THE SOLUTION OF CERTAIN DUAL INTEGRAL EQUATIONS

by W. E. WILLIAMS
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Summary

A class of dual integral equations involving Bessel functions is solved by formal application of Mellin transforms.

1. Introduction

Dual integral equations involving Bessel functions occur in the solution of axially-symmetric boundary value problems in elasticity and electrostatics. Noble (**1**) has recently obtained the solution of a general class of such integral equations. Noble's method is an extension of that developed by Copson (**2**) for the problem of the electrified disc. The approach is an indirect one and requires a knowledge of certain integrals of products of Bessel functions and considerable manipulation. In the present note a simple direct method is given for solving Noble's equations by a formal application of Mellin transforms. Mellin transforms were first used by Titchmarsh (**3**) to solve this type of dual integral equation but the present method differs from that of Titchmarsh.

Solutions of the Equations

The equations which we shall consider are

$$\int_0^\infty \xi^\alpha \psi(\xi) J_n(x\xi)d\xi = f(x), \quad 0<x<1, \qquad (1)$$

$$\int_0^\infty \psi(\xi) J_n(x\xi)d\xi = x^{-\alpha} g(x), \quad x>1. \qquad (2)$$

The left-hand sides of equations (1) and (2) represent functions of x for all values of x and we shall denote these functions by $F(x)$ and $G(x)$ respectively. We also denote the Mellin transform of a function $F(x)$ by $F^*(s)$ i.e.

$$F^*(s) = \int_0^\infty x^{s-1} F(x)dx.$$

From the convolution theorem for Mellin transforms (**4**) we have that

$$F^*(s) = J_n^*(s)\psi^*(1-s+\alpha), \qquad (3)$$

$$G^*(s) = J_n^*(s-\alpha)\psi^*(1-s+\alpha), \qquad (4)$$

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where

$$J_n^*(s) = 2^{s-1}\frac{\Gamma(\frac{1}{2}s+\frac{1}{2}n)}{\Gamma(1+\frac{1}{2}n-\frac{1}{2}s)}.$$

Equations (3) and (4) may be re-arranged to give.

$$\frac{2^{-\alpha}F^*(s)\Gamma(\frac{1}{2}n+1-\frac{1}{2}s)}{\Gamma(1+\frac{1}{2}n+\frac{1}{2}\alpha-\frac{1}{2}s)} = \frac{2^{s-\alpha-1}\Gamma(\frac{1}{2}s+\frac{1}{2}n)}{\Gamma(1+\frac{1}{2}n+\frac{1}{2}\alpha-\frac{1}{2}s)}\psi^*(1-s+\alpha), \quad \text{.........(5)}$$

$$\frac{G^*(s)\Gamma(\frac{1}{2}s+\frac{1}{2}n)}{\Gamma(\frac{1}{2}s+\frac{1}{2}n-\frac{1}{2}\alpha)} = \frac{2^{s-\alpha-1}\Gamma(\frac{1}{2}s+\frac{1}{2}n)}{\Gamma(1+\frac{1}{2}n+\frac{1}{2}\alpha-\frac{1}{2}s)}\psi^*(1-s+\alpha). \quad \text{.........(6)}$$

The convolution theorem shows that the right-hand side of equations (5) and (6) is the Mellin transform of the function $K(x)$ defined by

$$K(x) = (2x)^{-\frac{1}{2}\alpha}\int_0^\infty J_{n+\frac{1}{2}}(\xi x)\xi^{-\frac{1}{2}\alpha}\psi(\xi)d\xi. \quad \text{....................(7)}$$

In order to simplify the discussion it is convenient at this point to write down two Mellin transforms which will be used in the subsequent analysis. We have (4)

$$\left\{\frac{(1-x)^{\lambda-1}}{\Gamma(\lambda)}H(1-x)\right\}^* = \frac{\Gamma(s)}{\Gamma(s+\lambda)}, \quad \text{..........................(8)}$$

$$\left\{\frac{(x-1)^{\lambda-1}}{\Gamma(\lambda)}H(x-1)\right\}^* = \frac{\Gamma(1-\lambda-s)}{\Gamma(1-s)}, \quad \text{.....................(9)}$$

where $H(x)$ is Heaviside's unit function and $\lambda>0$. We thus see that the left-hand side of equation (5) is the product of the transforms of $F(x)$ and a function which vanishes for $x<1$, hence, since $F(x)$ is known for $x<1$, the convolution theorem shows that the function whose Mellin transform is the left-hand side of (5) is known for $x<1$. Similarly the function whose Mellin transform is the left-hand side of (6) is known for $x>1$. Thus the function $K(x)$ is known for all $x\geqq 0$ and hence $\psi(\xi)$ may be found by Hankel's inversion theorem. The actual form of the inverses of the left-hand sides of (5) and (6) depends on whether $-2<\alpha<0$ or $0<\alpha<2$ and we consider these two cases separately.

(a) $-2<\alpha<0$.

We have that

$$K(x) = x^{-(n-1)}\frac{d}{dx}\left(\frac{(2x)^{-\alpha}}{\Gamma(1+\frac{1}{2}\alpha)}\int_0^x (x^2-w^2)^{\frac{1}{2}\alpha}w^{n-1}f(w)dw\right)$$

$$= x^{-n-\alpha-1}\frac{d}{dx}\left(\frac{2^{-n}}{\Gamma(1+\frac{1}{2}\alpha)}\int_0^x (x^2-w^2)^{\frac{1}{2}\alpha}w^{n+1}f(w)dw\right), \quad 0<x<1$$

$$K(x) = \frac{2x^n}{\Gamma(-\frac{1}{2}\alpha)}\int_x^\infty w^{1-n}(w^2-x^2)^{-\frac{1}{2}\alpha-1}g(w)dw, \quad x>1.$$

Hankel's inversion theorem now gives

$$\psi(\xi) = \frac{2^{-\frac{1}{2}\alpha}\xi^{1-\frac{1}{2}\alpha}}{\Gamma(1+\frac{1}{2}\alpha)} \int_0^1 x^{-n-\frac{1}{2}\alpha} J_{n+\frac{1}{2}\alpha}(\xi x) \frac{d}{dx} \int_0^x (x^2-w^2)^{\frac{1}{2}\alpha} w^{n+1} f(w)dwdx$$
$$+ \frac{2^{1+\frac{1}{2}\alpha}\xi^{1-\frac{1}{2}\alpha}}{\Gamma(-\frac{1}{2}\alpha)} \int_1^\infty x^{n+\frac{1}{2}\alpha+1} J_{n+\frac{1}{2}\alpha}(\xi x) \int_x^\infty g(w)w^{1-n}(w^2-x^2)^{-\frac{1}{2}\alpha-1}dwdx. \quad \text{.........(10)}$$

(b) $0<\alpha<2$.

In this case

$$K(x) = \frac{2^{1-\alpha}x^{-\alpha-n}}{\Gamma(\frac{1}{2}\alpha)} \int_0^x w^{n+1}(x^2-w^2)^{\frac{1}{2}\alpha-1} f(w)dw, \quad 0<x<1$$

$$K(x) = \frac{-x^{n-1}}{\Gamma(1-\frac{1}{2}\alpha)} \frac{d}{dx} \int_x^\infty w^{1-n}(w^2-x^2)^{-\frac{1}{2}\alpha} g(w)dw, \quad x>1.$$

Hankel's inversion theorem then gives

$$\psi(\xi) = \frac{(2\xi)^{1-\frac{1}{2}\alpha}}{\Gamma(\frac{1}{2}\alpha)} \int_0^1 x^{-n-\frac{1}{2}\alpha+1} J_{n+\frac{1}{2}\alpha}(\xi x) \int_0^x f(w)w^{n+1}(x^2-w^2)^{\frac{1}{2}\alpha-1}dwdx$$
$$- \frac{2^{\frac{1}{2}\alpha}\xi^{1-\frac{1}{2}\alpha}}{\Gamma(1-\frac{1}{2}\alpha)} \int_1^\infty x^{n+\frac{1}{2}\alpha} J_{n+\frac{1}{2}\alpha}(\xi x) \frac{d}{dx} \int_x^\infty w^{1-n}(w^2-x^2)^{-\frac{1}{2}\alpha} g(w)dwdx. \quad \text{......(11)}$$

The above analysis is purely formal but it may be verified by direct substitution that equations (10) and (11) represent the solution of the integral equations for $n>0$. Equations (10) and (11) are in exact agreement with Noble's solution.

Solution for $F(x)$ and $G(x)$

In some problems one is more interested in the actual forms of $F(x)$ and $G(x)$ than in the function $\psi(\xi)$. Clearly F and G may be determined from ψ but it is possible to use a more direct approach not involving ψ. This approach was in fact the starting point of the present investigation. We shall consider only the case $-2<\alpha<0$, the case $0<\alpha<2$ may be treated similarly.

From (3) and (4) we obtain

$$\frac{F^*(s)}{G^*(s)} = \frac{2^\alpha\Gamma(\frac{1}{2}s+\frac{1}{2}n)\Gamma(\frac{1}{2}n+\frac{1}{2}\alpha+1-\frac{1}{2}s)}{\Gamma(\frac{1}{2}s+\frac{1}{2}n-\frac{1}{2}\alpha)\Gamma(\frac{1}{2}n+1-\frac{1}{2}s)}. \quad \text{..................(12)}$$

Thus, from equations (8), (9), (12) and the convolution theorem,

$$F(x) = \frac{2^{\alpha+2}x^{-n}}{\{\Gamma(-\frac{1}{2}\alpha)\}^2} \int_0^x (x^2-\xi^2)^{-\frac{1}{2}\alpha-1} J(\xi)d\xi \quad \text{.................(13)}$$

where

$$J(\xi) = \xi^{\alpha+2n+1} \int_\xi^\infty (t^2-\xi^2)^{-\frac{1}{2}\alpha-1} t^{\alpha+1-n} G(t)dt. \quad \text{...............(14)}$$

Then from equation (1)

$$\frac{2^{\alpha+2}}{\{\Gamma(-\frac{1}{2}\alpha)\}^2}\int_0^x (x^2-\xi^2)^{-\frac{1}{2}\alpha-1}J(\xi)d\xi = x^n f(x), \quad 0<x<1,$$

the above equation is a generalisation for Abel's integral equation and a solution has been given by Noble (**1**). We have that

$$J(\xi) = \frac{-2^{\alpha+1}\{\Gamma(-\frac{1}{2}\alpha)\}^2 \sin \frac{1}{2}\alpha\pi}{\pi}\frac{d}{d\xi}\int_0^\xi \frac{x^{n+1}f(x)dx}{(\xi^2-x^2)^{-\frac{1}{2}\alpha}}, \quad 0<\xi<1. \quad \text{......(15)}$$

$G(t)$ is known for $t>1$ and thus $J(\xi)$ is known for all ξ from equations (14) and (15). Hence

$$F(x) = \frac{-2}{\pi}\sin \tfrac{1}{2}\alpha\pi x^{-n}\int_0^1 (x^2-\xi^2)^{-\frac{1}{2}\alpha-1}\frac{d}{d\xi}\int_0^\xi \frac{t^{n+1}f(t)dt}{(\xi^2-t^2)^{-\frac{1}{2}\alpha}}d\xi$$

$$+\frac{2^{\alpha+2}x^{-n}}{\{\Gamma(-\frac{1}{2}\alpha)\}^2}\int_1^x (x^2-\xi^2)^{-\frac{1}{2}\alpha-1}\xi^{\alpha+2n+1}\int_\xi^\infty t^{1-n}(t^2-\xi^2)^{-\frac{1}{2}\alpha-1}g(t)dtd\xi,$$

$$x>1. \quad \text{.........(16)}$$

$G(t)$ in $0<t<1$ may be obtained in a similar fashion from equation (14).

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Department of Applied Mathematics
The University
Liverpool

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