

ON THE GENUS OF SOME MODULAR CURVES OF LEVEL N

CHANG HEON KIM AND JA KYUNG KOO

We estimate the genus of the modular curves $X_1(N)$.

INTRODUCTION

Let \mathfrak{h} be the complex upper half plane. Then $SL_2(\mathbb{Z})$ acts on \mathfrak{h} by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = (az + b)/(cz + d)$. Let \mathfrak{h}^* be the union of \mathfrak{h} and $\mathbb{P}^1(\mathbb{Q})$, and let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$ ($= \Gamma(1)$), which is a Fuchsian group of the first kind and contains a principal congruence subgroup $\Gamma(N)$ for some positive integer N . Then the modular curve $\Gamma \backslash \mathfrak{h}^*$ is a projective closure of the affine curve $\Gamma \backslash \mathfrak{h}$, which we denote by X_Γ , with genus g_Γ . In this article, we shall determine the genus $g(N)$ of the modular curve $X_1(N)$ ($= X_{\Gamma_1(N)}$) when $\Gamma = \Gamma_1(N)$ for $N = 1, 2, 3, \dots$. Here, we denote by $\Gamma_1(N)$ the group of elements in $\Gamma(1)$ which are congruent to $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N$.

THEOREM 1. *The genus $g(N)$ of $X_1(N)$ is given by*

$$g(N) = \begin{cases} 0, & \text{if } 1 \leq N \leq 4 \\ 1 + \frac{N^2}{24} \prod_{p|N} \left(1 - \frac{1}{p^2}\right) - \frac{1}{4} \sum_{d|N, d>0} \varphi(d)\varphi\left(\frac{N}{d}\right), & \text{otherwise} \end{cases}$$

where φ is the Euler's phi function.

We shall see later in §1 that $g(N) = 0$ only for the eleven cases $1 \leq N \leq 10$ and $N = 12$.

Throughout the article we adopt the following notation:

$\bar{\Gamma}$ is the inhomogeneous congruence group ($= \Gamma / \pm I$)

Γ_s is the isotropy group of s

$\Gamma(N) = \{\gamma \in SL_2(\mathbb{Z}) \mid \gamma \equiv I \pmod N\}$

$\Gamma_0(N)$ is the Hecke subgroup $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \mid c \equiv 0 \pmod N \right\}$

$\sigma_0(N)$ is the number of positive divisors of N .

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1. PROOF

Let μ be the index of $\bar{\Gamma}_1(N)$ in $\bar{\Gamma}(1)$. Let ν_2 (respectively ν_3) be the number of $\bar{\Gamma}_1(N)$ -inequivalent elliptic points of order 2 (respectively order 3) and ν_∞ be the number of $\bar{\Gamma}_1(N)$ -inequivalent cusps. It is well-known [1, p.68, 2, Chapter IV] or [3, Proposition 1.40] that

$$(*) \quad g = 1 + \frac{\mu}{12} - \frac{\nu_2}{4} - \frac{\nu_3}{3} - \frac{\nu_\infty}{2}.$$

Thus, in order to estimate g it is enough to know the explicit values of μ, ν_2, ν_3 and ν_∞ .

(i) μ :

For the congruence subgroup $\Gamma_0(N)$ of $\Gamma(1)$, we know [3, Proposition 1.43] that

$$(1.1) \quad [\bar{\Gamma}(1) : \bar{\Gamma}_0(N)] = N \cdot \prod_{p|N} \left(1 + \frac{1}{p}\right).$$

Note that $\Gamma_1(N)$ is the kernel of the surjective homomorphism f_N from $\Gamma_0(N)$ to $(\mathbb{Z}/N\mathbb{Z})^\times$ defined by $f_N \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = d \pmod N$. This yields

$$[\Gamma_0(N) : \Gamma_1(N)] = \varphi(N) = N \cdot \prod_{p|N} \left(1 - \frac{1}{p}\right).$$

Since $-1 \in \Gamma_0(N)$ and $-1 \notin \Gamma_1(N)$ except for $N = 1, 2$,

$$(1.2) \quad [\bar{\Gamma}_0(N) : \bar{\Gamma}_1(N)] = \begin{cases} N \cdot \prod_{p|N} \left(1 - \frac{1}{p}\right), & \text{if } N = 1, 2 \\ \frac{N}{2} \cdot \prod_{p|N} \left(1 - \frac{1}{p}\right), & \text{otherwise.} \end{cases}$$

By (1.1) and (1.2),

$$\mu = [\bar{\Gamma}(1) : \bar{\Gamma}_1(N)] = \begin{cases} 1, & \text{if } N = 1 \\ 3, & \text{if } N = 2 \\ \frac{N^2}{2} \cdot \prod_{p|N} \left(1 - \frac{1}{p^2}\right), & \text{otherwise.} \end{cases}$$

(ii) ν_2 and ν_3 :

Recall that $\gamma \in \Gamma(1)$ is an elliptic element if and only if $|tr(\gamma)| < 2$. If $\gamma \in \Gamma_1(N)$, then $\gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N$. Hence, $tr(\gamma)$ lies in $2 + N\mathbb{Z}$. Thus $\Gamma_1(N)$ has no elliptic

element unless $N = 1, 2, 3$. If $N = 1$, $\Gamma_1(1) = \Gamma(1)$ so that $\nu_2 = \nu_3 = 1$. If $N = 2$, $\Gamma_1(2) = \Gamma_0(2)$ and hence, by [3, Proposition 1.43], $\nu_2 = 1$ and $\nu_3 = 0$. If $N = 3$, then $\bar{\Gamma}_1(3) = \bar{\Gamma}_0(3)$. Again, by the same argument, $\nu_2 = 0$ and $\nu_3 = 1$. We summarise the above by

$$\nu_2 = \begin{cases} 1, & \text{if } N = 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\nu_3 = \begin{cases} 1, & \text{if } N = 1, 3 \\ 0, & \text{otherwise.} \end{cases}$$

(iii) ν_∞ :

First, we consider all pairs

$$(1.3) \quad \{c, d\} \text{ of positive integers satisfying } (c, d) = 1, d \mid N, 0 < c \leq N/d \\ \text{(or } c \text{ in any set of representatives for } \mathbb{Z} \text{ mod } (N/d)).$$

For each pair $\{c, d\}$, take a and b so that $ad - bc = 1$ and fix them. Then the elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying (1.3) form a set of representatives for $\Gamma_0(N) \backslash \Gamma(1)$. Also, the number of double cosets in $\Gamma_0(N) \backslash \Gamma(1) / \Gamma_s$ for any fixed cusp s gives the number of $\Gamma_0(N)$ -inequivalent cusps. Take s to be 0. Then we see that it is the number of pairs $\{c, d\}$ satisfying (1.3) modulo the equivalence \sim defined by $\{c, d\} \sim \{c', d'\}$ if $\begin{pmatrix} * & * \\ c' & d' \end{pmatrix} = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$ for some $m \in \mathbb{Z}$. From the last equality, we come up with $d = d'$ and $c' = c + dm$. Therefore, for fixed d

$$(1.4) \quad \text{there are exactly } \varphi((d, N/d)) \text{ inequivalent pairs.}$$

Now choose a pair $\{c, d\}$ satisfying (1.3) and $\xi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ from $\Gamma(1)$. Put $s = b/d$. Then $\xi \cdot 0 = s$. We want to estimate the index $[\bar{\Gamma}_0(N)_s : \bar{\Gamma}_1(N)_s]$. Suppose that $\pm \xi^{-1} \Gamma_0(N)_s \xi = \left\{ \pm \begin{pmatrix} 1 & 0 \\ h_1 & 1 \end{pmatrix}^n \right\}_{n \in \mathbb{Z}}$ for some $h_1 > 0$ and $\pm \xi^{-1} \Gamma_1(N)_s \xi = \left\{ \pm \begin{pmatrix} 1 & 0 \\ h_2 & 1 \end{pmatrix}^n \right\}_{n \in \mathbb{Z}}$ for some $h_2 > 0$. Recall that h_1 (respectively h_2) is the smallest positive integer h such that

$$(1.5) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 + bdh & -b^2h \\ d^2h & 1 - bdh \end{pmatrix} \in \pm \Gamma_0(N) \\ \text{(respectively } \pm \Gamma_1(N)).$$

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$ belongs to $-\Gamma_1(N)$, then by taking the trace we have $2 = -2 \pmod N$; hence N divides 4, that is, $N = 1, 2, 4$. In what follows, we assume that $N \neq 1, 2, 4$. The cases $N = 1, 2, 4$ will be dealt with separately. By (1.5), h_1 is the smallest positive integer h such that $d^2h \equiv 0 \pmod N$ and h_2 is the smallest positive integer h such that

$$(1.6) \quad d^2h \equiv 0 \equiv bdh \pmod N.$$

Clearly, $h_1 = N/(d^2, N)$. Let h'_1 be the smallest positive integer such that $bdh' \equiv 0 \pmod N$. Since $d \mid N$, we are forced to get

$$(1.7) \quad h'_1 = \frac{N/d}{(b, N/d)}.$$

Then $h_2 = \text{l.c.m.}(h_1, h'_1)$. Observe that $(d^2, N) = (d, N) \cdot ((d, N), N/(d, N)) = d \cdot (d, N/d)$ because N is divisible by d . Using this we are able to rewrite h_1 as

$$(1.8) \quad h_1 = \frac{N}{(d^2, N)} = \frac{N}{d \cdot (d, N/d)} = \frac{N}{d} \cdot \frac{1}{(d, N/d)}.$$

Since $(b, N/b) \mid b$, $(d, N/d) \mid d$ and $(b, d) = 1$, by (1.7) and (1.8) we have $h_2 = \text{l.c.m.}(h_1, h'_1) = N/d$. Thus

$$(1.9) \quad \begin{aligned} [\bar{\Gamma}_0(N)_s : \bar{\Gamma}_1(N)_s] &= [\pm\xi^{-1}\Gamma_0(N)_s\xi : \pm\xi^{-1}\Gamma_1(N)_s\xi] \\ &= [h_1\mathbb{Z} : h_2\mathbb{Z}] = \frac{h_2}{h_1} \\ &= \frac{N/d}{N/d \cdot 1/(d, N/d)} = (d, N/d). \end{aligned}$$

Now consider the natural projection $p : \bar{\Gamma}_1(N)\backslash\mathfrak{h}^* \rightarrow \bar{\Gamma}_0(N)\backslash\mathfrak{h}^*$. Let $p^{-1}(s) = \{s_1, \dots, s_h\}$ and let e_k be the ramification index of p at s_k . Then by [3, Proposition 1.37], $e_k = [\bar{\Gamma}_0(N)_{s_k} : \bar{\Gamma}_1(N)_{s_k}]$ for $k = 1, \dots, h$. Furthermore, $\bar{\Gamma}_1(N) \triangleleft \bar{\Gamma}_0(N)$ implies that $e_1 = \dots = e_h$ and

$$(1.10) \quad [\bar{\Gamma}_0(N) : \bar{\Gamma}_1(N)] = e_1h = (d, N/d) \cdot h$$

by (1.9). Here h is the number of elements of $p^{-1}(s)$ which is equal to the number of those in $p^{-1}(b/d)$ depending only on d . By (1.4), given d , there are $\varphi((d, N/d))$

$\bar{\Gamma}_0(N)$ -inequivalent cusps with the same d . Therefore, we have

$$\begin{aligned} \nu_\infty &= \sum_{d|N} \varphi((d, N/d))h \\ &= \sum_{d|N} \varphi((d, N/d))(d, N/d)^{-1} \varphi(N)/2 \text{ by (1.10)} \\ &= \sum_{d|N} \frac{\varphi(d)\varphi(N/d)}{\varphi(d \cdot (N/d))} \frac{\varphi(N)}{2} \text{ using the fact that } \varphi(n_1)\varphi(n_2) = \varphi(n_1n_2) \frac{\varphi((n_1, n_2))}{(n_1, n_2)} \\ &= \frac{1}{2} \sum_{d|N} \varphi(d)\varphi(N/d). \end{aligned}$$

Next, we deal with the cases $N = 1, 2, 4$. If $N = 1$, $\Gamma_1(1) = \Gamma(1)$; hence $\nu_\infty = 1$. If $N = 2$, $\Gamma_1(2) = \Gamma_0(2)$, and so by [3, Proposition 1.43], $\nu_\infty = 2$. If $N = 4$, $\bar{\Gamma}_1(4) = \bar{\Gamma}_0(4)$, and again by the same Proposition 1.43 in [3], $\nu_\infty = 3$. In summary,

$$\nu_\infty = \begin{cases} 1, & \text{if } N = 1 \\ 2, & \text{if } N = 2 \\ 3, & \text{if } N = 4 \\ \frac{1}{2} \sum_{d|N} \varphi(d)\varphi(N/d), & \text{otherwise.} \end{cases}$$

Substituting (i), (ii) and (iii) into the formula (*), we get the theorem.

PROPOSITION 2. For $N > 20$, $g(N) > 1$.

PROOF: It follows from Theorem 1 that $g(N) = 1 + (N^2/24) \prod_{p|N} (1 - 1/p^2) - (1/4) \sum_{d|N, d>0} \varphi(d)\varphi(N/d)$. Notice that $N \cdot \prod_{p|N} (1 - 1/p) = \varphi(N)$ and $\varphi(d)\varphi(N/d) = \varphi(N) \cdot (\varphi((d, N/d)))/((d, N/d)) \leq \varphi(N)$. Then $g(N) \geq 1 + (1/24) \left(N \cdot \prod_{p|N} (1 + 1/p) \cdot \varphi(N) - 6\sigma_0(N) \cdot \varphi(N) \right)$. We will show that for $N > 20$

$$(1.11) \quad N \cdot \prod_{p|N} \left(1 + \frac{1}{p} \right) \geq 6 \cdot \sigma_0(N),$$

where the equality holds if and only if N is square-free. Put $q(N) = \left(N \cdot \prod_{p|N} \left(1 + \frac{1}{p} \right) \right) / (\sigma_0(N))$ and $f_p(k) = (p^k + p^{k-1}) / (k + 1)$. We must show $q(N) \geq 6$. Then for $k \geq 1$,

$$\frac{d}{dk} f_p(k) = \frac{(p^k + p^{k-1})((\log p)(k + 1) - 1)}{(k + 1)^2} > 0$$

indicates that

$$(1.12) \quad f_p(k_1) < f_p(k_2) \quad \text{for} \quad k_1 < k_2.$$

Also it is easy to see that

$$(1.13) \quad f_{p_1}(k) < f_{p_2}(k) \quad \text{for} \quad p_1 < p_2.$$

For $1 \leq k \leq 5$, f_p has the following values:

k	f_2	f_3	f_5	f_7	f_{11}
1	1.5	2	3	4	6
2	2	4	10	$18\frac{2}{3}$	44
3	3	9	37.5	98	363
4	4.8	21.6	150	548.8	3194.4
5	8	54	625	$3201\frac{1}{3}$	29282

Let $N = p_1^{k_1} \cdots p_r^{k_r}$ be the prime factorisation. Then $q(N) = f_{p_1}(k_1) \cdots f_{p_r}(k_r)$. Let $\tau(N)$ be the number of distinct primes dividing N . If $\tau(N) \geq 3$,

$$q(N) \geq f_2(1)f_3(1)f_5(1) = 9 > 6 \quad \text{by (1.12), (1.13) and the table.}$$

If $\tau(N) = 1$ or 2 , we can check the inequality as follows:

- (i) $\tau(N) = 2, 2 \nmid N: q(N) > f_3(1)f_5(1) = 6.$
- (ii) $\tau(N) = 2, 2^3 \mid N:$ Since $\tau(N) = 2$, there exists an odd prime p dividing N . Then $q(N) \geq f_2(3)f_3(1) = 6$. In this case, N is not square-free and so we have strict inequality in (1.11).
- (iii) $\tau(N) = 2, 2 \mid N, (15, N) = 1:$ Since $\tau(N) = 2$ and $3 \nmid N, 5 \nmid N$, there exists an odd prime $p \geq 7$ dividing N . Then $q(N) > f_2(1)f_7(1) = 6$.
- (iv) $\tau(N) = 2, 2^2 \mid \mid N, 3^2 \mid N: q(N) \geq f_2(2)f_3(2) > 6.$
- (v) $\tau(N) = 2, 2^2 \mid \mid N, 5^2 \mid N: q(N) \geq f_2(2)f_5(2) > 6.$
- (vi) $\tau(N) = 2, 2 \mid \mid N, 3^3 \mid N: q(N) \geq f_2(1)f_3(3) > 6.$
- (vii) $\tau(N) = 2, 2 \mid \mid N, 5^2 \mid N: q(N) \geq f_2(1)f_5(2) > 6.$
- (viii) $\tau(N) = 1, N = p^k, p \geq 11: q(N) > f_{11}(1) \geq 6.$
- (ix) $\tau(N) = 1, N = 7^k, k \geq 2: q(N) \geq f_7(2) > 6.$
- (x) $\tau(N) = 1, N = 5^k, k \geq 2: q(N) \geq f_5(2) > 6.$
- (xi) $\tau(N) = 1, N = 3^k, k \geq 3: q(N) \geq f_3(3) > 6.$
- (xii) $\tau(N) = 1, N = 2^k, k \geq 5: q(N) \geq f_2(5) > 6.$

This completes the proof. □

For $N \leq 20$, Theorem 1 gives the following table :

N	μ	ν_∞	ν_2	ν_3	g
1	1	1	1	1	0
2	3	2	1	0	0
3	4	2	0	1	0
4	6	3	0	0	0
5	12	4	0	0	0
6	12	4	0	0	0
7	24	6	0	0	0
8	24	6	0	0	0
9	36	8	0	0	0
10	36	8	0	0	0
11	60	10	0	0	1
12	48	10	0	0	0
13	84	12	0	0	2
14	72	12	0	0	1
15	96	16	0	0	1
16	96	14	0	0	2
17	144	16	0	0	5
18	108	14	0	0	3
19	180	18	0	0	7
20	144	20	0	0	3

REMARK. From this table and Proposition 2, we conclude that $g(N) = 0$ if and only if $N = 1, \dots, 10$ and 12.

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Department of Mathematics
Korea Advanced Institute of Science and Technology
Taejon 305-701
Korea