times \{u_2\})$$

either ± 1. It is then clear that z is homologous in

$$P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{3} \mathrm{St}(u_i, A)\right] \times B\right)$$

to a 2-cycle z_1 in $(D \times \{u_1\} \cup (S \times (s_1 \cup s_2)) \cup (D \times \{u_2\})$. Since u_1 and u_2 are not in S, there is a sequence $r_1, r_2, \ldots, r_n$ of 1-simplexes in $\partial(\mathrm{St}(u_3, A))$ satisfying (5) and (6). Thus z_1 is homologous in $P(A^*)$ to a 2-cycle z_2 in

$$(D \times \{u_1\}) \cup \left(S \times \left(\bigcup_{j=1}^{n} r_j\right)\right) \cup (D \times \{u_2\}),$$

and it is clear that there is a 3-chain associated with

$$D \times \bigcup_{j=1}^{n} r_j$$

whose boundary is z_2.

Notation. Suppose v_1 and v_2 are vertices in different components of

$$\partial(\mathrm{St}(u_3, A)) - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A).$$

Let C_u denote the component of $A - \{u_3\}$ which contains u_1 and u_2, and for each i, let C_i be the component of $A - \{u_3\}$ which contains v_i. Also let $K = \bigcup\{\sigma|\ \sigma$ is a 2-simplex and there is a sequence $\sigma_1, \sigma_2, \ldots, \sigma_n$ of 2-simplexes in A with the property that $\sigma = \sigma_1$, u_1 is a vertex of σ_n, and $\sigma_j \cap \sigma_{j+1}$ is a 1-simplex for each $j\}$.

The author **(6)** defined a c-point as follows: A point $x \in A$ is called a *c-point* of A if there exist 2-simplexes, $\tau_1, \tau_2, \ldots, \tau_n$, of A and a simplex τ of A such that:

(a) τ is not a face of τ_i for any i,
(b) x is a vertex of τ and of τ_i for each i,
(c) $\tau_n \cap \tau_1$ is a 1-simplex,
(d) for each $i = 1, 2, \ldots, n-1$, $\tau_i \cap \tau_{i+1}$ is a 1-simplex, and
(e) $\tau_i \cap \tau_j = \{x\}$ unless i and j satisfy the conditions of either (c) or (d).

THEOREM 12. *Suppose v_1 and v_2 are vertices in different components of*

$$\partial(\mathrm{St}(u_3, A)) - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A).$$

For each $i = 1, 2$, let τ_i be the 1-simplex with vertices u_3 and v_i, and let z be a 1-cycle which assigns to each 1-simplex in

$$((\tau_1 \cup \tau_2) \times \{u_1\}) \cup (\{v_2\} \times s) \cup ((\tau_1 \cup \tau_2) \times \{u_2\}) \cup (\{v_1\} \times s)$$

either ± 1.

(a) *If $C_i \neq C_u$ for any i, then z bounds in*

$$P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{3} \mathrm{St}(u_i, A)\right] \times B\right).$$

(b) *If $C_i = C_u$ but $C_j \neq C_u$ $(i, j = 1, 2, i \neq j)$, then z bounds in*

$$P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{3} \mathrm{St}(u_i, A)\right] \times B\right)$$

if and only if either there is a vertex w in K such that $\partial(\mathrm{St}(w, K))$ contains a simple closed curve and w is a c-point of A or there is a 1-simplex in K which is a face of at least three 2-simplexes.

(c) *If $C_1 = C_2 = C_u$, then z bounds in*

$$P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{3} \mathrm{St}(u_i, A)\right] \times B\right)$$

if and only if either there is a vertex w in K such that $\partial(\mathrm{St}(w, K))$ contains a simple closed curve and w is a c-point of A, there is a 1-simplex in K which is a face of at least three 2-simplexes, or, for each i, v_1 and v_2 are in different components of $\partial(\mathrm{St}(u_3, A)) - \mathrm{St}(u_i, A)$.

Proof. It is clear that z is homologous in

$$P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{3} \mathrm{St}(u_i, A)\right] \times B\right)$$

to a 1-cycle z_1 which assigns to each 1-simplex in

$$((\tau_1 \cup \tau_2) \times \{u_1\}) \cup (\{v_2\} \times (s_1 \cup s_2))$$
$$\cup ((\tau_1 \cup \tau_2) \times \{u_2\}) \cup (\{v_1\} \times (s_1 \cup s_2))$$

either ± 1. Also it is clear that z bounds in

$$P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{3} \mathrm{St}(u_i, A)\right] \times B\right)$$

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if and only if z_1 bounds in $P(A^*)$. Let $r_1, r_2, \ldots, r_m$ be a sequence of 1-simplexes satisfying (5), and for each $j = 1, 2, \ldots, m$, let σ_j be the 2-simplex which has r_j as a face and u_3 as a vertex.

(a) It is clear that there is a 2-chain associated with

$$\left(\bigcup_{i=1}^{2} \tau_i \times \bigcup_{j=1}^{m} r_j\right) \cup \left(\bigcup_{i=1}^{2} \{v_i\} \times \bigcup_{j=1}^{m} \sigma_j\right)$$

whose boundary is z_1.

(b) We may assume without loss of generality that $C_1 = C_u$ but $C_2 \neq C_u$. It is clear that there is a 2-chain associated with

$$\left(\tau_2 \times \bigcup_{j=1}^{m} r_j\right) \cup \left(\{v_2\} \times \bigcup_{j=1}^{m} \sigma_j\right)$$

whose boundary is $z_1 - z_2$, where z_2 is a 1-cycle which assigns to each 1-simplex in

$$(\tau_1 \times \{u_1\}) \cup \left(\{u_3\} \times \bigcup_{j=1}^{m} r_j\right) \cup (\tau_1 \times \{u_2\}) \cup (\{v_1\} \times (s_1 \cup s_2))$$

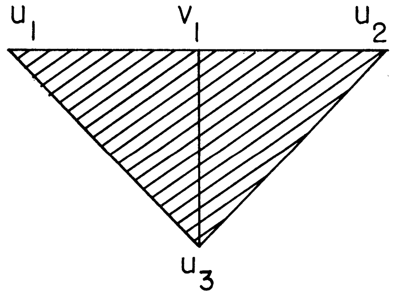
either ± 1. Now z_2 is a 1-cycle in $P(K^*)$.

If there is a 1-simplex in K which is a face of at least three 2-simplexes, then there is a subpolyhedron L of K such that $L = D \cup \mu$, where D and μ are homeomorphic to 2-simplexes and $D \cap \mu$ is a 1-simplex which is in the boundary of μ but not in the boundary of D, and

$$(\tau_1 \times \{u_1\}) \cup \left(\{u_3\} \times \bigcup_{j=1}^{m} r_j\right) \cup (\tau_1 \times \{u_2\}) \cup (\{v_1\} \times (s_1 \cup s_2)) \subset P(L^*).$$

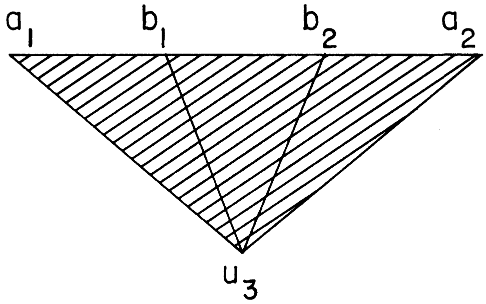
Since D^* has the homotopy type of a circle, $H_1(L^*) = 0$ by Theorem 9. Therefore z_2 bounds in $P(L^*)$ and hence in $P(A^*)$. Suppose there is no 1-simplex in K which is a face of more than two 2-simplexes, but there is a vertex w in K such that $\partial(\mathrm{St}(w, K))$ contains a simple closed curve and w is a c-point of A. Choose a 1-simplex μ in $A - K$ such that w is a vertex of μ. If $L = K \cup \mu$, then $H_1(L^*) = 0$ by Theorem 5. Therefore z_2 bounds in $P(L^*)$ and hence in $P(A^*)$.

Since $K \cap \overline{A - K}$ is a collection of vertices, if z_2 bounds in $P(A^*)$, then there is a 1-simplex μ in A such that if $L = K \cup \mu$, then z_2 bounds in $P(L^*)$. If there is no vertex w in K such that $\partial(\mathrm{St}(w, K))$ contains a simple closed curve and w is a c-point of A and there is a 1-simplex μ in A such that if $L = K \cup \mu$, then z_2 bounds in $P(L^*)$, then z_2 bounds in $P(K^*)$ by Theorem 5. If, in addition, there is no 1-simplex in K which is a face of more than two 2-simplexes, then K is homeomorphic to a disk. Thus in order to complete the proof of (b), it is sufficient to assume that K is the polyhedron



and show that z_2 does not bound in $P(K^*)$. But this is a routine verification.

(c) Since z_1 is a 1-cycle in $P(K^*)$, the proof of (b) applies to give us part of (c). For essentially the same reason as given in (b), in order to complete the proof of (c), it is sufficient to assume that K is the polyhedron



where each a_i and each b_i is either u_j or v_j, and show that z_1 bounds if and only if each b_i is a u_j and each a_i is a v_j. But again this is a routine verification.

COROLLARY 1. *Suppose v_1, v_2, and v_3 are vertices in different components of*

$$\partial(\mathrm{St}(u_3, A)) - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A).$$

For each $i = 1, 2, 3$, let τ_i be the 1-simplex with vertices u_3 and v_i, let z_1 be a 1-cycle which assigns to each 1-simplex in

$$((\tau_1 \cup \tau_2) \times \{u_1\}) \cup (\{v_2\} \times s) \cup ((\tau_1 \cup \tau_2) \times \{u_2\}) \cup (\{v_1\} \times s)$$

either ± 1, and let z_2 be a 1-cycle which assigns to each 1-simplex in

$$((\tau_1 \cup \tau_3) \times \{u_1\}) \cup (\{v_3\} \times s) \cup ((\tau_1 \cup \tau_3) \times \{u_2\}) \cup (\{v_1\} \times s)$$

either ± 1. If neither z_1 nor z_2 bounds in

$$P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{3} \mathrm{St}(u_i, A)\right] \times B\right),$$

then z_1 *is homologous in*

$$P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{3} \mathrm{St}(u_i, A)\right] \times B\right)$$

to either $\pm z_2$.

THEOREM 13. *Suppose* v_1 *and* v_2 *are vertices in different components of*

$$\partial(\mathrm{St}(u_3, A)) - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A).$$

For each $i = 1, 2$, *let* τ_i *be the* 1-*simplex with vertices* u_3 *and* v_i, *and let* z *be a* 1-*cycle which assigns to each* 1-*simplex in*

$$((\tau_1 \cup \tau_2) \times \{u_1\}) \cup (\{v_2\} \times (s_1 \cup s_2)) \\ \cup ((\tau_1 \cup \tau_2) \times \{u_2\}) \cup (\{v_1\} \times (s_1 \cup s_2))$$

either ± 1. *Then* z *is homologous in* $P(A^*)$ *to a* 1-*cycle in*

$$(\{u_1\} \times (\tau_1 \cup \tau_2)) \cup ((s_1 \cup s_2) \times \{v_2\}) \\ \cup (\{u_2\} \times (\tau_1 \cup \tau_2)) \cup ((s_1 \cup s_2) \times \{v_1\}).$$

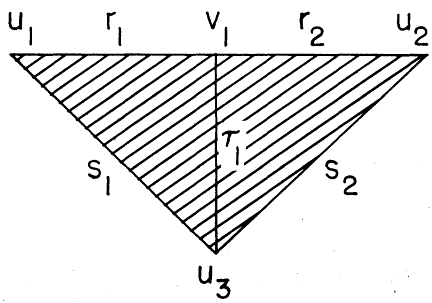
Proof. If z bounds in $P(A^*)$, then the theorem is obviously true. If z does not bound, then K is homeomorphic to a disk. Since v_1 and v_2 are in different components of

$$\partial(\mathrm{St}(u_3, A)) - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A),$$

we may assume that there is a sequence $r_1, r_2, \ldots, r_m$ of 1-simplexes satisfying (5) such that v_2 is not a vertex of r_j for any j. Then, as observed in the proof of Theorem 12, z is homologous in $P(A^*)$ to a 1-cycle z' which assigns to each 1-simplex in

$$(\tau_1 \times \{u_1\}) \cup \left(\{u_3\} \times \bigcup_{j=1}^{m} r_j\right) \cup (\tau_1 \times \{u_2\}) \cup (\{v_1\} \times (s_1 \cup s_2))$$

either ± 1. Thus it is sufficient to assume that A is the polyhedron



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and show that each 1-cycle in

$$(\tau_1 \times \{u_1\}) \cup (\{u_3\} \times (r_1 \cup r_2)) \cup (\tau_1 \times \{u_2\}) \cup (\{v_1\} \times (s_1 \cup s_2))$$

is homologous in $P(A^*)$ to a 1-cycle in

$$(\{u_1\} \times \tau_1) \cup ((r_1 \cup r_2) \times \{u_3\}) \cup (\{u_2\} \times \tau_1) \cup ((s_1 \cup s_2) \times \{v_1\}).$$

But this is a routine verification.

Definition 2. We define a number δ called the *simple* 2-*dimensional deleted product number* of A at u_3 with respect to u_1 and u_2 as follows: If $\gamma = 0$, there is a vertex w in K such that $\partial(\mathrm{St}(w, K))$ contains a simple closed curve, and w is a *c-point* of A, or there is a 1-simplex in K which is a face of at least three 2-simplexes, then $\delta = 0$. Otherwise $\delta = 1$.

THEOREM 14. *If δ is the simple 2-dimensional deleted product number of A at u_3 with respect to u_1 and u_2, then $H_k(X^*)$ is isomorphic to $H_k(A^*)$ for $k = 0$ and $k \geqslant 4$, $H_3(X^*)$ is isomorphic to the direct sum*

$$H_3(A^*) + H_1\left(\partial(\mathrm{St}(u_3, A)) - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right) + H_1\left(\partial(\mathrm{St}(u_3, A)) - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right),$$

$H_2(X^*)$ is the free abelian group on $\alpha + 2\gamma - \delta$ generators, and $H_1(X^*)$ is the free abelian group on $\beta - \delta$ generators.

Proof. Since

$$P(A^*) \cap \left(\left[A - \bigcup_{i=1}^{3} \mathrm{St}(u_i, A)\right] \times B\right) = \left[A - \bigcup_{i=1}^{3} \mathrm{St}(u_i, A)\right] \times (s_1 \cup s_2),$$

$$H_k\left(P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{3} \mathrm{St}(u_i, A)\right] \times B\right)\right)$$

is isomorphic to $H_k(P(A^*))$ for all K. Now

$$\begin{aligned}\left[P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{3} \mathrm{St}(u_i, A)\right] \times B\right)\right] &\\ \cap \left(\left[\overline{\mathrm{St}(u_3, A)} - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times s\right) &\\ = \left(\left[\overline{\mathrm{St}(u_3, A)} - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times \{u_1\}\right) &\\ \cup \left(\left[\overline{\mathrm{St}(u_3, A)} - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times \{u_2\}\right) &\\ \cup \left(\left[\partial(\mathrm{St}(u_3, A)) - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times s\right). &\end{aligned}$$

Since A^* is connected and

$$\dim\left(\left[\operatorname{St}(u_3, A) - \bigcup_{i=1}^{2} \operatorname{St}(u_i, A)\right] \times s\right) \leqslant 3,$$

$$H_k\Big(P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{3} \operatorname{St}(u_i, A)\right] \times B\right) \cup \left(\left[\operatorname{St}(u_3, A) - \bigcup_{i=1}^{2} \operatorname{St}(u_i, A)\right] \times s\right)\Big)$$

is isomorphic to $H_k(P(A^*))$ for $k = 0$ and $k \geqslant 4$. Now

$$H_2\Big(\left(\left[\overline{\operatorname{St}(u_3, A)} - \bigcup_{i=1}^{2} \operatorname{St}(u_i, A)\right] \times \{u_1\}\right) \cup \left(\left[\overline{\operatorname{St}(u_3, A)} - \bigcup_{i=1}^{2} \operatorname{St}(u_i, A)\right] \times \{u_2\}\right) \cup \left(\left[\partial(\operatorname{St}(u_3, A)) - \bigcup_{i=1}^{2} \operatorname{St}(u_i, A)\right] \times s\right)\Big)$$

is isomorphic to

$$H_1\left(\partial(\operatorname{St}(u_3, A)) - \bigcup_{i=1}^{2} \operatorname{St}(u_i, A)\right),$$

and each 2-cycle in

$$\left(\left[\overline{\operatorname{St}(u_3, A)} - \bigcup_{i=1}^{2} \operatorname{St}(u_i, A)\right] \times \{u_1\}\right) \cup \left(\left[\overline{\operatorname{St}(u_3, A)} - \bigcup_{i=1}^{2} \operatorname{St}(u_i, A)\right] \times \{u_2\}\right) \cup \left(\left[\partial(\operatorname{St}(u_3, A)) - \bigcup_{i=1}^{2} \operatorname{St}(u_i, A)\right] \times s\right)$$

bounds in

$$\left[\overline{\operatorname{St}(u_3, A)} - \bigcup_{i=1}^{2} \operatorname{St}(u_i, A)\right] \times s$$

and in

$$P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{3} \operatorname{St}(u_i, A)\right] \times B\right)$$

by Theorem 11. Therefore

$$H_3\Big(P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{3} \operatorname{St}(u_i, A)\right] \times B\right) \cup \left(\left[\overline{\operatorname{St}(u_3, A)} - \bigcup_{i=1}^{2} \operatorname{St}(u_i, A)\right] \times s\right)\Big)$$

is isomorphic to the direct sum

$$H_3(P(A^*)) + H_1\left(\partial(\mathrm{St}(u_3, A)) - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right).$$

Since

$$\overline{\mathrm{St}(u_3, A)} - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)$$

is contractible and

$$\begin{aligned} H_1\Bigg(\Bigg(\Bigg[\overline{\mathrm{St}(u_3, A)} - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\Bigg] \times \{u_1\}\Bigg) \\ \cup \Bigg(\Bigg[\overline{\mathrm{St}(u_3, A)} - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\Bigg] \times \{u_2\}\Bigg) \\ \cup \Bigg(\Bigg[\partial(\mathrm{St}(u_3, A)) - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\Bigg] \times s\Bigg)\Bigg) \end{aligned}$$

is isomorphic to

$$H_0\left(\partial(\mathrm{St}(u_3, A)) - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right),$$

it follows from Theorem 12 and Corollary 1 that

$$\begin{aligned} H_2\Bigg(P(A^*) \cup \Bigg(\Bigg[A - \bigcup_{i=1}^{3} \mathrm{St}(u_i, A)\Bigg] \times B\Bigg) \\ \cup \Bigg(\Bigg[\overline{\mathrm{St}(u_3, A)} - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\Bigg] \times s\Bigg)\Bigg) \end{aligned}$$

is the free abelian group on $\alpha + \gamma - \delta$ generators. Since, in addition,

$$\begin{aligned} \Bigg(\Bigg[\overline{\mathrm{St}(u_3, A)} - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\Bigg] \times \{u_1\}\Bigg) \\ \cup \Bigg(\Bigg[\overline{\mathrm{St}(u_3, A)} - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\Bigg] \times \{u_2\}\Bigg) \\ \cup \Bigg(\Bigg[\partial(\mathrm{St}(u_3, A)) - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\Bigg] \times s\Bigg) \end{aligned}$$

is connected,

$$\begin{aligned} H_1\Bigg(P(A^*) \cup \Bigg(\Bigg[A - \bigcup_{i=1}^{3} \mathrm{St}(u_i, A)\Bigg] \times B\Bigg) \\ \cup \Bigg(\Bigg[\overline{\mathrm{St}(u_3, A)} - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\Bigg] \times s\Bigg)\Bigg) \end{aligned}$$

is the free abelian group on $\beta - \delta$ generators. Since

$$\Big[P(A^*) \cup \Big(\Big[A - \bigcup_{i=1}^{3} \mathrm{St}(u_i, A)\Big] \times B\Big) \cup \Big(\Big[\overline{\mathrm{St}(u_3, A)} - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\Big] \times s\Big)\Big] \cap \Big(B \times \Big[A - \bigcup_{i=1}^{3} \mathrm{St}(u_i, A)\Big]\Big) = (s_1 \cup s_2) \times \Big[A - \bigcup_{i=1}^{3} \mathrm{St}(u_i, A)\Big],$$

$$H_k\Big(P(A^*) \cup \Big(\Big[A - \bigcup_{i=1}^{3} \mathrm{St}(u_i, A)\Big] \times B\Big) \cup \Big(\Big[\overline{\mathrm{St}(u_3, A)} - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\Big] \times s\Big) \cup \Big(B \times \Big[A - \bigcup_{i=1}^{3} \mathrm{St}(u_i, A)\Big]\Big)\Big)$$

is isomorphic to

$$H_k\Big(P(A^*) \cup \Big(\Big[A - \bigcup_{i=1}^{3} \mathrm{St}(u_i, A)\Big] \times B\Big) \cup \Big(\Big[\overline{\mathrm{St}(u_3, A)} - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\Big] \times s\Big)\Big)$$

for all k. Now

$$\Big[P(A^*) \cup \Big(\Big[A - \bigcup_{i=1}^{3} \mathrm{St}(u_i, A)\Big] \times B\Big) \cup \Big(\Big[\overline{\mathrm{St}(u_3, A)} - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\Big] \times s\Big) \cup \Big(B \times \Big[A - \bigcup_{i=1}^{3} \mathrm{St}(u_i, A)\Big]\Big)\Big] \cap \Big(s \times \Big[\overline{\mathrm{St}(u_3, A)} - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\Big]\Big)$$
$$= \Big(\{u_1\} \times \Big[\overline{\mathrm{St}(u_3, A)} - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\Big]\Big) \cup \Big(\{u_2\} \times \Big[\overline{\mathrm{St}(u_3, A)} - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\Big]\Big) \cup \Big(s \times \Big[\partial(\mathrm{St}(u_3, A)) - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\Big]\Big).$$

Therefore, for the same reason as before, $H_k(P(X^*))$ is isomorphic to $H_k(P(A^*))$ for $k = 0$ and $k \geqslant 4$, and $H_3(P(X^*))$ is isomorphic to the direct sum

$$H_3(P(A^*)) + H_1\left(\partial(\mathrm{St}(u_3, A)) - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right) + H_1\left(\partial(\mathrm{St}(u_3, A)) - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right).$$

By Theorem 13, each 1-cycle in

$$\left(\{u_1\} \times \left[\overline{\mathrm{St}(u_3, A)} - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right]\right) \cup \left(\{u_2\} \times \left[\overline{\mathrm{St}(u_3, A)} - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right]\right) \cup \left(s \times \left[\partial(\mathrm{St}(u_3, A)) - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right]\right)$$

is homologous in

$$P(A^*) \cup \left(\left[A - \bigcup_{i=1}^{3} \mathrm{St}(u_i, A)\right] \times B\right) \cup \left(B \times \left[A - \bigcup_{i=1}^{3} \mathrm{St}(u_i, A)\right]\right)$$

to a 1-cycle in

$$\left(\left[\overline{\mathrm{St}(u_3, A)} - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times \{u_1\}\right) \cup \left(\left[\overline{\mathrm{St}(u_3, A)} - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times \{u_2\}\right) \cup \left(\left[\partial(\mathrm{St}(u_3, A)) - \bigcup_{i=1}^{2} \mathrm{St}(u_i, A)\right] \times s\right).$$

Therefore $H_2(P(X^*))$ is the free abelian group on $\alpha + 2\gamma - \delta$ generators, and $H_1(P(X^*))$ is the free abelian group on $\beta - \delta$ generators.

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