

RESEARCH ARTICLE

ℓ -adic images of Galois for elliptic curves over \mathbb{Q} (and an appendix with John Voight)

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Abstract

We discuss the ℓ -adic case of Mazur’s ‘Program B’ over \mathbb{Q} : the problem of classifying the possible images of ℓ -adic Galois representations attached to elliptic curves E over \mathbb{Q} , equivalently, classifying the rational points on the corresponding modular curves. The primes $\ell = 2$ and $\ell \geq 13$ are addressed by prior work, so we focus on the remaining primes $\ell = 3, 5, 7, 11$. For each of these ℓ , we compute the directed graph of arithmetically maximal ℓ -power level modular curves X_H , compute explicit equations for all but three of them and classify the rational points on all of them except $X_{ns}^+(N)$, for $N = 27, 25, 49, 121$ and two-level 49 curves of genus 9 whose Jacobians have analytic rank 9.

Aside from the ℓ -adic images that are known to arise for infinitely many $\overline{\mathbb{Q}}$ -isomorphism classes of elliptic curves E/\mathbb{Q} , we find only 22 exceptional images that arise for any prime ℓ and any E/\mathbb{Q} without complex multiplication; these exceptional images are realised by 20 non-CM rational j -invariants. We conjecture that this list of 22 exceptional images is complete and show that any counterexamples must arise from unexpected rational points on $X_{ns}^+(\ell)$ with $\ell \geq 19$, or one of the six modular curves noted above. This yields a very efficient algorithm to compute the ℓ -adic images of Galois for any elliptic curve over \mathbb{Q} .

In an appendix with John Voight, we generalise Ribet’s observation that simple abelian varieties attached to newforms on $\Gamma_1(N)$ are of GL_2 -type; this extends Kolyvagin’s theorem that analytic rank zero implies algebraic rank zero to isogeny factors of the Jacobian of X_H .

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1. Introduction

To an elliptic curve E over a number field K , one can associate representations

$$\begin{aligned} \rho_{E,N} : G_K &\rightarrow \text{Aut } E[N] \simeq \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \\ \rho_{E,\ell^\infty} : G_K &\rightarrow \text{GL}_2(\mathbb{Z}_\ell) = \varprojlim_n \text{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z}) \\ \rho_E : G_K &\rightarrow \text{GL}_2(\widehat{\mathbb{Z}}) = \varprojlim_N \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \end{aligned}$$

of the absolute Galois group $G_K := \text{Gal}(\overline{K}/K)$. If E does not have complex multiplication (CM), then the images of these representations are open [Ser72], and thus of finite index. It is conjectured ([Zyw15a, Conjecture 1.12], [Sut16, Conjecture 1.1]) that $\rho_{E,\ell}$ is surjective for non-CM E/\mathbb{Q} for all $\ell > 37$ and that there is an absolute bound on the index of $\rho_E(G_{\mathbb{Q}})$.

Concordant with his watershed papers [Maz77b, Maz78] classifying rational points on the modular curves $X_1(N)$ and $X_0(N)$ of prime level, Mazur proposed the following program (labeled ‘Program B’) in his article [Maz77a]:

‘Given a number field K and a subgroup H of $\text{GL}_2(\widehat{\mathbb{Z}}) \simeq \prod_p \text{GL}_2(\mathbb{Z}_p)$, classify all elliptic curves E/K whose associated Galois representation on torsion points maps $\text{Gal}(\overline{K}/K)$ into $H \leq \text{GL}_2(\widehat{\mathbb{Z}})$.’

Mazur’s Program B has seen substantial progress over the past decade: for prime level, see [BP11, BPR13, Zyw15a, Sut16, BDM+19, BDM+21, LFL21]; for prime power level, see [RZB15, SZ17]; for multi-prime level, see [Zyw15b, DGJ, GJLR16, BJ16, Mor19, JM22, DLR, DM, DLRM21, Rak21, BS22]; and for CM curves, see [BC20, LR, CP, Lom17].

1.1. Main theorem

Our main theorem addresses almost all of the remaining groups of prime power level. For each open subgroup H of $GL_2(\widehat{\mathbb{Z}})$, we use X_H to denote the modular curve whose noncuspidal K -rational points classify elliptic curves E/K for which $\rho_E(G_K)$ is contained in H (see Section 2.3 for a precise definition).

Remark 1.1. We write $\rho_{E,\ell^\infty}(G_K) = H$ to indicate that H is conjugate to the inverse image of $\rho_{E,\ell^\infty}(G_K)$ under the projection $GL_2(\widehat{\mathbb{Z}}) \rightarrow GL_2(\mathbb{Z}_\ell)$. In general, we are only interested in the image of ρ_E up to conjugacy (the exact image depends on a choice of basis).

Definition 1.2. Let ℓ be a prime, let H be an open subgroup of $GL_2(\widehat{\mathbb{Z}})$ of ℓ -power level, and let K be a number field. We say that a point $P \in X_H(K)$ is exceptional if $X_H(K)$ is finite and P corresponds to a non-CM elliptic curve E/K . We call an element of K exceptional if it is the j -invariant of an exceptional point. If there is a non-CM elliptic curve E/K such that $j(E)$ is exceptional and $\rho_{E,\ell^\infty}(G_K) = H$ (as opposed to a subgroup of H), then we refer to H as K -exceptional; we may omit K when $K = \mathbb{Q}$.

For $K = \mathbb{Q}$, the known exceptional j -invariants of prime power level are listed in Table 1 along with the known exceptional points on X_H for exceptional subgroups H of prime power level that do not contain $-I$; the groups of prime level appear in both [Sut16] and [Zyw15a].

Remark 1.3. For the H and j listed in Table 1, if $-I \in H$, then all but finitely many elliptic curves with $j(E) = j$ have ℓ -adic image equal to H (rather than a proper subgroup). If $H' \subsetneq H$ is a proper subgroup such that $H = \langle H', -I \rangle$ and E/\mathbb{Q} is an elliptic curve with ℓ -adic image H and $j(E) \neq 1728$, then there are $[N_{GL_2(\mathbb{Z}_\ell)}(H) : N_{GL_2(\mathbb{Z}_\ell)}(H')]$ quadratic twists E' of E whose ℓ -adic image is H' ; see Section 10.

Remark 1.4. If $P \in X_H(K)$ corresponds to an elliptic curve E , then $\rho_E(G_K)$ may be a proper subgroup of H . For example, any elliptic curve E/\mathbb{Q} with j -invariant -11^2 admits a rational 11-isogeny, which forces $\rho_{E,11^\infty}(G_\mathbb{Q})$ to lie in the Borel group $B(11)$, but $\rho_{E,11^\infty}(G_\mathbb{Q})$ is generically the index-5 subgroup $\langle \begin{bmatrix} 10 & 1 \\ 0 & 10 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \rangle$ with label 11.60.1.4. In this example, -11^2 is an exceptional j -invariant and 11.60.1.4 is an exceptional group, but $B(11)$ is not an exceptional group, because no non-CM elliptic curve over \mathbb{Q} has 11-adic image $B(11)$. The group $B(11) = 11.12.1.1$ appears in Table 11 along with 13 other groups H of prime power level for which a similar phenomenon occurs.

Conjecture 1.5. Table 1 lists all rational exceptional j -invariants of prime power level.

Theorem 1.6. Let ℓ be a prime, let E/\mathbb{Q} be an elliptic curve without complex multiplication, and let $H = \rho_{E,\ell^\infty}(G_\mathbb{Q})$. Exactly one of the following is true:

- (i) the modular curve X_H is isomorphic to \mathbb{P}^1 or an elliptic curve of rank one, in which case H is identified in [SZ17, Corollary 1.6] and $\langle H, -I \rangle$ is listed in [SZ17, Tables 1–4];
- (ii) the modular curve X_H has an exceptional rational point, and H is listed in Table 1;
- (iii) $H \leq N_{ns}(3^3), N_{ns}(5^2), N_{ns}(7^2), N_{ns}(11^2)$ or $N_{ns}(\ell)$ for some $\ell \geq 19$;
- (iv) H is a subgroup of one of the groups labeled 49.147.9.1 or 49.196.9.1.

See Subsection 2.4 for a discussion of the group labels, and see Tables 2 and 3 for a summary of the groups H that appear in Theorem 1.6 for $\ell \leq 37$.

We conjecture that in cases (iii) and (iv), the curve X_H has no exceptional rational points; see Section 9 for a discussion of evidence for this conjecture. For the six groups in Table 2, the genus of X_H is 9, 12, 14, 69 or 511, and for each isogeny factor A of J_H , the analytic rank of A is equal to

Table 1. Known (and conjecturally all) exceptional groups and j -invariants of prime power level. When $-I \notin H$, we list models of elliptic curves E/\mathbb{Q} for which $\rho_{E,\ell^\infty}(G_{\mathbb{Q}}) = H$. The Magma script [checkimages.m](#) in the software repository associated with this paper [RSZB21] verifies each line of this table. Generators for the arithmetically maximal groups can be found in Appendix B.

| Label | Level | Notes | j -invariants/models of exceptional points |
|-------------|----------------|--|--|
| 16.64.2.1 | 2 ⁴ | $N_{\text{ns}}(16)$ | $-2^{18} \cdot 3 \cdot 5^3 \cdot 13^3 \cdot 41^3 \cdot 107^3 / 17^{16}$ $-2^{21} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13^3 \cdot 23^3 \cdot 41^3 \cdot 179^3 \cdot 409^3 / 79^{16}$ |
| 16.96.3.335 | 2 ⁴ | $H(4) \subsetneq N_{\text{sp}}(4)$ | $257^3 / 2^8$ |
| 16.96.3.338 | 2 ⁴ | $H(4) \subsetneq N_{\text{sp}}(4)$ | 2^{11} |
| 16.96.3.343 | 2 ⁴ | $H(4) \subsetneq N_{\text{sp}}(4)$ | $17^3 \cdot 241^3 / 2^4$ |
| 16.96.3.346 | 2 ⁴ | $H(4) \subsetneq N_{\text{sp}}(4)$ | $2^4 \cdot 17^3$ |
| 32.96.3.82 | 2 ⁵ | $H(4) \subsetneq N_{\text{sp}}(8)$ | $3^3 \cdot 5^6 \cdot 13^3 \cdot 23^3 \cdot 41^3 / (2^{16} \cdot 31^4)$ |
| 32.96.3.230 | 2 ⁵ | $H(4) \subsetneq N_{\text{sp}}(4)$ | $-3^3 \cdot 5^3 \cdot 47^3 \cdot 1217^3 / (2^8 \cdot 31^8)$ |
| 25.50.2.1 | 5 ² | $H(5) = N_{\text{ns}}(5)$ | $2^4 \cdot 3^2 \cdot 5^7 \cdot 23^3$ |
| 25.75.2.1 | 5 ² | $H(5) = N_{\text{sp}}(5)$ | $2^{12} \cdot 3^3 \cdot 5^7 \cdot 29^3 / 7^5$ |
| 7.56.1.2 | 7 | $H(7) \subsetneq N_{\text{sp}}(7)$ | $3^3 \cdot 5 \cdot 7^5 / 2^7$ |
| 7.112.1.2 | 7 | $-I \notin \langle \begin{bmatrix} 0 & 5 \\ 5 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 5 \\ 3 & 0 \end{bmatrix} \rangle$ | $y^2 + xy + y = x^3 - x^2 - 2680x - 50053$ $y^2 + xy + y = x^3 - x^2 - 131305x + 17430697$ |
| 11.60.1.3 | 11 | $\langle \begin{bmatrix} 10 & 1 \\ 0 & 10 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \rangle$ | $-11 \cdot 131^3$ |
| 11.60.1.4 | 11 | $\langle \begin{bmatrix} 10 & 1 \\ 0 & 10 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \rangle$ | -11^2 |
| 11.120.1.3 | 11 | $-I \notin \langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \rangle$ | $y^2 + xy = x^3 + x^2 - 3632x + 82757$ |
| 11.120.1.4 | 11 | $-I \notin \langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \rangle$ | $y^2 + xy + y = x^3 + x^2 - 305x + 7888$ |
| 11.120.1.8 | 11 | $-I \notin \langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \rangle$ | $y^2 + xy + y = x^3 + x^2 - 30x - 76$ |
| 11.120.1.9 | 11 | $-I \notin \langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \rangle$ | $y^2 + xy = x^3 + x^2 - 2x - 7$ |
| 13.91.3.2 | 13 | $S_4(13)$ | $2^4 \cdot 5 \cdot 13^4 \cdot 17^3 / 3^{13}$ $-2^{12} \cdot 5^3 \cdot 11 \cdot 13^4 / 3^{13}$ $2^{18} \cdot 3^3 \cdot 13^4 \cdot 127^3 \cdot 139^3 \cdot 157^3 \cdot 283^3 \cdot 929 / (5^{13} \cdot 61^{13})$ |
| 17.72.1.2 | 17 | $\langle \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \rangle$ | $-17 \cdot 373^3 / 2^{17}$ |
| 17.72.1.4 | 17 | $\langle \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \rangle$ | $-17^2 \cdot 101^3 / 2$ |
| 37.114.4.1 | 37 | $\langle \begin{bmatrix} 8 & 1 \\ 0 & 8 \end{bmatrix}, \begin{bmatrix} 6 & 0 \\ 0 & 9 \end{bmatrix} \rangle$ | $-7 \cdot 11^3$ |
| 37.114.4.2 | 37 | $\langle \begin{bmatrix} 8 & 1 \\ 0 & 8 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 0 & 6 \end{bmatrix} \rangle$ | $-7 \cdot 137^3 \cdot 2083^3$ |

Table 2. Arithmetically maximal groups of ℓ -power level with $\ell \leq 17$ for which $X_H(\mathbb{Q})$ is unknown; each has rank = genus, rational CM points, no rational cusps and no known exceptional points.

| Label | Level | Group | Genus |
|----------------|-----------------|---|-------|
| 27.243.12.1 | 3 ³ | $N_{\text{ns}}(3^3)$ | 12 |
| 25.250.14.1 | 5 ² | $N_{\text{ns}}(5^2)$ | 14 |
| 49.1029.69.1 | 7 ² | $N_{\text{ns}}(7^2)$ | 69 |
| 49.147.9.1 | 7 ² | $\langle \begin{bmatrix} 16 & 6 \\ 20 & 45 \end{bmatrix}, \begin{bmatrix} 20 & 17 \\ 40 & 36 \end{bmatrix} \rangle$ | 9 |
| 49.196.9.1 | 7 ² | $\langle \begin{bmatrix} 42 & 3 \\ 16 & 31 \end{bmatrix}, \begin{bmatrix} 16 & 23 \\ 8 & 47 \end{bmatrix} \rangle$ | 9 |
| 121.6655.511.1 | 11 ² | $N_{\text{ns}}(11^2)$ | 511 |

the dimension of A . We were not able to provably determine the rational points on these curves; see Section 9 for a more detailed discussion of these cases.

Remark 1.7. The recent groundbreaking work of [BDM+19, BDM+21] yields a complete classification of the possible 13-adic and 17-adic images for elliptic curves over \mathbb{Q} ; these are the only primes other than $\ell = 2$ for which this is currently known.

1.2. Outline of proof

The proof of Theorem 1.6 breaks down into the following steps:

Table 3. Numerical summary of open $H \leq \text{GL}_2(\widehat{\mathbb{Z}})$ of prime power level that can conjecturally occur as $\rho_{E,\ell^\infty}(G_{\mathbb{Q}})$ for E/\mathbb{Q} without CM. The starred primes depend on the conjecture that $N_{\text{ns}}(\ell)$ for $\ell > 17$ and the groups from Table 2 do not occur. See the file `sample.txt` in [RSZB21] for a list of elliptic curves over \mathbb{Q} that realise each of the subgroups in this table.

| ℓ | 2 | 3* | 5* | 7* | 11* | 13 | 17 | 37* | other* |
|--------------------------|------|----|-----|-----|-----|----|----|-----|--------|
| Subgroups | 1208 | 47 | 25 | 17 | 8 | 12 | 3 | 3 | 1 |
| Exceptional subgroups | 7 | 0 | 2 | 2 | 6 | 1 | 2 | 2 | 0 |
| Nonexceptional subgroups | 1201 | 47 | 23 | 15 | 2 | 11 | 1 | 1 | 1 |
| Max level | 32 | 27 | 25 | 7 | 11 | 13 | 17 | 37 | 1 |
| Max index | 96 | 72 | 120 | 112 | 120 | 91 | 72 | 114 | 1 |
| Max genus | 3 | 0 | 2 | 1 | 1 | 3 | 1 | 4 | 0 |

- We begin by computing the collection \mathcal{S} of arithmetically maximal subgroups of prime power level; see Section 3 and the tables in Appendix B.
- For $H \in \mathcal{S}$, we compute the isogeny decomposition of J_H and the analytic ranks of its simple factors; see Sections 5 and 6.
- For many $H \in \mathcal{S}$, we compute equations for X_H and $j_H : X_H \rightarrow X(1)$; see Section 7.
- For most $H \in \mathcal{S}$ with $-I \in H$, we determine $X_H(\mathbb{Q})$; see Sections 4, 5, 8.
- For $H \in \mathcal{S}$ with $-I \notin H$, we compute equations for the universal curve $E \rightarrow U$, where $U \subset X_H$ is the locus of points with $j \neq 0, 1728, \infty$; see Section 10.

Analyses of many of the relevant X_H already exist in the literature, which is summarised in Section 4; the remaining cases are listed in Table 5. Some are routine; for example, some of the genus 2 curves in Section 8 have rank zero Jacobians and are handled with no additional work by Magma’s Chabauty0 command, but many require novel arguments.

- We use the moduli enumeration of Section 5 to show that $X_H(\mathbb{F}_p)$ and therefore $X_H(\mathbb{Q})$ is empty in several cases without needing equations for X_H ; see Table 7.
- In Section 8, a few curves admit an involution ι such that the map $P \mapsto P - \iota(P)$ projects to the torsion subgroup of the Jacobian, and we ‘sieve’ with this map. In one case, we are only able to compute an upper bound on $J_H(\mathbb{Q})_{\text{tors}}$ but are then able to circumvent this via sieving.
- In Section 7, we compute canonical models of two high genus curves (genus 36 and 43); and in Section 8.6, we use these models to set up a nontrivial local computation to show that $X_H(\mathbb{Q}_p)$ is empty, where p divides the level of H and is thus the only possible prime of bad reduction.
- The curve X_H from Section 8.7 has genus 41, and we were not able to compute a model; X_H maps to $X_{\text{ns}}^+(11)$, which is a rank 1 elliptic curve, and we perform an ‘equationless’ Mordell–Weil sieve (via moduli) to show that $X_H(\mathbb{Q})$ is empty.
- Throughout, our approaches are heavily informed by the isogeny decompositions and analytic ranks computed in Section 6. In the end, for a rigorous proof, we often do not need to know that the analytic and algebraic ranks agree, but some cases in Section 8 rely on this and thus make crucial use of Appendix A.
- Appendix A contains an additional novelty: we make use of the Jacobian of a connected but geometrically disconnected curve.

1.3. Applications

In [Gre12], Greenberg shows that if ℓ is a prime number and E/\mathbb{Q} is an elliptic with a cyclic ℓ -isogeny, then $\rho_{E,\ell^\infty}(G_{\mathbb{Q}})$ is as large as possible given the existence of the isogeny if $\ell > 7$, or the existence of ℓ -power isogenies if $\ell = 5$. The $\ell = 7$ case is handled in [GRSS14]. As a consequence of our work, we obtain a classification of the 3-adic image for a non-CM elliptic curve E/\mathbb{Q} with a rational 3-isogeny.

Corollary 1.1. *Suppose that E/\mathbb{Q} is a non-CM elliptic curve with a rational 3-isogeny, and let $H = \langle \rho_{E,3^\infty}(G_{\mathbb{Q}}), -I \rangle$. Then exactly one of the following is true:*

- (i) $[\mathrm{GL}_2(\mathbb{Z}_3) : H] = 4$, and H has label 3.4.0.1;
- (ii) $[\mathrm{GL}_2(\mathbb{Z}_3) : H] = 12$, and H has label 3.12.0.1, 9.12.0.1, or 9.12.0.2;
- (iii) $[\mathrm{GL}_2(\mathbb{Z}_3) : H] = 36$, and H has label 9.36.0.1, 9.36.0.2, 9.36.0.3, 9.36.0.4, 9.36.0.5, 9.36.0.6, 9.36.0.7, 9.36.0.8, 9.36.0.9, or 27.36.0.1.

In each case, the relevant modular curve has genus zero and infinitely many rational points. Moreover, $\rho_{E,3^\infty}(G_{\mathbb{Q}})$ (which may have index two in H) contains all matrices congruent to the identity matrix modulo 27 and all matrices $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ with $\lambda \in \mathbb{Z}_3$ and $\lambda \equiv 1 \pmod{9}$.

This result plays a role in the classification of odd-degree isolated points on $X_1(N)$ given in [BGRW20]. In [LT, LT21], Lombardo and Tronto study the degree of the extension $\mathbb{Q}(E[N], N^{-1}\alpha)/\mathbb{Q}(E[N])$, where $\alpha \in E(\mathbb{Q})$ is a point of infinite order and the presence of scalars in $\rho_{E,\ell^\infty}(G_{\mathbb{Q}})$ is used to bound certain Galois cohomology groups. Corollary 1.1 gives precise information about the scalars present in the 3-adic image.

An additional application of Theorem 1.6 is a very fast ‘algebraic’ algorithm to compute all of the ℓ -adic Galois images of a given non-CM elliptic curve E/\mathbb{Q} ; this algorithm is described in Section 11. By exploiting the classification of ℓ -adic Galois images of CM elliptic curves obtained by Lozano-Robledo [LR], we can extend this algorithm to also handle CM elliptic curves over \mathbb{Q} (and, more generally, any CM elliptic curve over its minimal field of definition), as described in Section 12. An implementation of this algorithm is available in the GitHub repository associated to this paper [RSZB21], and we have applied it to the elliptic curves E/\mathbb{Q} in the L -functions and Modular Forms Database (LMFDB) [LMF] as well as two other large databases of elliptic curves over \mathbb{Q} [SW02, BHK+16]. This data has now been added to the LMFDB, where it extends existing 2-adic and mod- ℓ Galois image data that was previously available. As a result, the ℓ -adic images $\rho_{E,\ell^\infty}(G_{\mathbb{Q}})$ are now known for all elliptic curves E/\mathbb{Q} of conductor up to 500000 and every prime ℓ .

Other applications abound; examples include [JR10, CR21, LT, Guž21, BGRW20, GJN20, BEL+19, GJLR18, GJLR17, DLRNS18, Rei21, CLR21, BS22].

1.4. Code

We make extensive use of the computer algebra system Magma [BCP97]. Code verifying the computational claims made in this paper is available at the GitHub repository <https://github.com/AndrewVSutherland/ell-adic-galois-images>, which also includes related algorithms and data.

2. The Modular curves X_H

2.1. Notation

In this article, ℓ and p denote primes, and N is a positive integer. We define

$$\mathbb{Z}(N) := \mathbb{Z}/N\mathbb{Z}, \quad \mathrm{GL}_2(N) := \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}), \quad \mathrm{SL}_2(N) := \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}),$$

with the understanding that $\mathbb{Z}(1)$ is the zero ring and $\mathrm{GL}_2(1)$ and $\mathrm{SL}_2(1)$ are trivial groups. We then have $\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}(N)$ and $\mathrm{GL}_2(\widehat{\mathbb{Z}}) = \varprojlim \mathrm{GL}_2(N)$ and natural projection maps

$$\pi_N : \mathrm{GL}_2(\widehat{\mathbb{Z}}) \rightarrow \mathrm{GL}_2(N).$$

We view $\mathbb{Z}(N)^2$ as a $\mathbb{Z}(N)$ -module of column vectors equipped with a left action of $\mathrm{GL}_2(N)$ via matrix-vector multiplication.

For $N > 1$, we use $B(N)$ to denote the Borel subgroup of $\mathrm{GL}_2(N)$ consisting of upper triangular matrices, $C_{\mathrm{sp}}(N)$ to denote the split Cartan subgroup of diagonal matrices in $\mathrm{GL}_2(N)$ and $C_{\mathrm{ns}}(N)$ to denote the nonsplit Cartan subgroup of $\mathrm{GL}_2(N)$, which can be constructed by picking an imaginary quadratic order \mathcal{O} in which every prime $\ell|N$ is inert and embedding $(\mathcal{O}/N\mathcal{O})^\times$ in $\mathrm{GL}_2(N)$ via its action

on a $\mathbb{Z}(N)$ -basis for $\mathcal{O}/N\mathcal{O}$ (see Section 12), which is unique up to conjugacy. We use $N_{\text{sp}}(N)$ to denote the subgroup of the normaliser of $C_{\text{sp}}(N)$ in $\text{GL}_2(N)$ whose projection to $\text{GL}_2(M)$ contains $C_{\text{sp}}(M)$ with index 2 for all $M > 2$ dividing N , which is unique up to conjugacy and equal to the normaliser when $N > 2$ is a prime power; we similarly define $N_{\text{ns}}(N)$. If G is a subgroup of the projectivisation $\text{PGL}_2(N)$ of $\text{GL}_2(N)$, then we denote by $G(N)$ the preimage of G in $\text{GL}_2(N)$.

2.2. Group theory and Galois representations

Let $N \geq 1$ be an integer, let k be a perfect field of characteristic coprime to N , and let $G_k := \text{Gal}(\bar{k}/k)$. A basis (P_1, P_2) of $E[N](\bar{k})$ determines an isomorphism $E[N](\bar{k}) \xrightarrow{\sim} \mathbb{Z}(N)^2$ via the map $P_i \mapsto e_i$. This gives rise to an isomorphism $\iota: \text{Aut}(E[N](\bar{k})) \simeq \text{GL}_2(N)$ as follows: for $\phi \in \text{Aut}(E[N](\bar{k}))$ such that

$$\begin{aligned} \phi(P_1) &= aP_1 + cP_2 \\ \phi(P_2) &= bP_1 + dP_2, \end{aligned}$$

we define

$$\iota(\phi) := \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

which yields a Galois representation $G_k \rightarrow \text{GL}_2(N)$ given by $\sigma \mapsto \iota(\sigma_N)$, where σ_N is the automorphism of $E[N](\bar{k})$ induced by σ .

Remark 2.1. Our choice of ι corresponds to the left action of $\text{GL}_2(N)$ on $\mathbb{Z}(N)^2$. Many sources are ambiguous about the choice of left vs. right action; this ambiguity often makes no difference (see [RZB15, Remark 2.2]), but it does when $H \leq \text{GL}_2(N)$ is not conjugate to its transpose. Our choice here is consistent with [Sut16, SZ17], but differs from [RZB15], which uses right actions.

For each elliptic curve E/k , we fix a system of compatible bases for $E[N](\bar{k})$ for all $N \geq 1$ coprime to the characteristic of k , meaning if $N = N_1N_2$, then the bases for $E[N_1](\bar{k})$ and $E[N_2](\bar{k})$ are the images of the basis for $E[N](\bar{k})$ under multiplication by N_2 and N_1 , respectively. Let $\iota_E: \text{Aut}(E[N](\bar{k})) \xrightarrow{\sim} \text{GL}_2(N)$ denote the isomorphism determined by the basis for $E[N](\bar{k})$, and define the Galois representation $\rho_{E,N}: G_k \rightarrow \text{GL}_2(N)$ via $\rho_{E,N}(\sigma) := \iota_E(\sigma_N)$. When k has characteristic zero, we shall also use ι_E to denote the isomorphism $\iota_E: \text{Aut}(E_{\text{tor}}(\bar{k})) \xrightarrow{\sim} \text{GL}_2(\widehat{\mathbb{Z}})$ determined by this system of bases and use it to define the Galois representation $\rho_E: G_k \rightarrow \text{GL}_2(\widehat{\mathbb{Z}})$ via $\rho_E(\sigma) := \iota_E(\sigma)$. If k is a number field and \mathfrak{p} is a prime of good reduction for E , for each $N \geq 1$ coprime to $N(\mathfrak{p})$, we shall use the reduction of our chosen basis for $E[N](\bar{k})$ as a basis for $E_{\mathfrak{p}}[N](\overline{\mathbb{F}}_{\mathfrak{p}})$.

For a subgroup $H \leq \text{GL}_2(\widehat{\mathbb{Z}})$, we use $H(N)$ to denote the image $\pi_N(H)$ of H under π_N . An open subgroup $H \leq \text{GL}_2(\widehat{\mathbb{Z}})$ contains the kernel of $\pi_N: \text{GL}_2(\widehat{\mathbb{Z}}) \rightarrow \text{GL}_2(N)$ for some $N \geq 1$; the least such N is the level of H , in which case $H = \pi_N^{-1}(H(N))$ is completely determined by $\pi_N(H)$. We will typically work directly with subgroups of $\text{GL}_2(N)$ instead of $\text{GL}_2(\widehat{\mathbb{Z}})$, with the understanding that every subgroup of $\text{GL}_2(N)$ uniquely determines an open subgroup of $\text{GL}_2(\widehat{\mathbb{Z}})$ as its inverse image under π_N ; we may use the symbol H to denote both H and $\pi_N(H)$, which we note have the same index $[\text{GL}_2(\widehat{\mathbb{Z}}) : H] = [\text{GL}_2(N) : \pi_N(H)]$. We define the level of a subgroup $H \leq \text{GL}_2(N)$ to be the level of $\pi_N^{-1}(H)$.

We may also identify open subgroups of $\text{GL}_2(\mathbb{Z}_\ell)$ with their preimages in $\text{GL}_2(\widehat{\mathbb{Z}})$ under the projection map $\text{GL}_2(\widehat{\mathbb{Z}}) \rightarrow \text{GL}_2(\mathbb{Z}_\ell)$. For any $e \geq 0$, we may identify $H \leq \text{GL}_2(\ell^e)$ with the corresponding open subgroups of $\text{GL}_2(\mathbb{Z}_\ell)$ or $\text{GL}_2(\widehat{\mathbb{Z}})$ of the same index. In what follows, we shall only be interested in subgroups of $\text{GL}_2(\widehat{\mathbb{Z}})$ up to conjugacy and view each subgroup H as a representative of its conjugacy class. All inclusions $H_1 \leq H_2$ of subgroups of $\text{GL}_2(\widehat{\mathbb{Z}})$ should be understood to indicate that H_1 is conjugate to a subgroup of H_2 .

2.3. Modular curves

Let H be a subgroup of $\mathrm{GL}_2(N)$ of level N . We define the modular curve Y_H (respectively, X_H) to be the coarse moduli space of the stack \mathcal{M}_H^0 (respectively, \mathcal{M}_H), over $\mathrm{Spec} \mathbb{Z}[1/N]$, which parametrises elliptic curves (respectively, generalised elliptic curves) with H -level structure. Equivalently, by [DR73, IV-3.1] (see also [RZB15, Lemma 2.1]), X_H is isomorphic to the coarse space of the stack quotient $X(N)/H$, where $X(N)$ is the classical modular curve parametrising full level structures.

More precisely, an H -level structure on an elliptic curve E/\bar{k} is an equivalence class $[\iota]_H$ of isomorphisms $\iota: E[N](\bar{k}) \rightarrow \mathbb{Z}(N)^2$, where $\iota \sim \iota'$ if $\iota = h \circ \iota'$ for some $h \in H$. The set $Y_H(\bar{k})$ consists of equivalence classes of pairs $(E, [\iota]_H)$, where $(E, [\iota]_H) \sim (E', [\iota']_H)$ if there is an isomorphism $\phi: E \rightarrow E'$ for which the induced isomorphism $\phi_N: E[N] \rightarrow E'[N]$ satisfies $\iota \sim \iota' \circ \phi_N$. Equivalently, $Y_H(\bar{k})$ consists of pairs $(j(E), \alpha)$, where $\alpha = HgA_E$ is a double coset in $H \backslash \mathrm{GL}_2(N)/A_E$, where $A_E := \{\varphi_N : \varphi \in \mathrm{Aut}(E_{\bar{k}})\}$ with $\varphi_N := \iota_E(\varphi|_{E[N]})$.

Each $\sigma \in G_k$ induces an isomorphism $E^\sigma[N] \rightarrow E[N]$ defined by $P \mapsto \sigma^{-1}(P)$, which we denote σ^{-1} . We define a right G_k -action on $Y_H(\bar{k})$ via $(E, [\iota]_H) \mapsto (E^\sigma, [\iota \circ \sigma^{-1}]_H)$. The set of k -rational points $Y_H(k)$ consists of the elements in $Y_H(\bar{k})$ that are fixed by this G_k -action. Each $P \in Y_H(k)$ is represented by a pair $(E, [\iota]_H) \in Y_H(k)$ such that E is defined over k and for all $\sigma \in G_k$ there exist $\varphi \in \mathrm{Aut}(E_{\bar{k}})$ and $h \in H$ such that

$$\iota \circ \sigma^{-1} = h \circ \iota \circ \varphi_N, \tag{2.1}$$

as shown in [Zyw15b, Lemma 3.1].¹ Equivalently, a pair $(j(E), \alpha)$ with $\alpha = HgA_E$ lies in the set $Y_H(k)$ if and only if we have $j(E) \in k$ and $Hg\rho_{E,N}(\sigma)A_E = HgA_E$ for all $\sigma \in G_k$.

Remark 2.1. For an elliptic curve E/k , if $\rho_{E,N}(G_k) \leq H$, then there exists an isomorphism $\iota: E[N] \xrightarrow{\sim} \mathbb{Z}(N)^2$ for which $(E, [\iota]_H) \in Y_H(k)$. Conversely, assuming $\mathrm{char}(k) \neq 2, 3$, if $(E, [\iota]_H) \in Y_H(k)$, then for every twist E' of E , there is an isomorphism $\iota': E'[N] \xrightarrow{\sim} \mathbb{Z}(N)^2$ with $(E', [\iota']_H) \in Y_H(k)$, and for at least one E' , we have $\rho_{E',N}(G_k) \leq H$, as we now show.

If E' is a twist of E , then there is an isomorphism $\phi: E'_k \xrightarrow{\sim} E_{\bar{k}}$ that induces an isomorphism $\phi_N: E'[N](\bar{k}) \xrightarrow{\sim} E[N](\bar{k})$, and for $\iota' := \iota \circ \phi_N$ the pairs $(E, [\iota]_H)$ and $(E', [\iota']_H)$ represent the same point in $Y_H(\bar{k})$; thus if $(E, [\iota]_H)$ lies in $Y_H(k)$, then so does $(E', [\iota']_H)$.

If the point $(E, [\iota]_H) = (j(E), \alpha)$ lies in $Y_H(k)$, say for $\alpha = HgA_E$, then $\rho_{E,N}(G_k)$ must lie in $g^{-1}HgA_E$. In other words, up to conjugacy in $\mathrm{GL}_2(N)$, the subgroups $\rho_{E,N}(G_k)$ and H can differ only by elements of A_E , which for $\mathrm{char}(k) \neq 2, 3$ is cyclic of order 2, 4, 6, the last two occurring only for $j(E) = 0, 1728$. When $\#A_E = 2$, every twist of E is a quadratic twist, and [Sut16, Lemma 5.24] implies that there is a quadratic twist E' of E for which $\rho_{E',N}(G_k) \leq H$. Otherwise E has potential CM, and we may apply Proposition 12.1, which also addresses quartic and sextic twists. These results assume k is a number field, but the arguments apply whenever A_E is cyclic (including $\mathrm{char}(k) = 2, 3$ provided $j(E) \neq 0, 1728$).

Remark 2.2. The elliptic curves

$$E: y^2 = x^3 - 1 \quad \text{and} \quad E': y^2 = x^3 + 2$$

with $j(E) = j(E') = 0$ are cubic twists that correspond to the same point in $X_0(2)(\mathbb{Q})$, but E has a rational point of order 2 and $\rho_{E,2}(G_{\mathbb{Q}}) = B(2)$, while E' has no rational points of order 2 and $\rho_{E',2}(G_{\mathbb{Q}}) = \mathrm{GL}_2(2)$. In particular, [RZB15, Lemma 2.1] is incorrectly stated: it holds only up to twist.

The complement $X_H^\infty := X_H - Y_H$ is the set of cusps of X_H and corresponds to generalised elliptic curves with H -level structure. The set of k -rational cusps $X_H^\infty(k)$ can be alternatively described as follows. Let $U(N) := \langle \begin{bmatrix} 1 & \\ 0 & 1 \end{bmatrix}, -I \rangle \subseteq \mathrm{GL}_2(N)$. We define a right G_k -action on $H \backslash \mathrm{GL}_2(N)/U(N)$ via

¹The running assumption $-I \in H$ in [Zyw15b] is not used in the proof of [Zyw15b, Lemma 3.1]

$hgu \mapsto hg\chi_N(\sigma)u$, where $\chi_N(\sigma) := \begin{bmatrix} \sigma & 0 \\ 0 & 1 \end{bmatrix}$ is defined by $\sigma(\zeta_N) = \zeta_N^\sigma$. By [DR73, IV-5.3], $X_H^\infty(k)$ is in bijection with the subset of $H \backslash \text{GL}_2(N)/U(N)$ fixed by $\chi_N(G_k)$.

We define the genus $g(H)$ of H and $\pi_N^{-1}(H)$ to be the genus of each of the geometric connected components of the modular curve X_H , which can be directly computed from H via the usual formula

$$g(H) = g(\Gamma_H) = 1 + \frac{i(\Gamma_H)}{12} - \frac{\nu_2(\Gamma_H)}{4} - \frac{\nu_3(\Gamma_H)}{3} - \frac{\nu_\infty(\Gamma_H)}{2},$$

if we take $\Gamma_H := \pm H \cap \text{SL}_2(N)$, let $i(\Gamma_H) = [\text{SL}_2(N) : \Gamma_H]$, let ν_2 (respectively, ν_3) count the right cosets of Γ_H in $\text{SL}_2(N)$ that contain a conjugate of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (respectively, $\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$) and let $\nu_\infty(\Gamma_H)$ count the orbits of $\Gamma_H \backslash \text{SL}_2(N)$ under the right action of $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. The function `GL2Genus` in the file `g12.m` in [RSZB21] implements this computation.

We define the index $i(H)$ of H to be the integer $[\text{GL}_2(\widehat{\mathbb{Z}}) : H] = [\text{GL}_2(N) : H(N)]$.

Remark 2.3. For $H \leq \text{GL}_2(\widehat{\mathbb{Z}})$ of level N , the inverse image Γ of $\Gamma_H \leq \text{SL}_2(N)$ in $\text{SL}_2(\widehat{\mathbb{Z}})$ is a congruence subgroup of some level $M|N$, but M may be strictly smaller than N ; see [SZ17] for examples. In any case, the base change of X_H to $\mathbb{Q}(\zeta_N)$ breaks into connected components that are geometrically connected and are each isomorphic over \mathbb{C} to the modular curve X_Γ obtained by taking the quotient of the extended upper half plane by Γ .

Remark 2.4. For $H \leq \text{GL}_2(N)$, if $\det H = \mathbb{Z}(N)^\times$, then X_H is geometrically connected. We mostly consider the case where $\det H = \mathbb{Z}(N)^\times$ in this paper, although in Appendix A, we make use of the case $H = \{I\}$, where $X_H = X(N)$. When $\det H < \mathbb{Z}(N)^\times$ is a proper subgroup, X_H is still a curve defined over \mathbb{Q} , but it is not geometrically connected; its connected components are defined over $\mathbb{Q}(\zeta_N)^H$, indexed (via the Weil pairing on the level structures) by cosets of $\det H$ and isomorphic (over $\mathbb{Q}(\zeta_N)^H$) to X_Γ . See Appendix A where this is worked out in more detail for $X(N)$, and Example 6.3, where we discuss the isogeny decomposition of the ‘Jacobian’ of such an X_H .

2.4. Subgroup labels

To help organise our work, we introduce the following labeling convention for uniquely identifying (conjugacy classes of) open subgroups H of $\text{GL}_2(\widehat{\mathbb{Z}})$, which will also serve as labels of the corresponding modular curves X_H . All the subgroups we shall consider satisfy $\det(H) = \widehat{\mathbb{Z}}^\times$ and have labels of the form

N.i.g.n

where **N**, **i**, **g**, **n** are the decimal representations of integers N, i, g, n defined as follows:

- $N = N(H)$ is the level of H ;
- $i = i(H)$ is the index of H ;
- $g = g(H)$ is the genus of H ;
- $n \geq 1$ is an ordinal indicating the position of H among all subgroups of the same level, index and genus with $\det(H) = \widehat{\mathbb{Z}}^\times$ under the ordering defined below.

To define the ordinal n , we must totally order the set $S(N, i, g)$ of open subgroups of $\text{GL}_2(\widehat{\mathbb{Z}})$ of level N , index i and genus g with $\det(H) = \widehat{\mathbb{Z}}^\times$, and we want this ordering to be efficiently computable. With this in mind, we define the following invariants for each $H \in S(N, i, g)$:

- the parent list of H is the lexicographically ordered list of labels of the subgroups $G \leq \text{GL}_2(\widehat{\mathbb{Z}})$ of which H is a maximal subgroup;
- the orbit signature of H as the ordered list of triples (e, s, m) , where m counts orbits of $\mathbb{Z}(N)^2$ under the left-action of H of size s and exponent e (the least integer that kills every element of the orbit);²

²Magma efficiently computes the orbits of $\mathbb{Z}(N)^2$ under the right action of H ; one applies this function to the transpose of H to efficiently compute orbits under the left action of H .

- the class signature of H is the ordered set of tuples (o, d, t, s, m) , where m counts the conjugacy classes of elements of $H \leq \text{GL}_2(N)$ of size s whose elements have order o , determinant d and trace t ;
- the minimal conjugate of H is the least subgroup of $\text{GL}_2(N)$ conjugate to H , where subgroups of $\text{GL}_2(N)$ are ordered lexicographically as ordered lists of tuples of integers (a, b, c, d) with $a, b, c, d \in [0, N - 1]$ corresponding to the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in H$.

We now totally order the set $S(N, i, g, d)$ lexicographically by parent list, orbit signature, class signature and minimal conjugate; these invariants are intentionally ordered according to the difficulty of computing them, and in most cases the first two or three suffice to distinguish every element of $S(N, i, g, d)$, meaning we rarely need to compute minimal conjugates, which is by far the most expensive invariant to compute.

Example 2.1. Let $H = N_{\text{ns}}(11^2)$ be the normaliser of the nonsplit Cartan subgroup of $\mathbb{Z}(11^2)$, which has level $11^2 = 121$ and index $5 \cdot 11^3 = 6655$. There are just two subgroups $H \leq \text{GL}_2(11^2)$ of index 6655 (up to conjugacy), only one of which has $\det(H) = \widehat{\mathbb{Z}}^\times$, so computing $g(H) = 511$ suffices to determine the label $121.6655.511.1$ of $N_{\text{ns}}(11^2)$.

If we instead take $H = N_{\text{ns}}(11)$ with level 11 and index 55, we find there are two subgroups of $\text{GL}_2(11)$ of index 55; they have genus 1 and determinant index 1 and thus belong to the set $S(11, 55, 1, 1)$. These subgroups $H_1 := \langle \begin{bmatrix} 9 & 1 \\ 2 & 9 \end{bmatrix}, \begin{bmatrix} 7 & 0 \\ 0 & 4 \end{bmatrix} \rangle$ and $H_2 := \langle \begin{bmatrix} 5 & 10 \\ 9 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 7 \\ 10 & 10 \end{bmatrix} \rangle$ have parent list $(1.1.\mathbf{0}.1)$ and orbit signature $(1, 1, 1), (11, 120, 1)$. Their class signatures begin $(1, 1, 2, 1, 1), (2, 1, 9, 1, 1), (2, 10, 0, 12, 1)$ but differ in the fourth tuple, which is $(3, 1, 10, 2, 1)$ for H_1 but $(3, 1, 10, 8, 1)$ for H_2 , meaning H_1 contains a conjugacy class of size 2 whose elements have order 3, determinant 1, and trace 10, but H_2 does not. It follows that the labels of H_1 and H_2 are $11.55.1.1$ and $11.55.1.2$, and that $N_{\text{ns}}(11) = H_1$.

Remark 2.2. This labeling system can be extended to arbitrary open $H \leq \text{GL}_2(\widehat{\mathbb{Z}})$ using labels of the form $\mathbb{M}.d.m-N.i.g.n$, where $\mathbb{M}.d.m$ uniquely identifies $\det(H) \leq \widehat{\mathbb{Z}}^\times$ according to its level $M|N$, index $d := [\widehat{\mathbb{Z}}^\times : \det(H)]$ and an ordinal m that distinguishes $\det(H)$ among the open subgroups $D \leq \widehat{\mathbb{Z}}^\times$ of level M and index d .³ To define m , we order the index d subgroups $D \leq (\mathbb{Z}/M\mathbb{Z})^\times$ lexicographically according to the lists of labels $\mathbb{M}.a$ of Conrey characters $\chi_M(a, \cdot)$ of modulus M whose kernels contain D ; see [BBB+21, Section 3.2] for the definition of $\chi_M(a, \cdot)$. We then redefine N and i to be the relative level and index of H as a subgroup of the preimage G of $\det(H)$ in $\text{GL}_2(\widehat{\mathbb{Z}})$: this means N is the least positive integer for which $H = \pi_N^{-1}(\pi_N(H))$, where $\pi_N : G \rightarrow G(N)$ is the reduction map and $i = [G : H]$. This convention ensures that if E/K has Galois image $\rho_E(G_K) = H$, the values of N, i, g in the label of $\rho_E(G_L)$ will be the same for every finite extension L/K .

3. Arithmetically maximal subgroups

Definition 3.1. We define an open subgroup $H \leq \text{GL}_2(\widehat{\mathbb{Z}})$ to be arithmetically maximal if

- (i) $\det(H) = \widehat{\mathbb{Z}}^\times$,
- (ii) H contains an element conjugate to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$,
- (iii) $j(X_H(\mathbb{Q}))$ is finite and $j(X_K(\mathbb{Q}))$ is infinite for all $H < K \leq \text{GL}_2(\widehat{\mathbb{Z}})$.

Properties of the Weil pairing imply that if $H = \rho_E(G_{\mathbb{Q}})$, then $\det(H) = \mathbb{Z}^\times$, and more generally, that if E is an elliptic curve over a number field K and $H = \rho_E(G_K)$, then $[\mathbb{Z}^\times : \det(H)] = [K : K \cap \mathbb{Q}^{\text{cyc}}]$. In particular, condition (i) holds for every H that arises as $\rho_E(G_{\mathbb{Q}})$ for an elliptic curve E/\mathbb{Q} . Condition (ii) is a necessary and sufficient condition for X_H to have noncuspidal real points, by Proposition 3.5 of [Zyw15b], and it follows from Proposition 3.1 of [SZ17] that every genus zero H of prime power level $N = \ell^e$ that satisfies equations (i) and (ii) has infinitely many \mathbb{Q} -points (the assumption $-I \in H$ is not used in the proof of Proposition 3.1 in [SZ17]). For odd ℓ , the matrices in condition (ii) are conjugate in $\text{GL}_2(N)$, and only the first is needed. This definition differs slightly from that used in [RZB15, Definition 3.1] for $\ell = 2$.

³These can be viewed as labels of the finite abelian extensions K of \mathbb{Q} by identifying D with the subgroup of $(\mathbb{Z}/M\mathbb{Z})^\times \simeq \{\zeta_M \mapsto \zeta_M^a : a \in (\mathbb{Z}/M\mathbb{Z})^\times\} \simeq \text{Gal}(\mathbb{Q}(\zeta_M)/\mathbb{Q})$ whose fixed field is K , where $M = \text{cond}(K)$.

Table 4. Summary of arithmetically maximal $H \leq \text{GL}_2(\widehat{\mathbb{Z}})$ of ℓ -power level for $\ell \leq 37$. The level and index bounds are those implied by Lemma 3.2 and the paragraph following; the max level, index and genus rows are the maximum values realised by arithmetically maximal groups.

| ℓ | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 |
|---|-------|-----|-----|------|------|-----|-----|-----|-----|------|-----|------|
| Level bound | 64 | 81 | 125 | 49 | 121 | 169 | 17 | 19 | 23 | 29 | 31 | 37 |
| Index bound | 192 | 729 | 625 | 1372 | 6655 | 728 | 153 | 285 | 276 | 1015 | 496 | 2109 |
| Subgroups | 11091 | 469 | 111 | 144 | 141 | 54 | 18 | 25 | 17 | 64 | 45 | 100 |
| $\#\mathcal{S}_\ell^\infty(\mathbb{Q})$ | 1208 | 47 | 23 | 15 | 2 | 11 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\#\mathcal{S}_\ell(\mathbb{Q})$ | 130 | 19 | 14 | 10 | 6 | 10 | 3 | 4 | 3 | 4 | 3 | 4 |
| Max level | 32 | 27 | 125 | 49 | 121 | 169 | 17 | 19 | 23 | 29 | 31 | 37 |
| Max index | 96 | 729 | 625 | 1372 | 6655 | 182 | 153 | 285 | 276 | 1015 | 496 | 2109 |
| Max genus | 7 | 43 | 36 | 94 | 511 | 3 | 7 | 14 | 15 | 63 | 30 | 142 |

Let $\mathcal{S}_\ell(\mathbb{Q})$ be the set of arithmetically maximal subgroups of ℓ -power level, and let $\mathcal{S}_\ell^\infty(\mathbb{Q})$ be the set of open subgroups of $\text{GL}_2(\widehat{\mathbb{Z}})$ of ℓ -power level for which $j(X_H(\mathbb{Q}))$ is infinite.

The sets $\mathcal{S}_\ell^\infty(\mathbb{Q})$ are explicitly determined in [SZ17] (and in [RZB15] for $\ell = 2$) and have cardinalities 1208, 47, 23, 15, 2, 11 for $\ell = 2, 3, 5, 7, 11, 13$ and cardinality 1 for all $\ell > 13$.⁴ To enumerate the set $\mathcal{S}_\ell(\mathbb{Q})$, it would suffice to enumerate the maximal subgroups of all the groups in $\mathcal{S}_\ell^\infty(\mathbb{Q})$, but we take a slightly different approach and use our knowledge of $\mathcal{S}_\ell^\infty(\mathbb{Q})$ to derive an upper bound on the level of the groups in $\mathcal{S}_\ell(\mathbb{Q})$.

Lemma 3.2. *Let $e_\ell := 5, 3, 2, 1, 1, 1$ for $\ell = 2, 3, 5, 7, 11, 13$, respectively, and let $e_\ell := 0$ for all primes $\ell > 13$. Then ℓ^{e_ℓ} is an upper bound on the level of every $H \in \mathcal{S}_\ell^\infty(\mathbb{Q})$ and $\ell^{e_\ell+1}$ is an upper bound on the level of every $H \in \mathcal{S}_\ell(\mathbb{Q})$.*

Proof. The bound for $H \in \mathcal{S}_\ell^\infty(\mathbb{Q})$ follows from [SZ17, Corollary 1.6] and its associated data. The bound for $H \in \mathcal{S}_\ell(\mathbb{Q})$ follows from [SZ17, Lemma 3.7] for $\ell \leq 13$ (note $e_2 \geq 2$); and for $\ell > 13$, we can apply [Ser98, Lemma 3, pages IV-23], since $e_\ell = 0$. □

Lemma 3.2 reduces the problem of enumerating the sets $\mathcal{S}_\ell(\mathbb{Q})$ to a finite computation in $\text{GL}_2(\ell^{e_\ell+1})$. This computation can be further constrained by computing the maximal index i_ℓ of any maximal subgroup of $H_0 \in \mathcal{S}_\ell^\infty(\mathbb{Q})$ and then enumerating all (conjugacy classes of) subgroups of $\text{GL}_2(\ell^{e_\ell+1})$ of index at most i_ℓ , which can be efficiently accomplished using algorithms for subgroup enumeration that are implemented in Magma and other computer algebra systems. The advantages of this approach are that it enumerates subgroups up to conjugacy in $\text{GL}_2(\widehat{\mathbb{Z}})$, rather than up to conjugacy in H_0 , and it includes all subgroups that have the same level and index of any $H \in \mathcal{S}_\ell(\mathbb{Q})$, as well as all of the groups that contain them, which may be needed to compute the group labels defined in the previous section.

Table 4 summarises this computation. The ‘subgroups’ row counts the number of subgroups $H \leq \text{GL}_2(\widehat{\mathbb{Z}})$ of ℓ -power level with $\det(H) = \widehat{\mathbb{Z}}^\times$ that satisfy the level and index bounds in the first two rows, and the last three rows list the maximum level, index and genus that arise among the groups $H \in \mathcal{S}_\ell(\mathbb{Q})$.

4. Bookkeeping

By [Maz78], [Ser81, Section 8.4] and [BP11], if $\ell \geq 17$ and $\rho_{E,\ell}$ is not surjective, then either $\ell = 17$ or 37 and $j(E)$ is listed in Table 1, or the image $\rho_{E,\ell}(G_\mathbb{Q})$ is a subgroup of $N_{\text{ns}}(\ell)$. In the latter case, for $\ell \equiv 1 \pmod{3}$, the image must be equal to $N_{\text{ns}}(\ell)$ (not a proper subgroup); and for $\ell \equiv 2 \pmod{3}$, the image is either $N_{\text{ns}}(\ell)$ or the index-3 subgroup

$$G := \{a^3 : a \in C_{\text{ns}}(\ell)\} \cup \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot a^3 : a \in C_{\text{ns}}(\ell) \right\}$$

⁴Seven of the $H_0 \in \mathcal{S}_2^\infty(\mathbb{Q})$ never arise as $\rho_{E,2^\infty}(G_\mathbb{Q})$ because $\rho_{E,2^\infty}(G_\mathbb{Q})$ is strictly contained in H_0 for all the elliptic curves E that correspond to points in $X_{H_0}(\mathbb{Q})$; see [RZB15, Remark 6.3]. This explains the discrepancy between $\#\mathcal{S}_2^\infty(\mathbb{Q}) = 1208$ and the figure 1201 that appears in [SZ17, Corollary 1.6].

by [Zyw15a, Proposition 1.13]. Recent work by le Fourn and Lemos [LFL21, Theorem 1.2] shows that if $\ell > 1.4 \times 10^7$, then $\rho_{E,\ell}(G_{\mathbb{Q}})$ cannot be a proper subgroup of $N_{\text{ns}}(\ell)$.

Numerous individual cases remain. See Tables 12–17 in Appendix B for a list of all arithmetically maximal $H \leq \text{GL}_2(\widehat{\mathbb{Z}})$ of ℓ -power level for $\ell \leq 37$, and see Table 4 for a numerical summary of this data. Many of the corresponding modular curves X_H are already handled in the literature, and these account for most of the exceptional points listed in Table 1. Below is a summary of prior results.

1. Ligozat determined the rational points on $X_0(N)$ for $N = 27, 49, 11, 17, 19$ with labels 27.36.1.1, 49.56.1.1, 11.12.1.1, 17.18.1.1, 19.20.1.1 in [Lig75, 5.2.3.1]; there are exceptional points for $N = 11, 17$.
2. Ligozat addressed $X_{S_4}(11)$ with label 11.55.1.2 in [Lig77, Proposition 4.4.8.1].
3. Kenku determined the rational points on $X_0(13^2)$ (labeled 169.182.8.1) [Ken80] and on $X_0(125)$ (labeled 125.150.8.1) [Ken81]; the modular curve $X_0(5, 25)$ with label 25.150.8.1 is isomorphic to $X_0(125)$ and thus also handled by Kenku.
4. The curves $X_{\text{sp}}^+(25)$ and $X_{\text{sp}}^+(49)$ have labels 25.375.22.1 and 49.1372.94.1; they are isomorphic to $X_0^+(25^2)$ and $X_0^+(49^2)$ and are thus handled by Momose and Shimura [MS02, Theorems 0.1 and 3.14] and by Momose [Mom86, Theorem 3.6].
5. Momose also handled $X_{\text{sp}}^+(11)$ with label 11.66.2.1 in [Mom84, Theorem 0.1].
6. The curves $X_{\text{ns}}^+(13)$ and $X_{\text{sp}}^+(13)$ labeled 13.78.3.1 and 13.91.3.1 are handled via nonabelian Chabauty by Balakrishnan et al. in [BDM+19].
7. The curves $X_{S_4}(13)$ and $X_{\text{ns}}^+(17)$ labeled 13.91.3.2 and 17.136.6.1 are handled via nonabelian Chabauty by Balakrishnan et al. in [BDM+21].
8. Rouse and Zureick-Brown addressed all X_H of 2-power level in [RZB15].
9. The group with label 7.42.1.1 is the index 2 subgroup of $N_{\text{sp}}(7)$ containing the elements of $C_{\text{sp}}(7)$ with square determinant and the elements of $N_{\text{sp}}(7) - C_{\text{sp}}(7)$ with nonsquare determinant, handled by [Sut12a, Section 3] and [Zyw15a, Remark 4.3]. There is one exceptional j -invariant, which is interesting because elliptic curves with this mod 7 image admit a rational 7-isogeny locally everywhere but not globally; this is the unique example of this phenomenon over \mathbb{Q} for any prime ℓ .
10. Zywina addressed all remaining curves of prime level $\ell \leq 37$ except $N_{\text{ns}}(\ell)$ in [Zyw15a].
11. Elkies showed that the curve 9.81.1.1 has no rational points in [Elk06, p. 5].
12. Noncuspidal points on the modular curves labeled 49.168.12.1 and 49.168.12.2 are ‘7-exceptional’ in the sense of [GRSS14, Definition 1.1].⁵ By [GRSS14, page 3], the only 7-exceptional elliptic curves are CM curves with $j = -15^3$ and $j = 255^3$, which give rise to two points on the curve with label 49.168.12.2.
13. The groups with labels 25.150.4.1, 25.150.4.2, 25.150.4.5 and 25.150.4.6 correspond to elliptic curves E/\mathbb{Q} that admit a rational 5-isogeny with 5-adic image of index divisible by 25; their existence is ruled out by [Gre12, Theorem 2].
14. The genus 2 rank 2 curves 25.50.2.1 and 25.75.2.1 are handled by Balakrishnan et al. in [BDM+21, 4.3]. Each has a single exceptional j -invariant, and the minimal conductors for the two j -invariants are 396900 and 21175, respectively.

Of the 210 arithmetically maximal subgroups in Table 3, only 32 are not addressed above. Five are the groups $N_{\text{ns}}(\ell)$ for $19 \leq \ell \leq 37$; the remaining 27 are listed in Table 5.

5. Counting \mathbb{F}_q -points on X_H

For any prime power $q = p^e$ coprime to N and $H \leq \text{GL}_2(N)$, we may consider the modular curve X_H over the finite field \mathbb{F}_q . We can count \mathbb{F}_q -rational points on X_H via

$$\#X_H(\mathbb{F}_q) = \#X_H^{\infty}(\mathbb{F}_q) + \#Y_H(\mathbb{F}_q),$$

⁵These two curves have genus 12 but are not the genus 12 curve studied in [GRSS14, Theorem 5.4].

Table 5. The arithmetically maximal subgroups of ℓ -power level for $\ell \leq 17$ not addressed by previous results. Subgroups marked with asterisks are the subgroups H listed in Table 2 for which we are not able to determine $X_H(\mathbb{Q})$.

| Genus | Subgroups | Section |
|-------|---|---------|
| 1 | 9.12.1.1, 9.54.1.1, 27.36.1.2 | §8.1 |
| 2 | 9.36.2.1, 9.54.2.2, 27.36.2.1, 27.36.2.2 | §8.2 |
| 3 | 9.108.3.1, 27.36.3.1, 27.108.3.1 | §8.3 |
| 4 | 9.108.4.1, 27.108.4.1, 27.108.4.3, 27.108.4.5 | §8.4 |
| | 25.150.4.7, 25.150.4.8, 25.150.4.9 | §5.2 |
| 6 | 27.108.6.1 | §8.5 |
| 9 | 49.147.9.1*, 49.196.9.1* | §9.3 |
| 12 | 27.243.12.1* | §9.1 |
| 14 | 25.250.14.1* | §9.2 |
| 36 | 25.625.36.1 | §8.6 |
| 41 | 121.605.41.1 | §8.7 |
| 43 | 27.729.43.1 | §8.6 |
| 69 | 49.1029.69.1* | §9.2 |
| 511 | 121.6655.511.1* | §9.2 |

by counting fixed points of a right $G_{\mathbb{F}_q}$ -action on certain double coset spaces of $GL_2(N)$ defined in Subsection 2.3. In this section, we explain how to do this explicitly and efficiently via a refinement of the method proposed in [Zyw15b, Section 3]. Applications include:

- checking for p -adic obstructions to rational points (by checking if $\#X_H(\mathbb{F}_p) = 0$);
- computing the zeta function of X_H/\mathbb{F}_p (by computing $\#X_H(\mathbb{F}_{p^r})$ for $1 \leq r \leq g(H)$);
- determining the isogeny decomposition of $Jac(X_H)$;
- determining the analytic rank of $Jac(X_H)$.

The last two are described in Section 6. None require an explicit model for X_H .

5.1. Counting points

If we put $\overline{H} := \langle H, -I \rangle \leq GL_2(N)$, we can use the permutation representation provided by the right $G_{\mathbb{F}_q}$ -action on the coset space $[\overline{H} \backslash GL_2(N)]$ to compute

$$\#X_H^\infty(\mathbb{F}_q) = \#(H \backslash GL_2(N)/U(N))^{G_{\mathbb{F}_q}},$$

as the number of $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ -orbits of $[\overline{H} \backslash GL_2(N)]$ stable under the action of $\begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix}$, which we note depends only on $q \pmod N$; if one expects to compute $\#X_H^\infty(\mathbb{F}_q)$ for many finite fields \mathbb{F}_q (as when computing the L -function of X_H/\mathbb{Q} , for example), one can simply precompute a table of rational cusp counts indexed by $(\mathbb{Z}/N\mathbb{Z})^\times$.

To compute

$$\#Y_H(\mathbb{F}_q) = \sum_{j(E) \in k} \#(H \backslash GL_2(N)/A_E)^{G_{\mathbb{F}_q}}, \tag{5.1}$$

we first note that $\pm I \in A_E := \{\varphi_N : \varphi \in \text{Aut}(E_{\overline{k}})\}$, so we may replace H with \overline{H} and work with the quotient $A_E/\pm I$, which is trivial for $j(E) \neq 0, 1728$. It then suffices to count elements of $[\overline{H} \backslash GL_2(N)]$ fixed by $G_{\mathbb{F}_q}$ using the action of the Frobenius endomorphism π on $E[N]$ with respect to some basis; we are free to choose any basis we like, since the number of elements of $[\overline{H} \backslash GL_2(N)]$ fixed by a matrix in

$GL_2(N)$ is invariant under conjugation. By [DT02, Theorem 2.1], we can use the reduction modulo N of the integer matrix

$$A_\pi := A(a, b, \Delta) := \begin{bmatrix} \frac{a+bd}{2} & b \\ \frac{b(\Delta-d)}{4} & \frac{a-bd}{2} \end{bmatrix}, \tag{5.2}$$

where $a, b, \Delta, d \in \mathbb{Z}$ are defined as follows. Let $R_\pi := \text{End}(E) \cap \mathbb{Q}[\pi]$, let $\Delta := \text{disc}(R_\pi)$, let $a := \text{tr } \pi$, and let $b := [R_\pi : \mathbb{Z}[\pi]]$ if $\mathbb{Z}[\pi] \neq \mathbb{Z}$, and let $b := 0$ otherwise, so that we always have

$$4q = a^2 - b^2\Delta, \tag{5.3}$$

and let $d := \Delta \bmod 2 \in \{0, 1\}$. We then have $\text{tr } A_\pi \equiv \text{tr } \pi \pmod N$ and $\det A_\pi \equiv q \pmod N$. We note that A_π is uniquely determined by $\text{tr } \pi$ and Δ and represents an element of $GL_2(N)$ for every integer N coprime to q .

Let $\chi_{\overline{H}}: GL_2(N) \rightarrow \mathbb{Z}_{\geq 0}$ denote the character of the permutation representation given by the right $GL_2(N)$ -set $[\overline{H} \backslash GL_2(N)]$. Each term in equation (5.1) for $j(E) \neq 0, 1728$ can be computed as $\chi_{\overline{H}}(A_\pi)$. It follows from the theory of complex multiplication that for each imaginary quadratic discriminant Δ for which the norm equation (5.3) has a solution with $a > 0$ coprime to q , there are exactly $h(\Delta)$ ordinary j -invariants of elliptic curves over \mathbb{F}_q with $\text{disc}(R_\pi) = \Delta$, where $h(\Delta)$ is the class number; see [Sut11, Proposition 1], for example. We thus compute the number of \mathbb{F}_q -points on X_H corresponding to ordinary $j(E) \neq 0, 1728$ via

$$\#X_H^{\text{ord}'}(\mathbb{F}_q) = \sum_{\substack{0 < a < 2\sqrt{q} \\ \gcd(a, q) = 1}} \sum_{\substack{4q = a^2 - b^2\Delta \\ \Delta < -4}} h(\Delta) \chi_{\overline{H}}(A(a, b, \Delta)). \tag{5.4}$$

To count supersingular points in $X_H(\mathbb{F}_q)$ with $j \neq 0, 1728$, we rely on the following lemma.

Lemma 5.1. *Let $q = p^e$ be a prime power, let $H \leq GL_2(N)$ with $\gcd(q, N) = 1$, let $h' = \lfloor h(-4p)/2 \rfloor$, and let $s_0 = 1$ for $p \equiv 2 \pmod 3$ and $s_0 = 0$ otherwise. The number $\#X_H^{\text{ss}' }(\mathbb{F}_q)$ of \mathbb{F}_q -points on X_H corresponding to supersingular $j(E) \neq 0, 1728$ can be computed as follows.*

- If $p \leq 3$, then $\#X_H^{\text{ss}' }(\mathbb{F}_q) = 0$.
- If e is odd and $p \equiv 1 \pmod 4$, then

$$\#X_H^{\text{ss}' }(\mathbb{F}_q) = (h' - s_0) \chi_{\overline{H}}(A(0, p^{(e-1)/2}, -4p)).$$

- If e is odd and $p \equiv 3 \pmod 4$, then

$$\#X_H^{\text{ss}' }(\mathbb{F}_q) = h' \chi_{\overline{H}}(A(0, 2p^{(e-1)/2}, -p)) + (h' - s_0) \chi_{\overline{H}}(A(0, p^{(e-1)/2}, -4p)).$$

- If e is even then

$$\#X_H^{\text{ss}' }(\mathbb{F}_q) = \left(\frac{p - 6 + 2\left(\frac{-3}{p}\right) + 3\left(\frac{-4}{p}\right)}{12} \right) \chi_{\overline{H}}(A(2p^{e/2}, 0, 1)).$$

Proof. These formulas are derived from [Sch87, Theorem 4.6] by discarding \mathbb{F}_q -isomorphism classes with $j(E) = 0, 1728$ and dividing by 2 to count j -invariants rather than counting \mathbb{F}_q -isomorphism classes. For $p = 2, 3$, there are no supersingular $j(E) \neq 0, 1728$, and for $p > 3$ and $j(E) \neq 0, 1728$, we only need to consider supersingular \mathbb{F}_q -isomorphism classes with trace $a = 0$ and $\Delta = -0, -4p$ when e is odd, and with trace $a = 2p^{e/2}$ and $\Delta = 1$ when e is even. In each case, the number of supersingular \mathbb{F}_q -isomorphism classes with j -invariant 0, 1728 that we need to discard can be determined using Table 6 below. □

Table 6. \mathbb{F}_{p^e} -isomorphism classes of elliptic curves with j -invariant 0 or 1728.

| $\left(\frac{-3}{p}\right)$ | $e \bmod 2$ | $\#\text{Aut}(E)$ | $\text{rk}E[2](\mathbb{F}_{p^e})$ | $ a $ | b | Δ | m |
|---|-------------|-------------------|-----------------------------------|--------------------|-----------------------|----------|-----|
| -1 | 1 | 2 | 1 | 0 | $p^{(e-1)/2}$ | $-4p$ | 2 |
| -1 | 0 | 6 | 0 | $p^{e/2}$ | $p^{e/2}$ | -3 | 4 |
| -1 | 0 | 6 | 2 | $2p^{e/2}$ | 0 | 1 | 2 |
| 0 | 1 | 2 | 1 | 0 | $3^{(e-1)/2}$ | -12 | 1 |
| 0 | 1 | 6 | 2 | 0 | $2 \cdot 3^{(e-1)/2}$ | -3 | 1 |
| 0 | 1 | 6 | 0 | $3^{(e+1)/2}$ | $3^{(e-1)/2}$ | -3 | 2 |
| 0 | 0 | 4 | 1 | 0 | $3^{e/2}$ | -4 | 2 |
| 0 | 0 | 6 | 0 | $3^{e/2}$ | $3^{e/2}$ | -3 | 2 |
| 0 | 0 | 12 | 2 | $2 \cdot 3^{e/2}$ | 0 | 1 | 2 |
| +1 | * | 6 | 2 | u | v | -3 | 2 |
| +1 | * | 6 | 0 | $\frac{u+3v}{2}$ | $\frac{ u-v }{2}$ | -3 | 2 |
| +1 | * | 6 | 0 | $\frac{ u-3v }{2}$ | $\frac{u+v}{2}$ | -3 | 2 |
| $j(E) = 0, p \neq 2, 4p^e = u^2 + 3v^2$ with $u, v > 0$ even | | | | | | | |
| $\left(\frac{-4}{p}\right)$ | $e \bmod 2$ | $\#\text{Aut}(E)$ | $\text{rk}E[2](\mathbb{F}_{p^e})$ | $ a $ | b | Δ | m |
| -1 | 1 | 2 | 1 | 0 | $p^{(e-1)/2}$ | $-4p$ | 1 |
| -1 | 1 | 2 | 2 | 0 | $2p^{(e-1)/2}$ | $-p$ | 1 |
| -1 | 0 | 4 | 1 | 0 | $p^{e/2}$ | -4 | 2 |
| -1 | 0 | 4 | 2 | $2p^{e/2}$ | 0 | 1 | 2 |
| 0 | 1 | 2 | 0 | 0 | $2^{(e-1)/2}$ | -8 | 1 |
| 0 | 1 | 4 | 0 | $2^{(e+1)/2}$ | $2^{(e-1)/2}$ | -4 | 2 |
| 0 | 0 | 4 | 0 | 0 | $2^{e/2}$ | -4 | 1 |
| 0 | 0 | 6 | 0 | $2^{e/2}$ | $2^{e/2}$ | -3 | 4 |
| 0 | 0 | 24 | 0 | $2 \cdot 2^{e/2}$ | 0 | 1 | 2 |
| +1 | * | 4 | 2 | u | v | -4 | 2 |
| +1 | * | 4 | 1 | $2v$ | $\frac{u}{2}$ | -4 | 2 |
| $j(E) = 1728, p \neq 3, 4p^e = u^2 + 4v^2$ with $u, v > 0$ even | | | | | | | |

For $j(E) = 0, 1728$, rather than computing double cosets fixed by G_k , we instead compute a weighted sum of $\chi_{\overline{H}}(A_\pi)$ over k -isomorphism classes of elliptic curves with $j(E) = 0, 1728$ via the table below. If we extend the permutation character $\chi_{\overline{H}}$ to the group ring $\mathbb{Q}[\text{GL}_2(N)]$, we can compute the number of k -rational points of X_H above $j = 0, 1728 \in X(1)$ via

$$\#X_H^{j=0}(\mathbb{F}_q) = \chi_{\overline{H}} \left(\sum_{\substack{j(E)=0 \\ p \neq 2}} \frac{A(a, b, \Delta)}{\#\text{Aut}(E)} \right), \quad \#X_H^{j=1728}(\mathbb{F}_q) = \chi_{\overline{H}} \left(\sum_{\substack{j(E)=1728 \\ p \neq 3}} \frac{A(a, b, \Delta)}{\#\text{Aut}(E)} \right),$$

where the sums are over \mathbb{F}_q -isomorphism classes of elliptic curves E/\mathbb{F}_q , and the values of a, b, Δ are listed in Table 6. These values are derived from [Wat69] and [Sch87], and we note that for each triple $(|a|, b, \Delta)$ with $|a| > 0$ listed in the table with multiplicity m , the triples (a, b, Δ) and $(-a, b, \Delta)$ each occur with multiplicity $m/2$.

To sum up, for any $H \leq \text{GL}_2(N)$, we can compute $\#X_H(\mathbb{F}_q)$ as

$$\#X_H(\mathbb{F}_q) = \#X_H^\infty(\mathbb{F}_q) + \#X_H^{\text{ord}}(\mathbb{F}_q) + \#X_H^{\text{ss}'}(\mathbb{F}_q) + \#X_H^{j=0}(\mathbb{F}_q) + \#X_H^{j=1728}(\mathbb{F}_q), \tag{5.5}$$

where each term in the sum is computed using the permutation representation $[\overline{H} \backslash \text{GL}_2(N)]$: the term $\#X_H^\infty(\mathbb{F}_q)$ is computed by counting $\begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix}$ -stable $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ -orbits, and the remaining terms are weighted sums of values of the permutation character $\chi_{\overline{H}}$ on the matrices $A(a, b, \Delta)$ arising for elliptic curves E/\mathbb{F}_q . A Magma implementation of the formula in equation (5.5) is provided by the function `GL2PointCount` in the file `g12.m` in [RSZB21].

Example 5.2. Consider $H = \left\{ \begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix} \right\} \leq \mathrm{GL}_2(13)$ with label 13.168.2.1. The modular curve $X_H = X_1(13)$ parametrises elliptic curves with a rational point of order 13. Over \mathbb{F}_{37} , there are four such elliptic curves, up to \mathbb{F}_{37} -isomorphism, each with 12 points of order 13 with distinct j -invariants 0, 16, 26 = 1728, 35 (none are supersingular) and automorphism groups of order 6, 2, 4, 2, respectively. We thus expect $\#Y_H(\mathbb{F}_{37}) = 12/6 + 12/2 + 12/4 + 12/2 = 17$, and one can check that 6 of the 12 cusps of $X_1(13)$ are \mathbb{F}_{37} -rational (those of width of 13). This agrees with the fact that there are 23 rational points on the mod-37 reduction of the genus 2 curve $y^2 + (x^3 + x + 1)y = x^5 + x^4$, which is a $\mathbb{Z}[1/13]$ -model for $X_H = X_1(13)$.

To compute $\#X_H(\mathbb{F}_{37})$ using the algorithm proposed above, we apply equation (5.5), which does not require us to enumerate elliptic curves with a point of order 13. We compute $\#X_H^\infty(\mathbb{F}_{37}) = 6$ by noting that the coset space $[\overline{H} \backslash \mathrm{GL}_2(13)]$ has twelve $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ -orbits, six of size 1 and six of size 13, the latter of which are stable under the action of $\begin{bmatrix} 37 & 0 \\ 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} 11 & 0 \\ 0 & 1 \end{bmatrix}$. Enumerating integers $a \in [1, \lfloor 2\sqrt{37} \rfloor]$ yields 17 triples (a, b, Δ) with $\Delta < -4$ that satisfy the norm equation (5.3). Among these, only $A(1, 1, -147)$ fixes an element of $[\overline{H} \backslash \mathrm{GL}_2(13)]$, and from equation (5.4), we obtain

$$\#X_H^{\mathrm{ord}'}(\mathbb{F}_q) = h(\Delta)\chi_{\overline{H}}(A(a, b, \Delta)) = h(-147)\chi_{\overline{H}}(A(1, 1, -147)) = 2 \cdot 6 = 12.$$

To compute $\#X_H^{\mathrm{ss}'}(\mathbb{F}_{37})$, we apply Lemma 5.1 and find that $\chi_{\overline{H}}(A(0, 0, -4 \cdot 37)) = 0$, so $\#X_H^{\mathrm{ss}'}(\mathbb{F}_q) = 0$. Using Table 6, we compute

$$\begin{aligned} \#X_H^{j=0}(\mathbb{F}_{37}) &= \chi_{\overline{H}}(A(\pm 1, 7, -3)/6 + A(\pm 10, 4, -3)/6 + A(\pm 11, 3, -3)/6) = 2 + 0 + 0 = 2 \\ \#X_H^{j=1728}(\mathbb{F}_{37}) &= \chi_{\overline{H}}(A(\pm 2, 6, -4)/4 + A(\pm 12, 1, -4)/4) = 0 + 3 = 3, \end{aligned}$$

and equation (5.5) then yields

$$\begin{aligned} \#X_H(\mathbb{F}_{37}) &= \#X_H^\infty(\mathbb{F}_{37}) + \#X_H^{\mathrm{ord}'}(\mathbb{F}_{37}) + \#X_H^{\mathrm{ss}'}(\mathbb{F}_{37}) + \#X_H^{j=0}(\mathbb{F}_{37}) + \#X_H^{j=1728}(\mathbb{F}_{37}) \\ &= 6 + 12 + 0 + 2 + 3 = 23. \end{aligned}$$

In contrast to the method described in [Zyw15b, Section 3], in our computation above, we did not enumerate elliptic curves or j -invariants over \mathbb{F}_q , nor did we compute any Hilbert class polynomials (a commonly used but inefficient way to compute A_π). Instead, we enumerated possible Frobenius traces $a > 0$, solved norm equations, computed class numbers and computed values of the permutation character $\chi_{\overline{H}}$. This can be viewed as an algorithmic generalisation of the Eichler–Selberg trace formula that works for any open $H \leq \mathrm{GL}_2(\widehat{\mathbb{Z}})$. It has roughly the same asymptotic complexity as the trace formula (the dependence on q is $\tilde{O}(q^{1/2})$ if we ignore the cost of computing class numbers, which in practice we simply look up in a table such as those available for $|D| \leq 2^{40}$ in the LMFDB). But the constant factors are better, and even for groups that do not contain the GL_2 -analog of $\Gamma_1(N)$, we are always able to work at level N rather than level N^2 , as typically required by generalisations of the trace formula to handle congruence subgroups that do not contain $\Gamma_1(N)$.

5.2. Local obstructions

An easy application of this point counting algorithm is checking for local obstructions to rational points on the arithmetically maximal modular curves X_H . For X_H of genus g , it suffices to compute $\#X(\mathbb{F}_p)$ for $p < \lfloor 2g\sqrt{p} \rfloor$, since the Weil bounds force $\#X(\mathbb{F}_p) > 0$ for all larger primes p . Doing this for the 210 arithmetically maximal groups summarised in Table 4 yields Theorem 5.1, whose results are summarised in Table 7.

Theorem 5.1. *For $\ell \leq 37$, the arithmetically maximal $H \leq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ of ℓ -power level for which X_H has no \mathbb{F}_p -points for some prime $p \neq \ell$ are precisely those listed in Table 7.*

Table 7. Arithmetically maximal $H \leq \text{GL}_2(\widehat{\mathbb{Z}})$ of ℓ -power level for which X_H has no \mathbb{F}_p -points for some prime $p \neq \ell \leq 37$.

| Label | Level | Generators | p | Rank | Genus |
|-------------|-------|---|-------|------|-------|
| 16.48.2.17 | 2^4 | $\begin{bmatrix} 11 & 9 \\ 4 & 13 \end{bmatrix}, \begin{bmatrix} 13 & 5 \\ 4 & 11 \end{bmatrix}, \begin{bmatrix} 1 & 9 \\ 12 & 7 \end{bmatrix}, \begin{bmatrix} 1 & 9 \\ 0 & 5 \end{bmatrix}$ | 3, 11 | 0 | 2 |
| 27.108.4.5 | 3^3 | $\begin{bmatrix} 4 & 25 \\ 6 & 14 \end{bmatrix}, \begin{bmatrix} 8 & 0 \\ 3 & 1 \end{bmatrix}$ | 7, 31 | 0 | 4 |
| 25.150.4.2 | 5^2 | $\begin{bmatrix} 7 & 20 \\ 20 & 7 \end{bmatrix}, \begin{bmatrix} 22 & 2 \\ 13 & 22 \end{bmatrix}$ | 2 | 0 | 4 |
| 25.150.4.7 | 5^2 | $\begin{bmatrix} 24 & 24 \\ 0 & 18 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ 0 & 23 \end{bmatrix}$ | 3, 23 | 4 | 4 |
| 25.150.4.8 | 5^2 | $\begin{bmatrix} 8 & 4 \\ 0 & 23 \end{bmatrix}, \begin{bmatrix} 16 & 7 \\ 0 & 8 \end{bmatrix}$ | 2 | 0 | 4 |
| 25.150.4.9 | 5^2 | $\begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}, \begin{bmatrix} 3 & 18 \\ 0 & 14 \end{bmatrix}$ | 2 | 0 | 4 |
| 49.168.12.1 | 7^2 | $\begin{bmatrix} 39 & 6 \\ 36 & 24 \end{bmatrix}, \begin{bmatrix} 11 & 9 \\ 24 & 2 \end{bmatrix}$ | 2 | 3 | 12 |
| 13.84.2.2 | 13 | $\begin{bmatrix} 3 & 7 \\ 0 & 8 \end{bmatrix}, \begin{bmatrix} 12 & 4 \\ 0 & 12 \end{bmatrix}$ | 2 | 0 | 2 |
| 13.84.2.3 | 13 | $\begin{bmatrix} 9 & 2 \\ 0 & 7 \end{bmatrix}, \begin{bmatrix} 4 & 4 \\ 0 & 7 \end{bmatrix}$ | 3 | 0 | 2 |
| 13.84.2.4 | 13 | $\begin{bmatrix} 8 & 12 \\ 0 & 10 \end{bmatrix}, \begin{bmatrix} 8 & 3 \\ 0 & 9 \end{bmatrix}$ | 2 | 0 | 2 |
| 13.84.2.6 | 13 | $\begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 11 & 3 \\ 0 & 10 \end{bmatrix}$ | 3 | 0 | 2 |

6. Computing the isogeny decomposition and analytic rank of J_H

Let H be an open subgroup of $\text{GL}_2(\widehat{\mathbb{Z}})$. In this section, we explain how the point-counting algorithm described in the previous section can be used to determine the isogeny factors of the Jacobian $J_H := \text{Jac}(X_H)$ as modular abelian varieties; this decomposition can then be used to compute the analytic rank of J_H . To simplify the exposition, we shall assume that $\det(H) = \widehat{\mathbb{Z}}^\times$, which holds in all the cases of interest to us here, but the methods we describe can also be applied when $\det(H) \neq \widehat{\mathbb{Z}}^\times$; see Example 6.3 for details.

The Jacobian J_H is an abelian variety over \mathbb{Q} that has good reduction at all primes p that do not divide the level N of H . It follows from Theorem A.7 that the simple factors of J_H are each \mathbb{Q} -isogenous to the modular abelian variety associated to the Galois orbit of a (not necessarily new, normalised) eigenform of weight 2 for $\Gamma_1(N) \cap \Gamma_0(N^2)$. We may decompose this space of modular forms as

$$S_2(\Gamma_1(N) \cap \Gamma_0(N^2)) = \bigoplus_{\text{cond}(\chi) | N} S_2(\Gamma_0(N^2), [\chi]),$$

where $[\chi]$ denotes the Galois orbit of the Dirichlet character χ and the direct sum varies over Galois orbits of Dirichlet characters of modulus N^2 whose conductor divides N . The Galois orbit of a normalised eigenform $f \in S_2(\Gamma_0(N^2), [\chi])$ has an associated trace form $\text{Tr}(f)$ (see [BBB+21, Section 4.5]), which has an integral q -expansion

$$\text{Tr}(f)(q) := \sum_{n \geq 1} \text{Tr}_{\mathbb{Q}(f)/\mathbb{Q}}(a_n(f))q^n$$

with $a_1(\text{Tr}(f)) = \dim f := [\mathbb{Q}(f) : \mathbb{Q}]$; and for each prime p , the coefficient a_p is the Frobenius trace of the abelian variety A_f/\mathbb{Q} of dimension $\dim f$ associated to f via the Eichler–Shimura correspondence; equivalently, a_p is the p th (arithmetically normalised) Dirichlet coefficient of the L -function $L(A_f, s) = \prod_{f \in [f]} L(f, s)$.

Let $S(H) := \{[f_1], \dots, [f_m]\}$ be the Galois orbits of eigenforms $f \in S_2(\Gamma_1(N) \cap \Gamma_0(N^2))$ with $\dim f \leq \dim J_H = g(H)$. Theorem A.7 implies that there is a sequence of nonnegative integers $e(H) = (e_1, \dots, e_m)$ for which we have

$$L(J_H, s) = \prod_{i=1}^m L(A_{f_i}, s)^{e_i},$$

and this sequence also satisfies $\sum_i e_i \dim f_i = g(H)$. Now let $T(B)$ be the $n \times m$ integer matrix whose i th column is $[a_1(\text{Tr}(f_i)), a_2(\text{Tr}(f_i)), a_3(\text{Tr}(f_i)), a_5(\text{Tr}(f_i)), \dots, a_p(\text{Tr}(f_i)), \dots]$, where p varies over the $n - 1$ primes bounded by B that do not divide the level of H , and let $a(H; B)$ be the vector of integers $[g(H), a_2(H), a_3(H), \dots, a_p(H), \dots]$, where $a_p(H)$ denotes the trace of the Frobenius endomorphism of J_H at a prime $p \leq B$ not dividing the level of H , which we can compute as $a_p(H) = p + 1 - \#X_H(\mathbb{F}_p)$ via equation (5.5). For $B \geq 1$, we have

$$T(B)e(H) = a(H; B),$$

and it follows from an effective form of multiplicity 1 for the Selberg class [KP01, Sou04] that for all sufficiently large B , the matrix $T(B)$ and the vector $a(H, B)$ uniquely determine the vector $e(H)$. Indeed, there are only finitely many sequences of nonnegative integers that sum to $g(H)$, and the corresponding L -functions are all distinguished by their Dirichlet coefficients a_p on the set of primes $p \nmid N$ by strong multiplicity 1; for each pair of these L -functions, there is some smallest $p \nmid N$ for which the a_p differ, and we can take B to be the maximum of this p over all pairs. We note that L -functions of modular forms and all products of such L -functions not only lie in the Selberg class but also satisfy the polynomial Euler product condition used in [KP01] and the weaker condition required by [Sou04].

This suggests the following approach to determining the exponents in the factorisation $L(J_H, s) = \prod_i L(A_{f_i}, s)^{e_i}$, from which both the analytic rank of J_H and its isogeny-factor decomposition can be derived. We first determine the set $S(H)$, which we note depends only on the genus and level of H . We then determine a value of B for which the $\#S(H)$ columns of $T(B)$ become linearly independent over \mathbb{Q} : for example, by starting with a value of B that makes $T(B)$ a square matrix and successively increasing B by some constant factor $\alpha > 1$. Finally, we compute the vector $a(H; B)$ by computing $p + 1 - \#X_H(\mathbb{F}_p)$ for primes $p \leq B$ not dividing the level of H and determine $e(H)$ as the necessarily unique solution to the linear system $T(B)x = a(H; B)$.

This approach is remarkably effective in practice. For the 210 arithmetically maximal H arising for $\ell \leq 37$, the value of B never exceeds 3000, and the matrix $T(B)$ with linear independent columns typically has fewer than 100 rows. To compute $T(B)$, we use precomputed trace form data stored in the LMFDB along with additional data available in the file `cmfdata.txt` in [RSZB21] that we computed to handle cases where the set $S(H)$ contains Galois orbits of eigenforms that are not currently stored in the LMFDB, using the methods described in [BBB+21].

Remark 6.1. There are special cases in which we know the isogeny factors of J_H *a priori* and need not apply the approach described above in order to determine the vector $e(H)$. When H is the Borel subgroup of level N , we have $X_H = X_0(N)$ and the isogeny factors of J_H are the modular abelian varieties associated to the eigenforms $f \in S_2(\Gamma_0(N))$. When H is the normaliser of a nonsplit Cartan subgroup of prime power level ℓ^n , we have $X_H = X_{\text{ns}}^+(\ell^n)$ and the isogeny factors of J_H are the modular abelian varieties associated to the eigenforms in $f \in S_2^{\text{new}}(\Gamma_0(\ell^{2r}))$ with Fricke eigenvalue 1, for $1 \leq r \leq n$. And when H is the normaliser of a split Cartan subgroup of level ℓ^n , we have $X_H = X_{\text{sp}}^+(\ell^n)$ and the isogeny factors of J_H are the modular abelian varieties associated to the eigenforms in $f \in S_2(\Gamma_0(\ell^{2n}))$ with Fricke eigenvalue 1, as shown by Chen [Che04].

Having determined the vector $e(H) = [e_1, \dots, e_m]$, we compute the rank of J_H as $\sum_i e_i r_i$, where each r_i is the sum of the analytic ranks of the eigenforms in the i th Galois orbit $[f_i]$ (one expects them to have the same rank as f_i ; this is true for all of the modular forms in the LMFDB and for all of the modular forms that we computed). The method used to rigorously compute the analytic ranks of these modular forms is described in [BBB+21, Section 9].

Example 6.2. Consider the group $H = \langle \begin{bmatrix} 4 & 1 \\ 21 & 2 \end{bmatrix}, \begin{bmatrix} 11 & 0 \\ 0 & 19 \end{bmatrix} \rangle \leq \text{GL}_2(27)$ with label 27.36.3.1. Eleven Galois orbits of eigenforms $f \in S_2(\Gamma_1(27) \cap \Gamma_0(27^2))$ have $\dim f \leq g(H) = 3$, with LMFDB labels 243.2.a.a, 27.2.a.a, 243.2.a.b, 243.2.c.a, 243.2.a.c, 81.2.c.a, 243.2.a.d, 81.2.a.a, 243.2.c.b, 243.2.a.e, and 243.2.a.f. For $B = 73$ the 21×11 matrix $T(B)$

has independent columns. Computing $\#X_H(\mathbb{F}_p)$ for $p=2, 5, 7, 11, \dots, 73$ yields $a(H; B) = [3, 0, 0, 0, 0, -9, 0, -3, 0, 0, 9, 6, 0, 27, 0, 0, 0, 9, -9, 0, 15]^T$, and the linear system

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & 6 \\ -4 & -1 & 5 & -5 & -2 & 1 & 4 & 4 & 4 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 3 \\ -7 & 5 & 2 & -2 & 10 & -5 & -2 & -2 & 7 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -9 & 9 \\ -1 & -7 & 8 & 16 & -2 & -14 & -2 & 4 & -2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -12 & 12 \\ 11 & -4 & -7 & 7 & 10 & 4 & -2 & 16 & -11 & -12 & -12 \\ 10 & 11 & -1 & -2 & -2 & 22 & 16 & -14 & -20 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -3 \\ 5 & 8 & -13 & 13 & -2 & -8 & 22 & 4 & -5 & -12 & -12 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -18 & 18 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 21 & -21 \\ -1 & -1 & -1 & 1 & 4 & 1 & 10 & -14 & 1 & 6 & 6 \\ 5 & 5 & 5 & -5 & 16 & -5 & -14 & -20 & -5 & 6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & -9 \\ -7 & -7 & -7 & -14 & 4 & -14 & 22 & -14 & -14 & 6 & 6 \end{pmatrix} e(H) = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 0 \\ -9 \\ 0 \\ -3 \\ 0 \\ 0 \\ 9 \\ 6 \\ 0 \\ 27 \\ 0 \\ 0 \\ 0 \\ 0 \\ 9 \\ -9 \\ 0 \\ 0 \\ 15 \end{pmatrix}$$

has the unique solution $e(H) = [1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0]^T$ corresponding to the Galois orbits 243.2.a.a and 243.2.a.d of dimensions 1 and 2 with analytic ranks 1 and 0, respectively; thus J_H has analytic rank 1.

We used the method described above to compute the analytic ranks and the decomposition of J_H up to isogeny for all but one of the arithmetically maximal subgroups of ℓ -power level for $\ell \leq 37$. The sole exception is the group H with label 121.605.41.1 of level 121, where we were unable to compute all of the elements of $S(H)$ (we were able to do this for the group 121.6655.511.1 corresponding to $N_{\text{ns}}(121)$ by restricting to eigenforms $f \in S_2(\Gamma_0(\ell^{2n}))$ via the optimisation noted above). Computing all of the elements of $S(H)$ in this case would require decomposing the newspaces $S_2^{\text{new}}(\Gamma_0(11^4), [\chi])$ for all Dirichlet characters of modulus 11^4 and conductor dividing 11^2 , which we found impractical in the two cases where $\text{cond}(\chi) = 11^2$ (the corresponding newspaces have \mathbb{Q} -dimensions 10550 and 42200).

For the group 121.605.41.1, we apply the alternative strategy of verifying a candidate value for $e(H)$ by checking Fourier coefficients up to the Sturm bound. We first use the subset of $S(H)$ that we know to find a candidate $e(H) = [e_1, \dots, e_m]$ that works even for B much larger than necessary to ensure that the columns of $T(B)$ are linearly independent. To rigorously verify this candidate, it suffices to compare the Dirichlet coefficients $b_n, c_n \in \mathbb{Z}$ of the L -functions

$$L(J_H, s) = \sum_{n=1}^{\infty} b_n n^{-s} \quad \text{and} \quad \prod_i L(A_{f_i}, s)^{e_i} = \sum_{n=1}^{\infty} c_n n^{-s},$$

for n up to the Sturm bound for the space $S_2(\Gamma_1(N) \cap \Gamma_0(N^2))$, which is $B := 322102$. The integers c_n uniquely determine the Fourier coefficients a_n that appear in the q -expansion of the modular form $F := \sum_i \text{Tr}(f_i) = \sum_{n=1}^{\infty} a_n q^n \in S_2(\Gamma_1(N) \cap \Gamma_0(N^2))$ for $n \leq B$, which uniquely determines F .⁶

⁶The sequences $(c_n)_{n \leq B}$ and $(a_n)_{n \leq B}$ are not equal in general (although we do have $c_p = a_p$), but they uniquely determine each other, which is all we are using; see [BBB+21, Section 8.6].

Both L -functions are defined by an Euler product, so it suffices to check $b_q = c_q$ for prime powers $q \leq B$. We compute the c_q by computing Euler factors of $\prod_{f \in [f_i]} L(f, s)$ at all primes $p \leq B$ using the modular symbols functionality in Magma. For $11 \nmid q$, we compute the b_q by computing $\#X_H(\mathbb{F}_{p^r})$ for $1 \leq r \leq \log_p B$ via the point counting algorithm described in the previous section (these point counts determine enough coefficients of the degree-82 L -polynomial of J_H at p to determine the a_q for $q \leq B$). To compute the Euler factor of $L(J_H, s)$ at 11 (which determines the a_q for $11|q$), we use the GL_2 -modular symbols functionality in the Magma package `ModFrmGL2` recently developed by Assaf [Ass21]. The most time-consuming part of this process is computing the c_q for $q \leq B = 322\,102$, which takes 15–20 hours. It takes about 15 minutes to compute the b_q for $11 \nmid q$ and another 15 or 20 minutes to compute the Euler factor of $L(J_H, s)$ at 11. Magma scripts that reproduce all three computations and verify that the results match are available at [RSZB21].

Example 6.3. We remark that these methods apply to groups with nonsurjective determinant as well. While for such H , $X_H(\mathbb{Q})$ is empty (see the discussion after Definition 3.1) and X_H is connected but not geometrically connected, the sheafification of the functor $\text{Pic}^0 X_H$ is still representable by an abelian variety J_H over \mathbb{Q} ; see Remark 2.4 and Appendix A, where it is also proved that the isogeny factors of J_H are twists $A_{f \otimes \chi}$ of factors of $J_1(N^2)$, where N is the level of H and χ is a Dirichlet character of modulus N .

One can compute the decomposition of J_H via the methods of this section. At primes $p \nmid N$ whose reduction mod N is not an element of $\det(H)$, the point count $X_H(\mathbb{F}_p)$ is zero (again by the Weil pairing), and one can show that the trace of Frobenius acting on H^1 is also zero; for consistency, we also remark that since for such p none of the geometric connected components of X_{H, \mathbb{F}_p} are defined over \mathbb{F}_p , the trace of Frobenius acting on H^0 (and by duality, also on H^2) are zero.

For example, consider the group $H = \langle \begin{bmatrix} 8 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 0 & 8 \end{bmatrix} \rangle \leq GL_2(9)$ with label 9.3.1-9.108.1.1. One has $\det(H) = \{\pm 1\}$, which has index 3; X_H is connected but geometrically disconnected, and the geometric components are each isomorphic (over $\overline{\mathbb{Q}}$) to $E_{\overline{\mathbb{Q}}}$, where E/\mathbb{Q} is the elliptic curve with newform 27.2.a.a. Following the procedure outlined in Example 6.2, we find that over J_H decomposes (over \mathbb{Q}) as $E \times A$, where A is the simple abelian surface corresponding to the newform orbit 81.2.c.a. The abelian surface A is not geometrically simple; in fact, $A_{\overline{\mathbb{Q}}} \sim E_{\overline{\mathbb{Q}}}^2$, and the newform orbit 81.2.c.a is the twist of the newform 27.2.a.a by the Dirichlet character orbit 9.c.

7. Computing equations for X_H with $-I \in H$

We now discuss the computation of equations for X_H for arithmetically maximal H containing $-I$ of ℓ -power level for $3 \leq \ell \leq 13$; see Section 10 for the case $-I \notin H$.

In [RZB15], Rouse and Zureick-Brown construct the modular curves X_H as covers of $X_{\tilde{H}}$, where \tilde{H} is a supergroup of H for which $[\tilde{H} : H]$ is minimal. Because we are only recording equations for arithmetically maximal subgroups, in any case where we compute the covering $X_H \rightarrow X_{\tilde{H}}$, the genus of $X_{\tilde{H}}$ is zero or one. In all cases, we consider the genus of $X_{\tilde{H}}$ is zero, and we know a specific function $x \in \mathbb{Q}(X_{\tilde{H}})$ so that $\mathbb{Q}(X_{\tilde{H}}) = \mathbb{Q}(x)$. The models are computed by constructing elements of the function field of $\mathbb{Q}(X_H)$ built using products of weight 2 Eisenstein series whose Fourier expansions are simple to compute and for which it is easy to describe the action of $GL_2(N)$ on them. In this way, we find a generator y for $\mathbb{Q}(X_H)$ and then compute the action of H -coset representatives of \tilde{H} on it. This allows us to compute the minimal polynomial for y over $\mathbb{Q}(x)$, which gives a plane model for X_H . We use the method of Novocin and van Hoeij [vHN05] to reduce the resulting algebraic function field and obtain a simple model.

We now describe another approach for computing the canonical model of a high genus curve, which we use to compute equations of $X_{ns}^+(27)$ (label 27.243.12.1) and X_H , where H has label 27.729.43.1 or 25.625.36.1. Given $H \leq GL_2(N)$, there is a natural action of H on the field of modular functions for $\Gamma(N)$ with coefficients in $\mathbb{Q}(\zeta_N)$. In particular, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(N)$ acts via $f \mapsto f\left(\frac{az+b}{cz+d}\right)$, while the

matrix $h = \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}$ acts by sending $f = \sum a(n)q^n$ to $f|h = \sum \sigma_d(a(n))q^n$, where σ_d is the automorphism of $\mathbb{Q}(\zeta_N)$ sending $\sigma_d(\zeta_N) = \zeta_N^d$. One can extend this map to a natural map on $M_{2k}(\Gamma(N), \mathbb{Q}(\zeta_N))$ of the space of weight $2k$ modular forms for $\Gamma(N)$ with Fourier coefficients in $\mathbb{Q}(\zeta_N)$. To do so, given a form $F \in M_{2k}(\Gamma(N), \mathbb{Q}(\zeta_N))$, define $f = \frac{E_4(z)^k E_6(z)^k}{\Delta(z)^k} F$, a modular function for $\Gamma(N)$, and

$$(F|M) := (f|M) \cdot \frac{\Delta(z)^k}{E_4(z)^k E_6(z)^k}.$$

Here $E_4(z) := 1 + 240 \sum_{n=1}^\infty \sigma_3(n)q^n$ (with $q = e^{2\pi iz}$) and $E_6(z) := 1 - 504 \sum_{n=1}^\infty \sigma_5(n)q^n$ are the classical weight 4 and 6 Eisenstein series, respectively.

Given H , our goal is to find q -expansions for a basis for $S_2(\Gamma(N), \mathbb{Q}(\zeta_N))^H$. This problem has been considered before in [BC14] and recently in [Zyw21] and [Ass21]. Instead, we continue our approach from [RZB15] of first computing the space of weight 2 Eisenstein series for H . Compared to other methods, our method has the disadvantage of requiring a much larger number of Fourier coefficients. However, it only requires working with modular forms defined on H , and our method may be substantially more efficient if the index of $\Gamma(N)$ in H is large. The addition of modular forms functionality to PARI/GP in [BC18] uses the strategy of multiplying Eisenstein series.

The next task is to construct the Fourier coefficients of weight 2 cusp forms from those of the Eisenstein series. We do this by multiplying triples of weight 2 Eisenstein series together to obtain a large collection of weight 6 modular forms. Because we know how each weight 2 Eisenstein series transforms under matrices in $GL_2(N)$, we can keep track of the values of these products of the Eisenstein series at each cusp of X_H . There is no reason to expect that this is the entirety of the space of weight 6 modular forms because weight 2 modular forms must have a zero at each elliptic point of order 2 and a double zero at each elliptic point of order 3. One can use the Riemann–Roch theorem to compute that the dimension of the subspace of weight 6 modular forms with a zero of order ≥ 3 at each elliptic point of order 2 and zero of order ≥ 6 at each elliptic point of order 3 is $3v_\infty(\Gamma_H) + 5(g - 1)$. This allows one to avoid computing products of all triples of weight 2 Eisenstein series.

Next, one applies Hecke operators to compute the full space of weight 6 modular forms. In [Kat04], Kato defines a family of Hecke operators $T(n) \begin{bmatrix} n & 0 \\ 0 & 1 \end{bmatrix}^*$ for modular forms on $\Gamma(N)$ and gives their action on Fourier expansions for $\gcd(n, N) = 1$. As noted by Kato, this normalisation commutes with the action of $GL_2(N)$, but it does not preserve the connected components of $X(N)$.

Let w be the width of the cusp at ∞ for Γ_H , and suppose that $f(z) = \sum_{n=0}^\infty a(n)q^{n/w}$ is a modular form of weight k for Γ_H , $q = e^{2\pi iz}$. Then Kato’s formulas show that if p is a prime with $p \equiv \pm 1 \pmod{N}$

$$\left(\sum_{n=0}^\infty a(n)q^{n/w} \right) |T(p) = \sum_{n=0}^\infty a(pn)q^{n/w} + p^{k-1} \sum_{n=0 \pmod{p}}^\infty a(n/p)q^{n/w}.$$

The case $p \equiv -1 \pmod{N}$ requires $M := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in H$, so we conjugate H to contain M .

We choose the smallest prime $p \equiv \pm 1 \pmod{N}$ and apply the Hecke operator $T(p)$ to all the products of three weight 2 Eisenstein series. While we are not guaranteed to obtain the entire space $M_6(\Gamma(N), \mathbb{Q}(\zeta_N))^H$, in the cases under consideration we do. It is not difficult to see how the Hecke operator affects values at the cusps, and in this way, we obtain a basis for the space $S_6(\Gamma(N), \mathbb{Q}(\zeta_N))^H$. We apply the same tools to obtain a basis for the space $S_8(\Gamma(N), \mathbb{Q}(\zeta_N))^H$.

Let h be a meromorphic modular form on $\Gamma(N)$. We use the elementary observation that since the zero sets of $E_4(z)$ and $E_6(z)$ are disjoint, $h \in S_2(\Gamma(N), \mathbb{Q}(\zeta_N))^H$ if and only if $hE_4 \in S_6(\Gamma(N), \mathbb{Q}(\zeta_N))^H$ and $hE_6 \in S_8(\Gamma(N), \mathbb{Q}(\zeta_N))^H$. In this way, one obtains q -expansions for the weight 2 cusps forms for H . From these, we can easily obtain the canonical map for X_H and its image in \mathbb{P}^{g-1} and then check whether the curve is hyperelliptic. Let Ω be the sheaf of holomorphic 1-forms on the curve X_H . If $g(H) \geq 3$ and X_H is not hyperelliptic, the canonical ring $R = \bigoplus_{d \geq 0} H^0(X_H, \Omega^{\otimes d})$ is generated in degree 1 [VZB22, Chapter 2].

Finally, we seek to identify $j(z)$ as a ratio of two elements of the canonical ring. From a computational standpoint, it is easier to write

$$j(z) = \frac{1728E_4(z)^3}{E_4(z)^3 - E_6(z)^2}$$

and find representations of $E_4(z)$ and $E_6(z)$ as ratios of elements of the canonical ring. Let $h(z)$ be an element of the canonical ring of weight $k - 4$. Then $E_4(z) = \frac{f(z)}{h(z)}$ for some $f(z)$ in the canonical ring if and only if $h(z)E_4(z)dz^{k/2}$ is a holomorphic $k/2$ -form. Diamond and Shurman [DS05, p. 86] show that

$$\operatorname{div}(f(z)dz^{k/2}) = \operatorname{div}(f) - \sum \frac{k}{4}x_{2,i} - \sum \frac{k}{3}x_{3,i} - \sum \frac{k}{2}x_i,$$

where $\{x_{2,i}\}$ is the set of elliptic points of order 2, $\{x_{3,i}\}$ is the set of elliptic points of order 3 and $\{x_i\}$ is the set of cusps on X_H . Let $D = -\sum x_{2,i} - \sum x_{3,i} - \sum 2x_i$. Then $h(z)E_4(z)dz^{k/2}$ is holomorphic if and only if $h(z)dz^{(k-4)/2} \in L^{(k-4)/2}(D)$, the space of meromorphic $(k-4)/2$ -forms with poles at most at D . We have (by Lemma VII.4.11 from Miranda [Mir95]) that $L^{(k-4)/2}(D) \simeq L(D + ((k-4)/2)K)$, where K is the canonical divisor. Riemann–Roch now guarantees the existence of such a D , provided that

$$k \geq \frac{2\nu_\infty(\Gamma_H) + \nu_2(\Gamma_H) + \nu_3(\Gamma_H) + 5g(H) - 4}{g(H) - 1}.$$

A similar calculation shows that E_6 is a ratio of a holomorphic $(k/2)$ -form and a holomorphic $((k-6)/2)$ -form, provided that

$$k \geq \frac{3\nu_\infty(\Gamma_H) + \nu_2(\Gamma_H) + 2\nu_3(\Gamma_H) + 7g(H) - 6}{g(H) - 1}.$$

With these bounds, one can use the q -expansions of the weight 2 cusp forms to compute the q -expansions of the modular forms of weight k that correspond to elements of the canonical ring and use linear algebra to compute a ratio that is equal to $E_4(z)$ and $E_6(z)$.

This technique is used to compute the canonical models of $X_{\text{ns}}^+(25)$, $X_{\text{ns}}^+(27)$, a genus 43 curve with label 27.729.43.1 and a genus 36 curve with label 25.625.36.1. A variant of this technique was used to compute a model of a genus 3 modular curve defined over $\mathbb{Q}(\zeta_3)$ (see Subsection 9.1). We compute models for all of the arithmetically maximal modular curves with $3 \leq \ell \leq 13$ with the exceptions of $X_s^+(49)$ with label 49.1372.94.1, which is known not to have any noncuspidal non-CM rational points, 121.605.41.1 which is shown not to have any rational points in Section 8.7 and $X_{\text{ns}}^+(121)$ with label 121.6655.511.1. For the genus 69 curve $X_{\text{ns}}^+(49)$, we obtain a singular plane model, but we are unable to apply the method of Novocin and van Hoeij to simplify the equation.

Example 7.1. Consider the modular curve $X_H = X_{\text{ns}}^+(27)$ with label 27.243.12.1. We compute the Fourier expansions of the 8 Eisenstein series in $M_2(\Gamma(27), \mathbb{Q}(\zeta_{27}))^H$, which are elements of $\mathbb{Q}(\zeta_{27})[[q^{1/27}]]$, and multiply triples of these together. There are 120 such triples; however, the subspace of $M_6(\Gamma(27), \mathbb{Q}(\zeta_{27}))^H$ with triple zeros at the 19 elliptic points of order 2 has dimension 82. The first 82 triples that we try are linearly independent in $M_6(\Gamma(27), \mathbb{Q}(\zeta_{27}))^H$.

The smallest prime $p \equiv \pm 1 \pmod{27}$ is $p = 53$. We apply the Hecke operator $T(53)$ to the first 19 weight 6 forms we computed in the last step and find that these are linearly independent, while for the remaining 63 forms $g, g|T(53)$ is a linear combination of the forms we already have. This gives us a basis for $M_6(\Gamma(27), \mathbb{Q}(\zeta_{27}))^H$, which has dimension 101. We then identify the subspace of cusp forms, which has dimension 92.

Next we find a basis for $M_8(\Gamma(27), \mathbb{Q}(\zeta_{27}))^H$ by multiplying together 4-tuples of weight 2 Eisenstein series. We obtain 113 linearly independent forms this way and 38 additional forms by applying $T(53)$.

This gives us a basis for $M_8(\Gamma(27), \mathbb{Q}(\zeta_{27}))^H$, which has dimension 151. The subspace of cusp forms has dimension 142.

Next, we compute the subspace of $\mathbb{Q}(\zeta_{27})[[q^{1/27}]]$ spanned by the forms g_6/E_4 , where $g_6 \in S_6(\Gamma(27), \mathbb{Q}(\zeta_{27}))^H$, and intersect this with the subspace of $\mathbb{Q}(\zeta_{27})[[q^{1/27}]]$ spanned by the forms g_8/E_6 , where $g_8 \in S_8(\Gamma(27), \mathbb{Q}(\zeta_{27}))^H$. This subspace has dimension 12, so it must equal $S_2(\Gamma(27), \mathbb{Q}(\zeta_{27}))^H$. We compute an LLL-reduced basis for this latter space and an LLL-reduced basis for the space of quadratic relations and find that it has dimension 45. This shows that the image of the canonical map is not a rational normal curve (for in this case, the space of quadratic relations would have dimension 55), and hence X_H is not hyperelliptic and the curve X_H is given by the intersection of our 45 quadrics in \mathbb{P}^{11} .

The Riemann–Roch calculation described above shows that E_4 is a ratio of an element of $H^0(X_H, \Omega^5)$ and one in $H^0(X_H, \Omega^3)$, while E_6 is a ratio of elements in $H^0(X_H, \Omega^6)$ and one in $H^0(X_H, \Omega^3)$. We obtain Fourier expansions of the modular forms of weight $2r$ corresponding to elements of $H^0(X_H, \Omega^r)$ by multiplying together forms in $S_2(\Gamma(27), \mathbb{Q}(\zeta_{27}))^H$. A linear algebra calculation then finds (many) representations of E_4 and E_6 in the desired form. From this, the map $j: X_H \rightarrow \mathbb{P}^1$ can be constructed.

We do not reproduce the model in this paper (the file [modelfile27.243.12.1.txt](#) in [RSZB21] contains Magma code to compute it), but the coefficients are quite small: the largest absolute value of a coefficient is 4. Point searching on this model finds 8 rational points. Two of these 8 points are in the base locus of the map from $X_H \rightarrow \mathbb{P}^1$, and more care is needed to compute their image on the j -line (both map to $j = 1728$). The remaining 6 points map to $j = -3375$, $j = 287496$, $j = -884736$, $j = 16581375$, $j = -147197952000$ and $j = -262537412640768000$, all of which are CM points. The total time required for the computation of the model and the map to the j -line is about 12 minutes.

8. Analysis of rational points

In this section, we determine the rational points on $X_H(\mathbb{Q})$ for each of the 21 arithmetically maximal subgroups that appear in Table 5 without an asterisk.

8.1. Genus 1

The three curves of genus 1 listed in Table 5 have labels

$$9.12.1.1, \quad 9.54.1.1, \quad 27.36.1.2,$$

whose rational points are as follows:

- [9.12.1.1](#) is a degree 3 cover of $X_0(3)$ isomorphic to the elliptic curve $y^2 + y = x^3$. It has three rational points, two of which are cusps; the other is a CM point with $j = 0$.
- [9.54.1.1](#) also known as $X_{\text{sp}}^+(9)$ is also isomorphic to $y^2 + y = x^3$ (different j -map). In this case, the curve has one cusp and two CM points: $j = 8000$ and $j = -32768$.
- [27.36.1.2](#) is isomorphic to the elliptic curve $y^2 + y = x^3 + 2$; it has three rational points, two of which are cusps; the other is a CM point with $j = 0$.

The file [check-for-sporadic-points.m](#) in [RSZB21] verifies these computations.

8.2. Genus 2

The curves corresponding to the labels

$$9.36.2.1, \quad 27.36.2.1, \quad 27.36.2.2 \quad \text{and} \quad 9.54.2.2$$

have genus 2 and rank 0. The first two curves are isomorphic and have minimal Weierstrass model $y^2 + (x^3 + 1)y = -5x^3 - 7$, and $y^2 + (x^3 + 1)y = -2x^3 - 7$ is a minimal Weierstrass model for the third

curve. Using Magma’s RankBound command, it is straightforward to see that the Jacobians of these curves have rank zero; and using the command Chabauty0, one finds that each has two rational points. In each case, the rational points are cusps.

The fourth curve, with minimal Weierstrass model

$$y^2 + (x^2 + x + 1)y = 3x^6 + 2x^4 + 10x^3 + 6x^2 - 32x + 14,$$

has no 3-adic points. The file `genus-2-analysis.m` in [RSZB21] verifies these computations.

8.3. Genus 3

The curves corresponding to the labels

$$9.108.3.1, \quad 27.36.3.1 \quad \text{and} \quad 27.108.3.1$$

have genus 3 and are not hyperelliptic. The first has canonical model

$$X : x^3z - 6x^2z^2 + 3xy^3 + 3xz^3 + z^4 = 0.$$

Since X is a Picard curve (which is clear from the y coordinate), we can use Magma’s RankBound command to compute that the rank of $J_X(\mathbb{Q})$ is zero. In principle, one can now simply compute $J_X(\mathbb{Q}) = J_X(\mathbb{Q})_{\text{tors}}$ and then compute $X(\mathbb{Q})$ as the preimage of an Abel–Jacobi map $X(\mathbb{Q}) \hookrightarrow J_X(\mathbb{Q})$; see [DEvH+, Subsection 5.2] for a longer discussion of such computations.

In practice, we were not able to compute $J_X(\mathbb{Q})_{\text{tors}}$ exactly. By [Kat81, Appendix], reduction modulo an odd prime p gives an injection $J_X(\mathbb{Q})_{\text{tors}} \hookrightarrow J_X(\mathbb{F}_p)$. We compute (using Magma’s ClassGroup command) that $J_X(\mathbb{F}_5) \simeq \mathbb{Z}/198\mathbb{Z}$ and $J_X(\mathbb{F}_{13}) \simeq \mathbb{Z}/45\mathbb{Z} \times \mathbb{Z}/45\mathbb{Z}$. (See [DEvH+, Subsection 4.1] for a longer discussion of computing torsion on modular Jacobians). On the other hand, X has two visible rational points with coordinates $[0 : 1 : 0]$, $[1 : 0 : 0]$ whose difference is a point of order 3 in $J_X(\mathbb{Q})$. Thus the only two possibilities for $J_X(\mathbb{Q})$ are $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/9\mathbb{Z}$. A long search for low-degree points on X does not produce a point of order 9 on $J_X(\mathbb{Q})$, nor do further local computations (including refined techniques such as [OS19, Section 4]) give a better local bound than $\mathbb{Z}/9\mathbb{Z}$. This suggests a local to global failure for $J_X(\mathbb{Q})_{\text{tors}}$.

We circumvent this as follows. We enumerate $X(\mathbb{F}_2)$ and find that for any $P, Q \in X(\mathbb{F}_2)$, the order of $P - Q$ is not equal to 9. Since $J_X(\mathbb{Q})$ has rank 0 and odd order, the reduction map $J_X(\mathbb{Q})_{\text{tors}} \hookrightarrow J_X(\mathbb{F}_2)$ is injective by [Kat81, Appendix]. Thus, there also do not exist points $P, Q \in X(\mathbb{Q})$ such that the order of $P - Q$ is equal to 9 (but note that this does not exclude the possibility that $J_X(\mathbb{Q}) \simeq \mathbb{Z}/9\mathbb{Z}$).

Finally, we compute the preimage of the remaining points under an Abel–Jacobi map. Fix $P = [0 : 1 : 0]$, $P' = [1 : 0 : 0]$ and $D = P' - P$. Let $\iota_P : X \hookrightarrow J_X$ be the map $Q \mapsto Q - P$. By the previous paragraph, for any $Q \in X(\mathbb{Q})$, $\iota_P(Q)$ has order 1 or 3. By definition, $\iota_P(P) = 0 \cdot D$, and $\iota_P(Q) = 1 \cdot D$. Using Magma’s RiemannRoch command, we find that the linear system $|2 \cdot D + P|$ is empty, so there is no $Q \in X(\mathbb{Q})$ such that $\iota_P(Q) = 2 \cdot D$. Since this exhausts $J_X(\mathbb{Q})_{\text{tors}}$, we have computed $X(\mathbb{Q})$. Using our equations for the j -map, we determine that the two known points are both CM points with $j = 0$.

The second curve has canonical model

$$X : -x^3y + x^2y^2 - xy^3 + 3xz^3 + 3yz^3 = 0$$

and is also a Picard curve (which is clear from the z coordinate). It has an automorphism $\iota : X \rightarrow X$ that swaps x and y . The quotient is the elliptic curve $E : y^2 + y = x^3 + 20$, which has rank one and corresponding newform 243.2.a.a. Applying Magma’s RankBound command reveals that the rank of $J_X(\mathbb{Q})$ is also one. In particular, J_X is isogenous to $E \times A$, where A is the abelian surface A corresponding to the newform 243.2.a.d; see Example 6.2. The analytic rank of $A(\mathbb{Q})$ is zero; by Corollary A.8, the algebraic rank is also zero.

One could in principle apply Chabauty’s method, but this is not directly implemented in Magma for nonhyperelliptic curves. We present an alternative argument. For any point $P \in X(\mathbb{Q})$, P and $\iota(P)$ have the same image in $E(\mathbb{Q})$, so $P - \iota(P)$ has trivial image in $E(\mathbb{Q})$. Thus, since $A(\mathbb{Q})$ is torsion, the image of $P - \iota(P)$ in $E(\mathbb{Q}) \times A(\mathbb{Q})$ is also torsion; since the map $J(\mathbb{Q}) \rightarrow E(\mathbb{Q}) \times A(\mathbb{Q})$ is finite, we conclude that $P - \iota(P)$ is an element of $J_X(\mathbb{Q})_{\text{tors}}$.

It thus suffices to compute $J_X(\mathbb{Q})_{\text{tors}}$ and the preimage of the map

$$\begin{aligned} \tau: X(\mathbb{Q}) &\rightarrow J_X(\mathbb{Q})_{\text{tors}} \\ P &\mapsto P - \iota(P). \end{aligned}$$

Local computations show that $J_X(\mathbb{Q})_{\text{tors}}$ is isomorphic to a subgroup of $(\mathbb{Z}/3\mathbb{Z})^2$, and differences of rational points generate a group of order 9.

To compute the preimage of τ , we first note that the τ is injective away from the fixed points of ι . Indeed, if $P - \iota(P) = Q - \iota(Q)$ as divisors, then either $P = Q$, or $P = \iota(P)$ and $Q = \iota(Q)$. Now suppose that $P - \iota(P) \neq Q - \iota(Q)$ as divisors, but $P - \iota(P) \sim Q - \iota(Q)$. If $P = \iota(P)$ but $Q \neq \iota(Q)$, this equivalence gives a g_1^1 , which is a contradiction since X is not rational (and similarly if $Q = \iota(Q)$ but $P \neq \iota(P)$). If $Q \neq \iota(Q)$ and $P \neq \iota(P)$, then instead this gives a g_2^1 on a nonhyperelliptic curve, which is also a contradiction. Thus τ is injective away from the fixed points of ι .

Next, differences of rational points give an explicit basis D_1, D_2 for $J_X(\mathbb{Q})_{\text{tors}}$. For each $a, b \in \mathbb{Z}/3\mathbb{Z}$ such that $(a, b) \neq (0, 0)$, we either find an explicit point P for which we have $P - \iota(P) = aD_1 + bD_2$, or we find a prime p such that there is no $P \in X(\mathbb{F}_p)$ such that $P - \iota(P) = aD_1 + bD_2$ in $J_X(\mathbb{F}_p)$ (essentially, ‘sieving’) and thus no such $P \in X(\mathbb{Q})$. If $(a, b) = (0, 0)$, then $P - \iota(P) = aD_1 + bD_2$ if and only if $P = \iota(P)$: that is, if and only if P is a fixed point of the involution ι ; there are finitely many such points, and they are straightforward to compute. Using our equations for the j -map, we find that two of the known points are cusps and one is a CM point with $j = 0$.

The third curve is isomorphic to the first, which is easy to verify with the explicit models. The 2 rational points are not exceptional; they are both CM points with $j = 0$.

The files [9.108.3.1.m](#), [27.36.3.1.m](#), [27.108.3.1.m](#) in [RSZB21] verify these computations.

8.4. Genus 4

The curves corresponding to the labels

$$\text{9.108.4.1, 27.108.4.1, 27.108.4.3, and 27.108.4.5}$$

have genus 4. The fourth curve (with label [27.108.4.5](#)) has no \mathbb{F}_7 points (see Section 5), and computations with Magma show that the other 3 are all isomorphic. Using the method of [vHN05], we find that they are all isomorphic to the Picard curve

$$X: t(t + 1)y^3 = t^3 - 3t^2 - 6t - 1.$$

(We note that while [vHN05] is unpublished and thus has not been refereed, in our example it is straightforward to verify that our curve is isomorphic to this one.) The Jacobian of the curve X is isogenous to the modular abelian variety [81.2.c.b](#), which has analytic rank 0.

From here the argument is identical to those in Subsection 8.3. Local computations show that $J_X(\mathbb{Q})_{\text{tors}}$ is isomorphic to a subgroup of $(\mathbb{Z}/3\mathbb{Z})^2$, and differences of rational points generate a group of order 9. Using Magma’s `RiemannRoch` command, we compute the preimage of $J(\mathbb{Q})$ under an Abel–Jacobi map and find that $\#X(\mathbb{Q}) = 3$; via Section 5, we compute a priori that for each of the 3 curves, there are 3 rational cusps.

The files [9.108.4.1.m](#), [27.108.4.1.m](#), and [27.108.4.3.m](#) in [RSZB21] verify these computations.

8.5. Genus 6

Let H be the subgroup labeled 27.108.6.1. The curve X_H has genus 6; it is a Picard curve and admits an automorphism ι of order 2 defined over \mathbb{Q} . The quotient curve Y has genus 3 and appears to have 11 rational points; its Jacobian J_Y is geometrically simple and is isogenous to the modular abelian variety 243.2.a.f, which has analytic rank 3. The Jacobian J_H of X_H is isogenous to $J_Y \times A$, where A is the geometrically simple modular abelian variety 243.2.a.e, which has analytic rank 0.

We are thus in a similar situation to the group with label 27.36.3.1 from Subsection 8.3: every rational point $P \in X_H(\mathbb{Q})$ has the property that $P - \iota(P)$ is torsion in $J(X_H)$. Local computations show that $J_H(\mathbb{Q})_{\text{tors}}$ is isomorphic to a subgroup of $(\mathbb{Z}/3\mathbb{Z})^2$. (In fact, we can show that $J_H(\mathbb{Q})_{\text{tors}} \simeq (\mathbb{Z}/3\mathbb{Z})^2$, but we will not need this fact.)

As in Subsection 8.3, the map

$$\begin{aligned} \tau : X(\mathbb{Q}) &\rightarrow J_H(\mathbb{Q})_{\text{tors}} \\ P &\mapsto P - \iota(P). \end{aligned}$$

is injective away from the fixed points of ι . Additionally, since $\#J_H(\mathbb{Q})_{\text{tors}}$ is odd, the reduction map

$$J_H(\mathbb{Q})_{\text{tors}} \rightarrow J_H(\mathbb{F}_2)$$

is injective [Kat81, Appendix]. The involution ι has two fixed points P and Q , both of which are rational. Let $U = X_H - \{P, Q\}$, and consider the commutative diagram

$$\begin{array}{ccc} U(\mathbb{Q}) & \xrightarrow{\tau} & J_H(\mathbb{Q})_{\text{tors}} \\ \text{red}_{U,2} \downarrow & & \downarrow \text{red}_{J,2} \\ U(\mathbb{F}_2) & \xrightarrow{\tau_2} & J_H(\mathbb{F}_2). \end{array}$$

Since τ and $\text{red}_{J,2}$ are injective, $\text{red}_{U,2}$ is also injective. We compute in Magma that $\#U(\mathbb{F}_2) = 2$ and that both points lift to \mathbb{Q} . We conclude that $\#X(\mathbb{Q}) = 4$.

Finally, using our equations for the j -map, we compute that the four known points are CM with $j = 0$ and $j = 54000$.

The file 27.108.6.1.m in [RSZB21] verifies these computations.

8.6. Genus 36 and 43

Let H be the group with label 27.729.43.1. The curve X_H has genus 43 and is a degree 27 cover of the curve with label 9.27.0.1, the modular curve that is the subject of the paper [Eik06]. If E/\mathbb{Q} is an elliptic curve and the image of $\rho_{E,3^\infty}$ is contained in H , then $\mathbb{Q}(E[27]) = \mathbb{Q}(E[3], \zeta_{27})$. This remarkable situation is incompatible with most known possibilities for $\rho_{E,3^\infty}(G_{\mathbb{Q}_3})$, which suggests that X_H has no 3-adic points.

We verify this as follows. Using the methods described in Section 7, we compute the canonical model for X_H in \mathbb{P}^{42} . We successfully compute a basis for the space of weight 6 and weight 8 modular forms for H (which have dimensions 318 and 438, respectively). In general, a nonhyperelliptic curve of genus g contains $\binom{g-2}{2}$ quadrics as generators for the canonical ideal, and these generate the canonical ideal provided $g \geq 4$, the curve is not trigonal, and, if $g = 6$, the curve is not a smooth plane quintic [Pet23], [GL85, Corollary 1.7, Remark 1.9]. Indeed, the curve X_H is not trigonal, so our canonical model is given by the vanishing of $\binom{41}{2} = 820$ quadrics in \mathbb{P}^{42} .

The model computation takes about 10 hours. We are able to represent E_4 as a ratio of a degree 4 and a degree 2 element in the canonical ring and represent E_6 as a ratio of a degree 5 and a degree 2 element in the canonical ring. However, memory limitations do not allow us to use these to represent

j as a ratio of elements in the canonical ring. Using the expressions for E_4 and E_6 to compute j gives a ratio of elements each of degree 18. Note that a generic homogeneous degree 18 polynomial in 43 variables is a linear combination of $\binom{60}{18} = 925029565741050$ monomials.

We use this model to show that X_H has no 3-adic points using a suggestion of David Zywina. First we find that the reduction mod 3 of our model has 19 points over \mathbb{F}_3 ; this reduction is not one-dimensional. For each point $(a_0 : \dots : a_{42})$ on $X_H(\mathbb{F}_3)$, we plug $a_i + 3x_i$ into the defining equations for X_H/\mathbb{Z} ; here a_i and x_i represent the first two digits in the 3-adic expansion of the i th coordinate. Dividing by 3 and reducing modulo 3 represents the mod 9 lifts of $(a_0 : \dots : a_{42})$ as a system of linear equations in x_0, \dots, x_{42} . We find that in all cases, this system of linear equations has no solutions, so $X_H(\mathbb{Z}/9\mathbb{Z})$ is empty. The total running time to verify this is approximately 20 seconds. This technique computes the fibres of the map $\text{Gr}_2(X_H)(\mathbb{F}_p) \rightarrow X_H(\mathbb{F}_p)$, where Gr_2 denotes the level 2 Greenberg functor. For a short introduction to the Greenberg transform, see [Poo01, Section 3].

The group H with label 25.625.36.1 is similarly remarkable and has the following properties: $\#H(25) = 5 \cdot \#H(5) = \#\text{GL}_2(\mathbb{F}_5)$ (i.e., the kernel reduction map $H(25) \rightarrow H(5)$ has order 5), the image of $H(5)$ in $\text{PGL}_2(\mathbb{F}_5)$ is S_4 , and if E/\mathbb{Q} is an elliptic curve such that $\rho_{E,5^\infty}(G_{\mathbb{Q}}) \leq H$, then $\mathbb{Q}(E[25]) = \mathbb{Q}(E[5], \zeta_{25})$ or $\zeta_{25} \in \mathbb{Q}(E[5])$. The procedure described above for the group 27.729.43.1 also works here and shows that $X_H(\mathbb{Z}/25\mathbb{Z})$ is empty.

The files 27.729.43.1.m and 25.625.36.1.m in [RSZB21] verify these computations.

8.7. Genus 41: an equationless Mordell–Weil sieve

Let H be the subgroup with label 121.605.41.1. The associated curve X_H has genus 41, and its Jacobian has analytic rank 41. We were unable to compute a model for X_H but will show that $X_H(\mathbb{Q})$ is empty by Mordell–Weil sieving without using equations (in other words, sieving ‘via moduli’).

The reduction $H(11) \bmod 11$ is a subgroup of $N_{\text{ns}}(11)$, so there is a map $\pi: X_H \rightarrow X_{\text{ns}}^+(11)$. The curve $X_{\text{ns}}^+(11)$ is an elliptic curve with Mordell–Weil group $X_{\text{ns}}^+(11)(\mathbb{Q}) \simeq \mathbb{Z}$. We will perform a Mordell–Weil sieve (see [BS10]) in the following sense: for a set S of primes of good reduction (i.e., $11 \notin S$), the diagram

$$\begin{array}{ccc} X_H(\mathbb{Q}) & \xrightarrow{\pi} & X_{\text{ns}}^+(11)(\mathbb{Q}) \\ \alpha \downarrow & & \downarrow \beta \\ \prod_{p \in S} X_H(\mathbb{F}_p) & \xrightarrow{\pi_S} & \prod_{p \in S} X_{\text{ns}}^+(11)(\mathbb{F}_p). \end{array}$$

is commutative. We claim that for $S = \{13, 307\}$, the intersection of the images of the maps π_S and β is empty, and in particular $X_H(\mathbb{Q})$ is empty.

In principle, this is a straightforward computation to implement. In practice, we do not have explicit equations for the map $\pi: X_H \rightarrow X_{\text{ns}}^+(11)$. Nonetheless, for $p \neq 11$, we can determine whether a point $P \in X_{\text{ns}}^+(11)(\mathbb{F}_p)$ is the image under π of some point of $X_H(\mathbb{F}_p)$ by working directly with the (generalised) elliptic curve E corresponding to P , as follows. The mod 11 image $\rho_{E,11}(G_{\mathbb{F}_p})$ is a subgroup of $N_{\text{ns}}^+(11)$. To prove there does not exist a point $Q \in X_H(\mathbb{F}_p)$ such that $\pi(Q) = P$, it suffices to show that the mod 121 image $\rho_{E,121}(G_{\mathbb{F}_p})$ is not a subgroup of H . These images are cyclic groups generated by the image of Frobenius.

The reduction mod N of the matrix A_π from equation (5.2) is the matrix of Frobenius acting on $E[N](\mathbb{F}_p)$ (with respect to some basis) and is computed by the function `GL2nTorsionFrobenius` in the file g12.m in [RSZB21]. We compute this and check whether it (or a conjugate) is contained in H ; if not, then P is not in the image of $\pi \pmod p$.

We can thus perform the Mordell–Weil sieve ‘at the level of moduli’. For our chosen generator R of $X_{\text{ns}}^+(11)(\mathbb{Q})$, we find that, for $p = 13$, a hypothetical point of $X_H(\mathbb{Q})$ maps to $n \cdot R$, where n is 1 or 5

mod 7; and for $p = 307$, any point of $X_H(\mathbb{Q})$ maps to $n \cdot R$, where n is 2, 3, 4, 7, 10 or 13 mod 14; we conclude that $X_H(\mathbb{Q})$ is empty.

The file [121.605.41.1.m](#) in [RSZB21] verifies these computations.

9. Comments on the remaining cases

In this section, we discuss our (unsuccessful) efforts to determine the rational points on the six curves marked with asterisks in Table 5, which also appear in Table 2. Each of these curves has no rational cusps and at least one rational CM point; our analysis suggests that there are no exceptional points on any of these curves, but we are unable to prove this.

9.1. $X_{ns}^+(27)$

While we are not able to determine the rational points on $X = X_{ns}^+(27)$ (labeled 27.243.12.1), we will describe our attempts. As described in Section 7, we compute the canonical model of $X_{ns}^+(27)$, which has genus 12.

Rather than try to work directly with X , we work with a modular curve X_H for which $\det H$ is not surjective and whose geometric connected components are defined over $\mathbb{Q}(\zeta_3)$ rather than \mathbb{Q} . Although $N_{ns}(27)$ is a maximal subgroup of index 9 in $N_{ns}(9)$, if we let $D = \{g \in \text{GL}_2(\mathbb{Z}_3) : \det(g) \equiv 1 \pmod{3}\}$, then there is a nontrivial subgroup between $N_{ns}(27) \cap D$ and $N_{ns}(9) \cap D$. This is the subgroup $H = \left\langle \begin{bmatrix} 0 & 26 \\ 4 & 6 \end{bmatrix}, \begin{bmatrix} 10 & 1 \\ 25 & 26 \end{bmatrix} \right\rangle \leq \text{GL}_2(27)$ with label 3.2.1-27.81.3.1. Let X'_H denote one of the connected components of $X_{H, \mathbb{Q}(\zeta_3)}$ (the components are all isomorphic). The curve X'_H has genus 3, and we use a variant of the method from Section 7 to compute a model. We find that X'_H is the smooth plane quartic

$$a^4 + (\zeta_3 - 1)a^3b + (3\zeta_3 + 2)a^3c - 3a^2c^2 + (2\zeta_3 + 2)ab^3 - 3\zeta_3ab^2c + 3\zeta_3abc^2 - 2\zeta_3ac^3 - \zeta_3b^3c + 3\zeta_3b^2c^2 + (-\zeta_3 + 1)bc^3 + (\zeta_3 + 1)c^4 = 0.$$

There is a degree 3 morphism $X \rightarrow X'_H$ defined over $\mathbb{Q}(\zeta_3)$, so every $\mathbb{Q}(\zeta_3)$ point of X maps to a $\mathbb{Q}(\zeta_3)$ point of X'_H . There are at least 13 points in $X'_H(\mathbb{Q}(\zeta_3))$. These include the images of the 8 rational points on $X_{ns}^+(27)$, a CM point with discriminant -43 , three more CM points with discriminant -4 and a non-CM point corresponding to an elliptic curve with j -invariant $2^3 \cdot 5^3 \cdot 19^{-27} \cdot (-15 - 2\zeta_3)^3 \cdot (452 + 431\zeta_3)^3 \cdot (53515427 + 41314914\zeta_3)^3$. The restriction of scalars $\text{Res}_{\mathbb{Q}(\zeta_3)/\mathbb{Q}} J(X'_H)$ is isogenous to J_H , and Theorem A.7 shows that J_H is isogenous to the \mathbb{Q} -simple abelian variety associated to 729.2.a.c with analytic rank 6.

We searched for points on $X_{ns}^+(27)$ by enumerating rational points on $X_{ns}^+(9) \cong \mathbb{P}^1$ and seeing if they lift to X'_H . We tested all points on $X_{ns}^+(9)$ with height ≤ 1000 , and the only points we found that lifted were 8 CM points (two with discriminant $D = -4$ and one each for $D = -7 - 16, -19, -28, -67, -163$). The computation takes approximately two hours.

It is not hard to find a divisor D supported at 4 of the $\mathbb{Q}(\zeta_3)$ points of X'_H whose image in $J(X'_H)$ has order 3. Using this point, one can construct a family $\{D_\delta\}$ of 9 étale triple covers of X'_H . Counting points on these genus 7 curves over finite fields strongly suggests that each of these genus 7 curves D_δ maps to an elliptic curve $E/\mathbb{Q}(\zeta_3)$. One can search for elliptic curves defined over $\mathbb{Q}(\zeta_3)$ with bad reduction only at $\langle 1 - \zeta_3 \rangle$ and use this to predict, for each twist D_δ , which elliptic curve it maps to. For 8 of the 9 twists D_δ , the elliptic curve in question has rank 0 or 1, which should in principle make an elliptic curve Chabauty computation feasible.

Six of the 13 known $\mathbb{Q}(\zeta_3)$ points of X'_H lift to the final twist D_1 . Since the cover $D_1 \rightarrow X'_H$ is étale, each of those six points on X'_H has 3 $\mathbb{Q}(\zeta_3)$ -rational preimages on X_K , so D_1 has at least 18 $\mathbb{Q}(\zeta_3)$ -rational points. For D_1 , the elliptic curve in question is $E : y^2 = x^3 - 48$, and for this curve $E(\mathbb{Q}(\zeta_3)) \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}^2$. We explicitly compute the map from $D_1 \rightarrow E$ over $\mathbb{Q}(\zeta_3)$ as follows. We attempt to find the composition $D_1 \rightarrow E \rightarrow \mathbb{P}^1$ given by the x -coordinate map. If the degree of the map

from $D_1 \rightarrow E$ is d , then the map $x: D_1 \rightarrow \mathbb{P}^1$ has degree $2d$ and each pole of x has even multiplicity. First we pick $\mathfrak{p} = \langle 1 - 2\zeta_3 \rangle$ to be one of the prime ideals above 7 in $\mathbb{Q}(\zeta_3)$, and we enumerate effective divisors D of degree d on D_1/\mathbb{F}_7 . We search for divisors such that the Riemann–Roch space of $2D$ is nontrivial. If $f \in L(2D)$ is nonconstant, then we build the function field of D_1/\mathbb{F}_7 as an extension of $\mathbb{F}_7(f)$ and search to see if there are intermediate fields. If so, this produces a nontrivial morphism from D_1 to a lower genus curve. This technique succeeds for $d = 3$, and we find a map from $D_1 \rightarrow E$ defined over \mathbb{F}_7 . The x -coordinate map can be written as a ratio of degree 1 elements of the canonical ring for D_1 , and the y -coordinate map can be written as a ratio of degree 2 elements.

From here, we write down a scheme X that parametrises maps from $D_1 \rightarrow E$ of the form we found, with the additional constraint that one particular $\mathbb{Q}(\zeta_3)$ -rational point on D_1 maps to $(0 : 1 : 0)$. The map from $D_1 \rightarrow E$ gives rise to a point on X , and we use Hensel’s lemma to lift this point to a point modulo $\langle 1 - 2\zeta_3 \rangle^{512}$. We then use Magma’s `Reconstruct` function to find simple elements of $\mathbb{Q}(\zeta_3)$ that reduce modulo $\langle 1 - 2\zeta_3 \rangle^{512}$ to the point found on X . This yields the desired map to E .

Using the map $\phi: D_1 \rightarrow E$, we can create an étale triple cover of D_1 via the fibre product

$$\begin{array}{ccc} D_1 \times_E E & \longrightarrow & D_1 \\ \downarrow & & \downarrow \phi \\ E & \xrightarrow{\psi} & E \end{array}$$

where $\psi: E \rightarrow E$ is a cyclic 3-isogeny defined over $\mathbb{Q}(\zeta_3)$ (which exists because E has complex multiplication). The 18 known $\mathbb{Q}(\zeta_3)$ rational points become equidistributed among the nine twists of $D_1 \times_E E$. Our hope was that we could find a map from $D_1 \times_E E$, which has genus 19, to a lower genus curve. We can compute the zeta function of $D_1 \times_E E$ over \mathbb{F}_4 by counting points on it over \mathbb{F}_{4^k} for $1 \leq k \leq 12$, since we already know a 7-dimensional factor (namely $J(D_1)$) of $J(D_1 \times_E E)$. We find that the dimension 12 ‘new’ part of the Jacobian of $D_1 \times_E E$ is irreducible. We have not been able to provably determine $D_1(\mathbb{Q}(\zeta_3))$ or $X_{\text{ns}}^+(27)(\mathbb{Q})$.

9.2. $X_{\text{ns}}^+(\ell^2)$, $\ell = 5, 7, 11$

Using the techniques of Section 7, we have computed the canonical model of $X_{\text{ns}}^+(25)$, which has genus 14. We searched for points on $X_{\text{ns}}^+(25)$ by enumerating points on $X_{\text{ns}}^+(5) \cong \mathbb{P}^1$ with height ≤ 1000 , writing down the corresponding elliptic curve E and testing if the pairs $(p + 1 - \#E(\mathbb{F}_p) \bmod 25, p \bmod 25)$ are in $\{(\text{tr } g, \det g) : g \in N_{\text{ns}}(25)\}$. This process rules out all non-CM j -invariants in the range we tested and takes approximately 6 hours.

Point searching finds 8 rational points, all of which are CM with discriminants $-3, -7, -12, -27, -28, -43, -67$ and -163 . The Jacobian has three 2-dimensional factors, and this raises the possibility that $X_{\text{ns}}^+(25)$ might map to a genus 2 hyperelliptic curve; two genus 2 curves X_H whose Jacobians are factors of $J(X_{\text{ns}}^+(25))$ are those with labels 25.50.2.1 and 25.75.2.1. The automorphism group of $X_{\text{ns}}^+(25)$ is trivial over \mathbb{Q} .

If $f: X_{\text{ns}}^+(25) \rightarrow C$ were a map to a genus 2 curve, it seems plausible that the induced map on differentials $\Omega_C \rightarrow \Omega_{X_{\text{ns}}^+(25)}$ should respect the action of the Hecke operators. We have identified the two-dimensional Hecke stable subspaces of $\Omega_{X_{\text{ns}}^+(25)}$ (canonically identified with weight 2 cusp forms) and checked that these do not give rise to a map to a genus 2 curve.

For $X_{\text{ns}}^+(49)$, we applied the same technique to search for points as for $X_{\text{ns}}^+(25)$ and found no non-CM rational points.

For $X_{\text{ns}}^+(121)$ with genus 511, we did not compute a model. However, $X_{\text{ns}}^+(11)$ is isomorphic to the elliptic curve $E: y^2 + y = x^3 - x^2 - 7x + 10$ with Mordell–Weil group $E(\mathbb{Q}) \cong \mathbb{Z}$ generated by $P = (4 : 5 : 1)$. We use the isomorphism computed in [CC04]. We used this to Mordell–Weil sieve (as in Section 8.7). As a result, if $Q \in X_{\text{ns}}^+(121)$ is a rational point and its image on $X_{\text{ns}}^+(11)$ is kP , then either $k \in \{-5, -4, -3, -2, -1, 0, 1\}$, which implies that Q is CM, or $|k| \geq \text{lcm}(1, 2, 3, \dots, 233)$. This

implies that if Q is a non-CM point on $X_{\text{ns}}^+(121)$, then the height of $j(Q)$ would need to be roughly on the order of $10^{7 \cdot 92 \cdot 10^{202}}$ (or larger).

9.3. The curves 49.147.9.1 and 49.196.9.1

The curve 49.147.9.1 is a degree 7 cover of $X_{\text{ns}}^+(7)$, which has genus zero. Using this, we have a singular degree 7 plane model of the modular curve. We searched for points on our plane model with height up to 10^7 and found a unique rational point (above $j = 0$). The Jacobian is irreducible and has trivial torsion subgroup over \mathbb{Q} . The automorphism group of the modular curve is also trivial over \mathbb{Q} . We have not made any further progress with this curve.

The modular curve 49.196.9.1 is a degree 7 cover of $X_{\text{s}}^+(7)$, which also has genus zero. We likewise have a singular degree 7 plane model for it and searched for points with height up to 10^7 . Again, it appears to have a unique rational point (above $j = 0$) on it. The curve does not have any nontrivial automorphisms over \mathbb{Q} , and the torsion subgroup of its Jacobian is trivial. We have not made any further progress with this curve.

10. The cases with $-I \notin H$

We now suppose that $H \leq \text{GL}_2(N)$ has genus 0, $-I \notin H$ and $X_{\langle H, -I \rangle} \cong \mathbb{P}^1$. Let U be the complement of the cusps and preimages on X_H of $j = 0, 1728$. Then $U \subseteq \text{Spec } \mathbb{Q}[t]$ is an affine scheme, and the moduli problem of H -level structures is rigid. Therefore, by [KM85, Scholie 4.7.0, page 111], the moduli problem there is representable and there is a universal curve $\mathcal{E} \rightarrow U$ such that an elliptic curve E/K with $j(E) \neq 0, 1728$ has $\text{im } \rho_{E,N} \leq H$ if and only if $E \simeq \mathcal{E}_t$ for some (possibly more than one) $t \in U(K)$. The following is [RZB15, Lemma 2.4].

Lemma 10.1. *Let $f: \mathcal{E} \rightarrow U$ be as above, and assume that $U \subset \mathbb{A}^1$. Then there exists a closed immersion $\mathcal{E} \hookrightarrow \mathbb{P}_U^2$ given by a homogeneous polynomial*

$$Y^2Z - X^3 - aXZ^2 - bZ^3,$$

where $a, b \in \mathbb{Z}[t]$.

In [RZB15, Section 5.2], the authors proved [RZB15, Lemma 5.1] that if E/K is an elliptic curve for which $\text{im } \rho_{E,N} \leq \langle H, -I \rangle$, then there is a unique quadratic twist E_d of E so that the image of $\rho_{E_d,N}$ is contained in H (where $\rho_{E_d,N}$ is defined using a basis for $E_d[N]$ that is compatible with the choice for E). See also [Sut16, Section 5.6]. It follows that if E has ℓ -adic image $\langle H, -I \rangle$, then the number of quadratic twists E_d for which the ℓ -adic image is H is equal to the number of index two subgroups of $\langle H, -I \rangle$ that are $\text{GL}_2(\mathbb{Z}_\ell)$ -conjugate to H . It is straightforward to see that this is $[N_{\text{GL}_2(\mathbb{Z}_\ell)}(\langle H, -I \rangle) : N_{\text{GL}_2(\mathbb{Z}_\ell)}(H)]$. If $j \neq 1728$, then $E_d \cong E_{d'}$ if and only if $d/d' \in (K^\times)^2$.

The authors of [RZB15] gave a procedure (based on a general theory of resolvent polynomials from [DD13]) to represent $\mathcal{E} \rightarrow U$ as an elliptic curve over $\mathbb{Q}(t)$.

This technique works in general, and the same techniques are used to compute models for the universal elliptic curves over X_H for all arithmetically maximal H for which $X_{\langle H, -I \rangle}(\mathbb{Q})$ is infinite. In all such cases, $X_{\langle H, -I \rangle} \simeq \mathbb{P}^1$. As an example, if E/\mathbb{Q} is an elliptic curve with $E[5] \simeq \mathbb{Z}/5\mathbb{Z} \times \mu_5$ as Galois modules, then there is a $t \in \mathbb{Q}$ with $t \neq 0$ so that $E \simeq E_t$, where

$$E_t : y^2 = x^3 + (-27t^{20} - 6156t^{15} - 13338t^{10} + 6156t^5 - 27)x + (54t^{30} - 28188t^{25} - 540270t^{20} - 540270t^{10} + 28188t^5 + 54).$$

This corresponds to the 5-adic image with label 5.120.0.1.

These models allow us to efficiently compute the ℓ -adic image of Galois for non-CM elliptic curves E/\mathbb{Q} ; see Section 11 for details.

11. Computing ℓ -adic images of non-CM elliptic curves over \mathbb{Q}

In this section, we present the algorithm promised in the introduction, which takes as input an elliptic curve E/\mathbb{Q} without complex multiplication and outputs a list of labels that identify all groups of nontrivial index that arise as $\rho_{E,\ell^\infty}(G_{\mathbb{Q}})$ for some prime ℓ . Serre's open image theorem guarantees that this list is finite, and we use an extension of an algorithm due to Zywna [Zyw22, Algorithm 1.1] to compute a finite set S that is guaranteed to contain all the primes ℓ , we must consider. The set S consists of the primes $\ell = 2, 3, 5, 7, 13$, any prime $\ell = 11, 17, 37$ for which $j(E)$ is exceptional and listed in Table 1, and possibly a small number of additional primes. While this 'small number' can be arbitrarily large, it is typically zero unless E has bad or supersingular reduction at many small primes, and in fact it is zero for all non-CM E/\mathbb{Q} of conductor up to 500000. The function `PossiblyNonsurjectivePrimes` in the file `g12.m` in [RSZB21] implements this computation.

11.1. Computing Frobenius matrices

The next step of the algorithm is to compute a sequence of matrices $A_p \in \mathbb{Z}^{2 \times 2}$ for primes $p \leq B$ of good reduction for E whose reductions modulo any integer N coprime to p give the action of the Frobenius endomorphism π of $E_p := E \bmod p$ on a basis for $E_p[N]$. From equation (5.2), we may take $A_p = A_\pi = A(a, b, \Delta)$, where a is the trace of Frobenius, Δ is the discriminant of the ring $R_\pi = \text{End}(E_p) \cap \mathbb{Q}[\pi]$ and $b = [R_\pi : \mathbb{Z}[\pi]]$ when $\mathbb{Z}[\pi] \neq \mathbb{Z}$, and $b = 0$ otherwise.

In [Cen16], Centeleghe gives an algorithm to compute $A(a, b, \Delta)$ using Hilbert class polynomials that is implemented in the Magma package `IntegralFrobenius`. The time and space complexity of this algorithm are both superlinear in p , which means the cost of computing A_p for good primes $p \leq B$ grows quadratically with B , making the algorithm unsuitable for our intended application. Here we describe a more efficient approach whose complexity is subexponential in $\log p$ under the generalised Riemann hypothesis (GRH), meaning that in practice, we can compute A_p for all good primes $p \leq B$ in quasi-linear time.

For $j(E_p) = 0, 1728$, the matrix $A(a, b, \Delta)$ can be read from Table 6 after computing $\text{Aut}(E_p)$ and $\#E_p[2]$, which takes time $(\log p)^{2+o(1)}$. If $j(E_p) \neq 0, 1728$ and E_p is supersingular (which can be determined in expected time $(\log p)^{3+o(1)}$ and heuristically in time $(\log p)^{2+o(1)}$ on average for $p \leq B$ via [Sut12b]), then $a = 0$, and $D = a^2 - 4p = -4p$ implies that either $b = 1$ and $\Delta = -4p$ (which must occur if $p \equiv 1 \pmod{4}$) or $b = 2$ and $\Delta = -p$, which occurs precisely when E_p is on the surface of its 2-volcano (the connected components of its 2-isogeny graph over \mathbb{F}_p ; see [Sut13] for details). We can determine which case applies in $(\log p)^{2+o(1)}$ time by checking whether $\#E_p[2] = 4$.

In all remaining cases, E_p is ordinary and $j(E_p) \neq 0, 1728$, which occurs for 100 percent of the primes $p \leq B$ asymptotically. In this situation, our first step is to compute the trace of Frobenius $a := a_p$, which can be accomplished using Schoof's algorithm [Sch85] in $(\log p)^{5+o(1)}$ time [SS15, Corollary 11]. Alternatively, one can apply the Schoof-Elkies-Atkin (SEA) algorithm [Sch95, Elk98], which takes $(\log p)^{4+o(1)}$ expected time on average under the generalised Riemann hypothesis (GRH) [SS15, Corollary 14], or the average polynomial-time algorithm of [HS16], which takes $(\log p)^{4+o(1)}$ time on average unconditionally. For small values of p , it is practically faster to use Mestre's algorithm (see [Sch95, Sections 2–3]); the Magma function `TraceOfFrobenius` switches between Mestre's algorithm and the SEA algorithm depending on the size of p .

Having computed a , it suffices to compute $b = [\text{End}(E_p) : \mathbb{Z}[\pi]]$, since $\Delta = (a^2 - 4p)/b^2$. The integer b is computed by the algorithm in [BS11] in subexponential expected time $\exp((\log p)^{1/2+o(1)})$ under heuristic assumptions that can be reduced to GRH via [Bis11]. The first step of this algorithm involves an isogeny volcano computation to determine the ℓ -adic valuation of b at any small primes ℓ that divide $v := [\mathcal{O}_K : \mathbb{Z}[\pi]]$, where \mathcal{O}_K is the ring of integers of $K := \mathbb{Q}(\sqrt{a^2 - 4p}) \simeq \text{End}(E) \otimes \mathbb{Q}$. To compute v and its prime divisors ℓ , we factor $D := a^2 - 4p$ to determine the fundamental discriminant $D_0 := \text{disc } K$ for which $D = v^2 D_0$, which can be accomplished in subexponential expected time $\exp((\log p)^{1/2+o(1)})$ via [LJP92] (there are also heuristically faster algorithms).

The ℓ -adic valuation of b is equal to the distance from $j(E)$ to the floor of its ℓ -volcano, which can be computed using the `FINDSHORTESTPATHTOFLOOR` algorithm in [Sut13]. In the typical case where D is not divisible by ℓ^4 , this amounts to checking whether the instantiated modular polynomial $\Phi_\ell(j(E), x) \in \mathbb{F}_p[x]$ splits completely in $\mathbb{F}_p[x]$. Typically v will be quite small. Indeed, for a random integer $-D \in [1, 4p]$, the average size of the positive v for which $D = v^2 D_0$ with D_0 squarefree is $\frac{3}{\pi^2} \log(4p)$, and when v is $O(\log p)$, we are happy to compute the ℓ -adic valuation of b for every prime $\ell|v$ in this fashion. This yields $b \in \mathbb{Z}$ in expected time bounded by $(\log p)^{5+o(1)}$, a bound that includes the time to compute the modular polynomial $\Phi_\ell(x, y)$ via [BLS12]. We may gain a large constant factor speedup (up to 1728 in the best case) by using alternative modular polynomials such as those provided by the `AtkinModularPolynomial` function in `Magma` or one of the modular polynomials considered in [BLS12]. This complexity bound can be improved to $(\log p)^{4+o(1)}$ via [Koh96, Lemma 25], but with the constant factor improvements noted above, it is typically not worth doing this when v is $O(\log p)$ (in [Koh96] v may be as large as $p^{1/6}$).

Even if v is not small, as long as its prime factors ℓ are all small, say $\ell = O(\log p)$, we can achieve the same complexity bound for computing b , which is comparable to the time required to compute a . In the uncommon case that v is divisible by a large prime ℓ , we instead use the volcano climbing algorithm of [Sut11, Section 4.1] to obtain an isogenous elliptic curve E'_p whose endomorphism ring is typically isomorphic to \mathcal{O}_K or a suborder of index ℓ (when there are multiple large prime divisors of v , one needs to consider additional suborders). These cases can be distinguished by searching for smooth relations in the ideal class group of \mathcal{O} and testing them in the isogeny graph of E_p as described in [BS11]. As an additional optimisation, when $|D_0|$ is particularly small, one may instead simply test whether $j(E_p)$ is a root of the Hilbert class polynomial $H_{D_0}(x)$ modulo p (or, better, use the class polynomials provided by the `WeberClassPolynomial` function in `Magma` or one of the class polynomials considered in [ES10]).

The function `EndomorphismRingData` in `gl2.m` in [RSZB21] computes the triple (a, b, Δ) using a simple `Magma` implementation of the algorithm described above that is optimised for the case where v is small and provides reasonably good performance in the range $p \in [1, 2^{30}]$, which is more than sufficient for our intended application. This is used by the function `GL2FrobeniusMatrices` in `gl2.m` in [RSZB21] to compute the matrices A_p for primes p of good reduction for a given elliptic curve E/\mathbb{Q} up to a specified bound B . An asymptotically fast implementation of the algorithm in [BS11] is available in the C library `smoothrelation`.

Table 8 compares the times to compute the Frobenius trace a_p and the Frobenius matrix A_p for all primes $p \leq B$ of good reduction for a non-CM elliptic curve E/\mathbb{Q} using the function `GL2FrobeniusMatrices` described above, and using the `Magma` package `IntegralFrobenius`, which implements the algorithm in [Cen16]; both algorithms use the built-in `Magma` function `TraceOfFrobenius` to compute a_p .

In Table 8, one sees the quasi-linear time complexity of the functions `TracesOfFrobenius` and `FrobeniusMatrices` versus the super-quadratic time complexity of `IntegralFrobenius`. In fact, the

Table 8. Time to compute a_p and A_p for non-CM E/\mathbb{Q} at good $p \leq 2^n$. Times in CPU-seconds on a 4.40GHz Intel i9-9960X core, averaged over the first 100 nonisogenous elliptic curves of conductor greater than 250000 (just the first 10 curves for times over 100s and just the first curve for times over 1000s)

| | 2^8 | 2^{10} | 2^{12} | 2^{14} | 2^{16} | 2^{18} |
|--------------------------------|----------|----------|----------|----------|----------|----------|
| <code>TracesOfFrobenius</code> | 0.000 | 0.001 | 0.006 | 0.026 | 0.118 | 0.52 |
| <code>FrobeniusMatrices</code> | 0.011 | 0.031 | 0.089 | 0.315 | 1.162 | 4.24 |
| <code>IntegralFrobenius</code> | 0.036 | 0.226 | 2.82 | 63.4 | 1910 | 59200 |
| | 2^{20} | 2^{22} | 2^{24} | 2^{26} | 2^{28} | 2^{30} |
| <code>TracesOfFrobenius</code> | 2.34 | 10.6 | 48.1 | 223 | 1010 | 4690 |
| <code>FrobeniusMatrices</code> | 15.8 | 60.9 | 238 | 842 | 3510 | 15700 |
| Relative cost | 6.75 | 5.75 | 4.95 | 3.78 | 3.48 | 3.35 |

running times for FrobeniusMatrices grows sublinearly up to $B = 2^{26}$, and its performance relative to TracesOfFrobenius continues to improve even up to $B = 2^{30}$.

11.2. Proving surjectivity

Having computed an initial set of primes $\ell \in S$, where ρ_{E,ℓ^∞} might be nonsurjective via PossiblyNonsurjectivePrimes, we use FrobeniusMatrices to compute Frobenius matrices A_p for good primes p up to an initial bound B_{\min} (we used $B_{\min} = 256$ in our computations). Let $e_\ell = 3, 2$ for $\ell = 2, 3$, respectively, and let $e_\ell = 1$ for all $\ell \geq 5$. Then ρ_{E,ℓ^∞} is surjective if and only if $\rho_{E,\ell^{e_\ell}}$ is surjective; for $\ell = 2, 3$ this follows from [SZ17, Lemma 3.7] and for $\ell \geq 5$, we may apply [Ser98, Lemma 3, p. IV-23]. By reducing each A_p modulo ℓ^{e_ℓ} , we obtain a set of $\text{GL}_2(\ell^{e_\ell})$ -conjugacy classes C_ℓ that we can typically use to prove that ρ_{E,ℓ^∞} is surjective in any case where this is true. For every maximal subgroup $H \leq \text{GL}_2(\ell^{e_\ell})$, there are several $\text{GL}_2(\ell^{e_\ell})$ -conjugacy classes that do not intersect H . If any of these lie in C_ℓ , then $\rho_{E,\ell^{e_\ell}}(G_\mathbb{Q})$ cannot lie in H ; and if this holds for every maximal subgroup H , then ρ_{E,ℓ^∞} must be surjective. By precomputing sets of conjugacy classes for the maximal subgroups of $\text{GL}_2(\ell^{e_\ell})$ for $\ell \leq 13$, this test can be performed very efficiently (the set S typically contains no primes $\ell > 13$). We remove from S any primes ℓ for which we can prove ρ_{E,ℓ^∞} is surjective using the sets of conjugacy classes C_ℓ .

11.3. Determining nonsurjective ℓ -adic images

Having reduced S to a set of primes ℓ for which we expect ρ_{E,ℓ^∞} to be nonsurjective, we first check whether $j(E)$ is one of the 20 exceptional j -invariants listed in Table 1, and if so, whether E is isomorphic to any of the corresponding exceptional models listed in Table 1. Whenever this applies, we can read the image of ρ_{E,ℓ^∞} from the appropriate row of the table.

Otherwise, we expect that $j(E)$ is not exceptional (we conjecture that this is always true), in which case it corresponds to a point on a modular curve X_H for which $j(X_H(\mathbb{Q}))$ is infinite (possibly more than one such H). Except for the four 2-adic groups with labels

$$8.96.1.9, \quad 8.96.1.18, \quad 8.96.1.75, \quad 8.96.1.102,$$

which were shown to have no noncuspidal rational points in [RZB15], there is no open subgroup $H \leq \text{GL}_2(\widehat{\mathbb{Z}})$ of prime power level for which $j(X_H(\mathbb{Q}))$ is infinite that has a maximal subgroup $H' \leq H$ for which $j(X_{H'}(\mathbb{Q}))$ is finite that contains the same set of $\text{GL}_2(\ell^e)$ -conjugacy classes as H , where ℓ^e is the level of H' . It follows that if we compute sufficiently many Frobenius matrices A_p , for each $\ell \in S$, we should be able to prove that $j(E)$ is not exceptional and obtain a small list of groups H of ℓ -power level for which $j(E)$ is one of infinitely many elements of $j(X_H(\mathbb{Q}))$. Typically, the initial set of Frobenius matrices A_p for good $p \leq B = B_{\min}$ is already sufficient; but if not, we successively double B and compute more Frobenius matrices A_p until this condition holds or we reach a preset maximum B_{\max} (we used $B_{\max} = 2^{20}$ in our computations), at which point we are convinced that we must have found a new exceptional point and compute $\rho_{E,\ell^\infty}(G_\mathbb{Q})$ directly by explicitly computing the torsion fields $\mathbb{Q}(E[\ell^n])$ for sufficiently many powers of ℓ (we must eventually have $[\mathbb{Q}(E[\ell^{n+1}]) : \mathbb{Q}(E[\ell^n])] = \ell^4$ with $\ell > 2$ or $n > 1$, at which point we can stop, by [SZ17, Lemma 3.7]). We conjecture that this will never happen, and it has not happened for any non-CM elliptic curve E/\mathbb{Q} whose ℓ -adic Galois images we have computed; as described in Section 11.4, this includes more than 380 million elliptic curves.

Assuming we do not reach $B = B_{\max}$, for each remaining prime $\ell \in S$, we have a small list of candidate groups H of ℓ -power level for which $j(X_H(\mathbb{Q}))$ is infinite that we know includes the image of ρ_{E,ℓ^∞} . These groups H can have genus at most 1, and we have precomputed explicit models (\mathbb{P}^1 or a positive rank elliptic curve over \mathbb{Q}) and maps $j_H: X_H \rightarrow X(1)$ for each of them, as described in Sections 7 and 10. There is necessarily a unique H of maximal index containing $-I$ for which $j(E) \in j_H(X_H(\mathbb{Q}))$, which we can determine by testing whether $j(E)$ lies in the image of $j_H: X_H \rightarrow X(1)$. If H has genus 0, then $j_H \in \mathbb{Q}(t)$ and $j(E)$ lies in $j_H(X_H(\mathbb{Q}))$ if and only if the numerator of $j_H(t) - j(E) \in \mathbb{Q}(t)$

has a rational root or $j_H(1/t)(0) = j(E)$. When H has genus 1, we have a plane Weierstrass model $X_H: y^2 + h(x)y = f(x)$ and a rational map $j_H(x, y) : X_H \rightarrow X(1)$, and we can use the equations $j_H(x, y) = j(E)$ and $y^2 + h(x) = f(x)$ to solve for y as a function $g(x)$ of x ; for each root x_0 of $j_H(x, g(x)) = j(E)$, we then check whether $g(x_0)^2 + h(x_0) = f(x_0)$ holds. These computations are implemented in the functions `OnGenusZeroCurve`, `OnGenusOneCurve` in the file `gl2data.m` in [RSZB21].

It may happen that the unique H of maximal index containing $-I$ with $j(E) \in j_H(X_H(\mathbb{Q}))$ actually has index 1, in which case we have proved that the Galois representation ρ_{E, ℓ^∞} is surjective; this is rare but may occasionally happen when the initial set of Frobenius matrices A_p for $p \leq B_{\min}$ are not sufficient to rule out every maximal subgroup of $GL_2(\ell^{e\ell})$ as described in Section 11.2. When we encounter this case, we remove ℓ from S and continue.

Having determined $H = \langle \rho_{E, \ell^\infty}(G_{\mathbb{Q}}), -I \rangle$, it remains only to check the index-2 subgroups $H' \leq H$ for which $\langle H', -I \rangle = H$. For each H' , we need to test whether E is isomorphic to the universal elliptic curve E_t for H' instantiated at a value of t for which $j(E_t) = j(E)$ (there are no such H' for any H of genus 1, so we may assume the universal elliptic curve has a single parameter). This amounts to finding the rational roots of an integer polynomial and checking the isomorphism of elliptic curves over \mathbb{Q} , a computation that is implemented in the function `OnGenusZeroCurveTwist` in `gl2data.m` in [RSZB21]. There will necessarily be at most one subgroup H' for which this holds. If there is one, then we have proved that $\rho_{E, \ell^\infty}(G_{\mathbb{Q}}) = H'$, and if not, we have proved that $\rho_{E, \ell^\infty}(G_{\mathbb{Q}}) = H$.

The function `GL2EllAdicImages` in `gl2data.m` in [RSZB21] implements the algorithm we have described in this section. It takes as input a non-CM elliptic curve E/\mathbb{Q} and a collection of precomputed data for the relevant subgroups H , models X_H and maps j_H that can be loaded using the function `GL2Load` in `gl2data.m` in [RSZB21]. It outputs a list of labels that uniquely identify (up to conjugacy) all of the groups H corresponding to the image of a nonsurjective ℓ -adic Galois representation $\rho_{E, \ell^\infty} : G_{\mathbb{Q}} \rightarrow GL_2(\mathbb{Z}_\ell)$ attached to E .

11.4. Database computations

We used the function `GL2EllAdicImages` to compute ℓ -adic images of Galois for the non-CM elliptic curves over \mathbb{Q} in three large databases:

- o The L -functions and Modular Forms Database (LMFDB) [LMF], which includes all elliptic curves E/\mathbb{Q} of conductor up to 500000, of which approximately 3 million do not have complex multiplication (the LMFDB contains additional E/\mathbb{Q} whose ℓ -adic images we also computed, but the results are not included in Table 9).

Table 9. Summary of ℓ -adic image data for non-CM elliptic curves E/\mathbb{Q} in the LMFDB, Stein–Watkins (SW) and Balakrishnan–Ho–Kaplan–Spicer–Stein–Weigandt (BHKSSW) databases. Nonsurjective counts are inclusive and may include curves that are also nonsurjective at another prime.

| Database | Nonsurjective primes | | | | | | | | | Total |
|----------|----------------------|---------|--------|-------|------|------|-----|------|-----------|-----------|
| | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 37 | None | |
| LMFDB | 1357468 | 266426 | 20238 | 3984 | 156 | 536 | 40 | 80 | 1467623 | 3058813 |
| SW | 35598552 | 3671444 | 181224 | 43966 | 2048 | 7444 | 368 | 1024 | 109142150 | 148168204 |
| BHKSSW | 242540 | 8750 | 400 | 108 | 0 | 2 | 44 | 2 | 238447364 | 238698578 |

| Database | Nonsurjective pairs and triples of primes | | | | | | | | |
|----------|---|--------|--------|---------|---------|--------|--------|-----------|-----------|
| | {2, 3} | {2, 5} | {2, 7} | {2, 11} | {2, 13} | {3, 5} | {3, 7} | {2, 3, 5} | {2, 3, 7} |
| LMFDB | 53168 | 3354 | 800 | 148 | 44 | 788 | 240 | 564 | 240 |
| SW | 424566 | 38790 | 11044 | 2048 | 640 | 10832 | 3272 | 7904 | 3272 |
| BHKSSW | 382 | 154 | 62 | 2 | 22 | 42 | 16 | 32 | 16 |

- The Stein–Watkins Database [SW02], which consists of a nearly complete set of elliptic curves E/\mathbb{Q} of absolute discriminant at most 10^{12} that have conductor at most 10^8 or prime conductor at most 10^{10} , which has been filtered to include just one representative of each isogeny class and each class of quadratic twists; it contains more than 148 million elliptic curves without complex multiplication.
- The Balakrishnan–Ho–Kaplan–Spicer–Stein–Weigandt Database [BHK+16], which includes all elliptic curves E/\mathbb{Q} of naive height up to 2.7×10^{10} , including nearly 239 million elliptic curves without complex multiplication.

The total time to compute all ℓ -adic Galois images of every non-CM elliptic curve over \mathbb{Q} in the LMFDB was less than 45 hours, an average of about 42 milliseconds per curve. Most of this time is spent on elliptic curves whose ℓ -adic Galois representation is nonsurjective for one or more primes ℓ . The method given in Section 11.2 for proving nonsurjectivity is very efficient, and for elliptic curves whose ℓ -adic Galois representations are surjective at every prime, it typically takes less than a millisecond to prove this. As can be seen in Table 9, such curves make up almost all of the Balakrishnan–Ho–Kaplan–Spicer–Stein–Weigandt database, and it did not take much longer to compute the ℓ -adic image data for this dataset than it did for the LMFDB. The Stein–Watkins database took substantially longer, nearly two core-years (about five days of parallel computation on 128 cores).

12. Computing ℓ -adic images of CM elliptic curves over \mathbb{Q}

Let E be an elliptic curve over the number field $F = \mathbb{Q}(j(E))$ with potential CM by \mathcal{O} , by which we mean that $\text{End}(E_{\overline{F}}) \simeq \mathcal{O}$ is an order in an imaginary quadratic field K . The possible images of ρ_{E, ℓ^∞} are determined by Lozano-Robledo in [LR]. In this section, we give an efficient algorithm to compute these images. We are primarily interested in the case $F = \mathbb{Q}$, where \mathcal{O} is one of the 13 imaginary quadratic orders of class number one, but we shall treat the more general case of a CM elliptic curve defined over its minimal field of definition, as this requires little additional effort.

12.1. Adelic Cartan groups

For each imaginary quadratic order \mathcal{O} , we define

$$C_{\mathcal{O}} := \varprojlim_N (\mathcal{O}/N\mathcal{O})^\times,$$

where N varies over positive integers ordered by divisibility. We embed $C_{\mathcal{O}}$ in $\text{GL}_2(\widehat{\mathbb{Z}})$ as follows. Let $f := [\mathcal{O}_K : \mathcal{O}]$, let $D := \text{disc } \mathcal{O} = f^2 \text{disc } \mathcal{O}_K$, and let $\phi := f$ if D is odd, with $\phi := 0$ otherwise. Let

$$\omega := \frac{\phi + \sqrt{D}}{2} \quad \text{and} \quad \delta := \frac{D - \phi^2}{4},$$

so that $\mathcal{O} = \mathbb{Z}[1, \omega]$ with $\omega^2 - \phi\omega - \delta = 0$. We now define the Cartan subgroup $\mathcal{C}_{\mathcal{O}} := C_{\mathcal{O}}(\widehat{\mathbb{Z}})$ as the inverse limit of the groups

$$\mathcal{C}_{\mathcal{O}}(N) := \left\{ \begin{bmatrix} a + b\phi & b \\ \delta b & a \end{bmatrix} : a, b \in \mathbb{Z}(N), a^2 + ab\phi - \delta b^2 \in \mathbb{Z}(N)^\times \right\} \subseteq \text{GL}_2(N).$$

It is a closed subgroup of $\text{GL}_2(\widehat{\mathbb{Z}})$ that is isomorphic to $C_{\mathcal{O}}$ via the map induced by

$$a + b\omega \mapsto \begin{bmatrix} a + b\phi & b \\ \delta b & a \end{bmatrix}. \tag{12.1}$$

It follows from the theory of complex multiplication that if the CM field $K := \mathcal{O} \otimes \mathbb{Q}$ lies in the field of definition of an elliptic curve E with CM by \mathcal{O} , then the image of ρ_E lies in $\mathcal{C}_{\mathcal{O}} \leq \text{GL}_2(\widehat{\mathbb{Z}})$ (recall that

we view subgroups of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ as defined only up to conjugation); see [ST68, Theorem 5], for example. This is not true when E is defined over its minimal field of definition $F = \mathbb{Q}(j(E))$ or over any field that does not contain K . In this case, the image of ρ_E lies in the group $\mathcal{N}_{\mathcal{O}} \leq \mathrm{GL}_2(\widehat{\mathbb{Z}})$, which we define as the inverse limit of the groups

$$\mathcal{N}_{\mathcal{O}}(N) := \langle \mathcal{C}_{\mathcal{O}}(N), \begin{bmatrix} -1 & 0 \\ \phi & 1 \end{bmatrix} \rangle.$$

The group $\mathcal{N}_{\mathcal{O}}$ is nonabelian and contains the abelian group $\mathcal{C}_{\mathcal{O}}$ with index 2. We shall use $\mathcal{C}_{\mathcal{O}}(\mathbb{Z}_{\ell})$ and $\mathcal{N}_{\mathcal{O}}(\mathbb{Z}_{\ell})$ to denote the images of these groups in $\mathrm{GL}_2(\mathbb{Z}_{\ell})$ under the projection from $\mathrm{GL}_2(\widehat{\mathbb{Z}}) \simeq \prod_{\ell} \mathrm{GL}_2(\mathbb{Z}_{\ell})$.

Remark 12.1. Our definition of $\mathcal{C}_{\mathcal{O}}$ above differs slightly from that in [LR] for odd N when $D \equiv 1 \pmod{4}$, but the groups are conjugate. This difference simplifies our presentation here and has no impact on our results or those in [LR] on which we rely.

Remark 12.2. The group $\mathcal{N}_{\mathcal{O}}$ is sometimes referred to as the ‘normaliser’ of the Cartan subgroup $\mathcal{C}_{\mathcal{O}}$ in the literature. The group $\mathcal{N}_{\mathcal{O}}(\mathbb{Z}_{\ell})$ is the normaliser of $\mathcal{C}_{\mathcal{O}}(\mathbb{Z}_{\ell})$ in $\mathrm{GL}_2(\mathbb{Z}_{\ell})$, by [LR, Proposition 5.6(2)], but $\mathcal{N}_{\mathcal{O}}$ is not the normaliser of $\mathcal{C}_{\mathcal{O}}$ in $\mathrm{GL}_2(\widehat{\mathbb{Z}})$; indeed, $\mathcal{C}_{\mathcal{O}}$ has infinite index in its normaliser. The group $\mathcal{N}_{\mathcal{O}}(N)$ is properly contained in the normaliser of $\mathcal{C}_{\mathcal{O}}(N)$ in $\mathrm{GL}_2(N)$ for most values of N . This can occur even when N is a prime (if it divides disc \mathcal{O}) and for infinitely many powers of the same prime; see [LR, Proposition 5.6] for examples.

Proposition 12.3. *Let E be an elliptic curve over $F = \mathbb{Q}(j(E))$ with potential CM by \mathcal{O} . There is a choice of $\iota_E: \mathrm{Aut}(E_{\mathrm{tor}}(\overline{F})) \xrightarrow{\sim} \mathrm{GL}_2(\widehat{\mathbb{Z}})$ for which the induced Galois representation $\rho_E: G_F \rightarrow \mathrm{GL}_2(\widehat{\mathbb{Z}})$ satisfies the following:*

- (i) $\rho_E(G_F)$ is an open subgroup of $\mathcal{N}_{\mathcal{O}}$,
- (ii) $[\mathcal{N}_{\mathcal{O}} : \rho_E(G_F)]$ divides $\#\mathcal{O}^{\times}$,
- (iii) $[\rho_E(G_F) : \rho_E(G_F) \cap \mathcal{C}_{\mathcal{O}}] = 2$.

Proof. Parts (i) and (ii) follow from [LR, Theorem 1.1]. For (iii), we note that, as in [Zyw15a, Lemma 7.2], if we pick $\beta \in \mathcal{O} - \mathbb{Z}$ with $\beta^2 \in \mathbb{Z}$ squarefree, then for any $\sigma \in G_F - G_{FK}$, we have $\sigma(\beta P) = \sigma(\beta)\sigma(P) = -\beta\sigma(P)$ for all $P \in E[N]$ and all $N \geq 1$. It follows that $\rho_E(\sigma)\pi(\beta) = -\pi(\beta)\rho_E(\sigma)$, where π is the natural map $\mathcal{O} \rightarrow \mathcal{C}_{\mathcal{O}} \xrightarrow{\sim} \mathcal{C}_{\mathcal{O}}$, thus $\rho_E(G_F)$ is nonabelian. It therefore contains the abelian group $\rho_E(G_G) \cap \mathcal{C}_{\mathcal{O}}$ with index at least 2, and (i) implies that the index is at most 2, since $[\mathcal{N}_{\mathcal{O}} : \mathcal{C}_{\mathcal{O}}] = 2$. □

Proposition 12.4. *Let E be an elliptic curve over $F = \mathbb{Q}(j(E))$ with potential CM by \mathcal{O} , let ℓ be a prime, and let*

$$e = \begin{cases} 4 & \text{if } \ell = 2, \\ 3 & \text{if } \ell = 3, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\rho_{E, \ell^{\infty}}(G_F)$ is the inverse image of $\rho_{E, \ell^e}(G_{\ell})$ under the projection $\mathcal{N}_{\mathcal{O}}(\mathbb{Z}_{\ell}) \rightarrow \mathcal{N}_{\mathcal{O}}(\ell^e)$.

Proof. This follows from Theorems 1.2, 1.4, 1.5, 1.6, 1.7 in [LR] once one notes that for every imaginary quadratic order \mathcal{O} (including $\mathbb{Z}[\zeta_3], \mathbb{Z}[i]$), the possibilities for $\rho_{E, \ell^{\infty}}(G_F) \leq \mathcal{N}_{\mathcal{O}}(\mathbb{Z}_{\ell})$ enumerated in these theorems are nonconjugate modulo ℓ^e . □

Remark 12.5. The values of e in Proposition 12.4 are best possible, even if one restricts to $F = \mathbb{Q}$. If one excludes $\mathcal{O} = \mathbb{Z}[\zeta_3]$, one can take $e = 1$ for $\ell = 3$, but $e = 4$ is necessary for $\ell = 2$ and infinitely many imaginary quadratic orders \mathcal{O} .

Proposition 12.4 implies that to identify $\rho_{E, \ell^{\infty}}(G_F)$, it suffices to specify the order \mathcal{O} (via its discriminant) and the subgroup $\rho_{E, \ell^e}(G_F) \leq \mathcal{N}_{\mathcal{O}}(\ell^e) \leq \mathrm{GL}_2(\ell^e)$ (via its label defined in Section 2.4).

Note that in the CM setting, these labels identify the inverse image of a subgroup of $\mathcal{N}_{\mathcal{O}}(\ell^e) \leq \mathrm{GL}_2(\ell^e)$ in $\mathcal{N}_{\mathcal{O}}(\mathbb{Z}_{\ell})$ rather than in $\mathrm{GL}_2(\mathbb{Z}_{\ell})$.

Remark 12.6. For each prime ℓ , there are infinitely many nonconjugate $\mathcal{N}_{\mathcal{O}}(\mathbb{Z}_{\ell}) \leq \mathrm{GL}_2(\mathbb{Z}_{\ell})$ as \mathcal{O} varies, even though there are only finitely many possibilities for $\mathcal{N}_{\mathcal{O}}(\ell^e)$. It is the pair $(\mathcal{O}, \rho_{E, \ell^e}(G_F))$ that uniquely determines $\rho_{E, \ell^{\infty}}(G_F) \leq \mathcal{N}_{\mathcal{O}}(\mathbb{Z}_{\ell}) \leq \mathrm{GL}_2(\mathbb{Z}_{\ell})$. We should note that $\mathcal{N}_{\mathcal{O}}(\mathbb{Z}_{\ell})$ and $\mathcal{N}_{\mathcal{O}'}(\mathbb{Z}_{\ell})$ may be conjugate even when $\mathcal{O} \neq \mathcal{O}'$; this happens when $\mathrm{disc} \mathcal{O} / \mathrm{disc} \mathcal{O}' = u^2$ with $u \in \mathbb{Z}_{\ell}^{\times}$ (to see this, conjugate equation (12.1) by $\begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix}$).

For any number field F , we say that an elliptic curve E/F with potential CM by \mathcal{O} has maximal ℓ -adic image if $\rho_{E, \ell^{\infty}}(G_F)$ contains $\mathrm{SL}_2(\mathbb{Z}_{\ell}) \cap \mathcal{C}_{\mathcal{O}}(\mathbb{Z}_{\ell})$.

Proposition 12.7. *Let E be an elliptic curve over $F = \mathbb{Q}(j(E))$ with potential CM by $\mathcal{O} \neq \mathbb{Z}[\zeta_3]$. Then E has maximal ℓ -adic image for all primes $\ell \nmid 2 \mathrm{disc} \mathcal{O}$.*

Proof. This is [LR, Theorem 1.2(4)]. □

The following lemma allows us to immediately compute the label of $\rho_{E, \ell}(G_F)$ when E has maximal ℓ -adic image at a prime $\ell > 3$ not dividing $\mathrm{disc} \mathcal{O}$.

Lemma 12.8. *Let E be an elliptic curve over $F = \mathbb{Q}(j(E))$ with potential CM by \mathcal{O} and maximal ℓ -adic image at a prime $\ell > 3$ not dividing $D := \mathrm{disc} \mathcal{O}$. Then $H := \rho_{E, \ell}(G_F)$ has index $i = 2\ell(\ell + (\frac{D}{\ell}))$ in $\mathrm{GL}_2(\ell)$ and genus*

$$g = \begin{cases} (11 + (\ell - 8)\ell - 4(\frac{-3}{\ell}))/24 & \text{if } (\frac{D}{\ell}) = +1, \\ (23 + (\ell - 10)\ell + 6(\frac{-1}{\ell}) + 4(\frac{-3}{\ell}))/24 & \text{if } (\frac{D}{\ell}) = -1. \end{cases}$$

The subgroup H has label $\ell.i.g.1$ under the labeling system defined in §2.4.

Proof. When E has maximal ℓ -adic image $\rho_{E, \ell}(G_F)$ contains an index-2 Cartan subgroup of $\mathrm{GL}_2(\ell)$ that is split if $(\frac{D}{\ell}) = +1$ and nonsplit if $(\frac{D}{\ell}) = -1$. We have $(\mathcal{O}/\ell\mathcal{O})^{\times} \simeq \mathbb{F}_{\ell}^{\times} \times \mathbb{F}_{\ell}^{\times}$ of order $(\ell - 1)^2$ in the split case and $(\mathcal{O}/\ell\mathcal{O})^{\times} \simeq \mathbb{F}_{\ell^2}^{\times}$ of order $\ell^2 - 1$ in the nonsplit case, which implies the index formula, since $\#\mathrm{GL}_2(\ell) = \ell(\ell - 1)(\ell^2 - 1)$. The genus formulas for these groups appear on page 117 of [Maz77a]. For $\ell > 31$, there is a unique conjugacy class of subgroups of index $i = 2(\ell - 1)(\ell \pm 1)$ in $\mathrm{GL}_2(\ell)$ (this follows from Dickson’s classification [Dic58]), and this implies that the last number in the label of H must be 1. A direct calculation shows that this is also true for $3 \leq \ell \leq 31$. □

12.2. CM twists

For each imaginary quadratic order \mathcal{O} , we define

$$z_{\mathcal{O}} := \begin{cases} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} & \text{if } \mathrm{disc} \mathcal{O} = -3, \\ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \text{if } \mathrm{disc} \mathcal{O} = -4, \\ \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} & \text{otherwise,} \end{cases}$$

so that $\langle z_{\mathcal{O}} \rangle$ is the image of $\mathcal{O}^{\times} \leq \mathcal{C}_{\mathcal{O}}$ under the isomorphism defined in equation (12.1). We note that $\mathrm{Aut}(E_{\overline{F}}) \simeq \mathcal{O}^{\times} \simeq \langle z_{\mathcal{O}} \rangle$ via the restriction of $\iota_E: \mathrm{Aut}(E_{\mathrm{tor}}(\overline{F})) \simeq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ in Proposition 12.3. We also note that $\langle z_{\mathcal{O}} \rangle$ is a normal subgroup of $\mathcal{N}_{\mathcal{O}}$.

Recall that a twist of an elliptic curve E/F is an elliptic curve E'/F for which $E'_F \simeq E_{\overline{F}}$, equivalently, for which $j(E') = j(E)$.

Proposition 12.1. *Let E be an elliptic curve over $F = \mathbb{Q}(j(E))$ with potential CM by \mathcal{O} . There is a choice of $\iota_E: \mathrm{Aut}(E_{\mathrm{tor}}(\overline{F})) \xrightarrow{\sim} \mathrm{GL}_2(\widehat{\mathbb{Z}})$ for which the induced Galois representation $\rho_E: G_F \rightarrow \mathrm{GL}_2(\widehat{\mathbb{Z}})$ has image $H = \rho_E(G_F)$ satisfying $\mathcal{N}_{\mathcal{O}} = \langle H, z_{\mathcal{O}} \rangle$. Moreover, every $H \leq \mathcal{N}_{\mathcal{O}}$ with $\langle H, z_{\mathcal{O}} \rangle = \mathcal{N}_{\mathcal{O}}$ occurs as $\rho_{E'}(G_F)$ for some twist E' of E and choice of ι'_E .*

In particular, for every positive integer N , the images $\rho_{E',N}(G_F) \leq \mathcal{N}_{\mathcal{O}}(N)$ realised by twists E' of E are the subgroups $H \leq \mathcal{N}_{\mathcal{O}}(N)$ for which $\langle H, z_{\mathcal{O}} \rangle = \mathcal{N}_{\mathcal{O}}(N)$.

Proof. It follows from [LR, Corollary 4.3] that if we choose ι_E as in Proposition 12.3, then $\langle H, z_{\mathcal{O}} \rangle = \mathcal{N}_{\mathcal{O}}$, which proves the first statement in the proposition.

By [LR, Theorem 6.5], we may assume $\rho_E(G_F) = \mathcal{N}_{\mathcal{O}}$ (replace E by an appropriate twist and choose $\iota_E: \text{Aut}(E_{\text{tor}}(\overline{F})) \rightarrow \text{GL}_2(\widehat{\mathbb{Z}})$ as above) with $\iota_E(\text{Aut}(E_{\overline{F}})) = \langle z_{\mathcal{O}} \rangle \triangleleft \mathcal{N}_{\mathcal{O}}$. Let H be any subgroup of $\mathcal{N}_{\mathcal{O}}$ for which $\langle H, z_{\mathcal{O}} \rangle = \mathcal{N}_{\mathcal{O}}$. Let $A = \langle a \rangle$ be the cyclic subgroup of $\langle z_{\mathcal{O}} \rangle$ of order $m := [\mathcal{N}_{\mathcal{O}} : H]$ so that $\mathcal{N}_{\mathcal{O}} = A \rtimes H$ (note that A is a subgroup of a cyclic normal subgroup, hence normal), and define the character $\chi: G_F \rightarrow \mu_m = \mathcal{O}^\times[m]$ by $\sigma \mapsto \zeta_m^{e_\sigma}$, where $e_\sigma \in [0, \dots, m-1]$ is uniquely determined by $\rho_E(\sigma) \in h_\sigma a^{e_\sigma}$, with $h_\sigma \in H$. Now consider the twist E' of E by χ^{-1} ; it is isomorphic to E over the fixed field of the kernel of χ , which is a cyclic extension of F of degree m . For all $\sigma \in G_F$, we have $\rho_{E'}(\sigma) = \rho_E(\sigma)\iota_E(\chi(\sigma)) = h_\sigma a^{e_\sigma} a^{-e_\sigma} = h_\sigma$, and it follows that $\rho_{E'}(G_F) = H$. \square

In view of Proposition 12.1, we shall call the subgroups $H \leq \mathcal{N}_{\mathcal{O}}$ for which $\langle H, z_{\mathcal{O}} \rangle = \mathcal{N}_{\mathcal{O}}$ twists of $\mathcal{N}_{\mathcal{O}}$, and similarly for $\mathcal{N}_{\mathcal{O}}(\mathbb{Z}_\ell)$ and $\mathcal{N}_{\mathcal{O}}(N)$. The proposition states that the images of Galois that arise among the twists of an elliptic curve $E/\mathbb{Q}(j(E))$ with potential CM by \mathcal{O} are precisely the twists of $\mathcal{N}_{\mathcal{O}}$. It will also be convenient to distinguish twists H of $\mathcal{N}_{\mathcal{O}}$ for which $\langle H, -I \rangle = \mathcal{N}_{\mathcal{O}}$ as quadratic twists, and similarly for $\mathcal{N}_{\mathcal{O}}(N)$. If $\mathcal{O} \neq \mathbb{Z}[\zeta_3], \mathbb{Z}[i]$, then every twist of $\mathcal{N}_{\mathcal{O}}$ is a quadratic twist.

Proposition 12.2. *Let E be an elliptic curve over $F = \mathbb{Q}(j(E))$ with potential CM by \mathcal{O} , and let $\ell > 3$ be a prime divisor of disc \mathcal{O} . All but two F -isomorphism classes of quadratic twists of E have maximal ℓ -adic image $\mathcal{N}_{\mathcal{O}}(\ell^\infty)$. The two that do not have mod- ℓ image equal to one of the two index-2 subgroups $H_1, H_2 \leq \mathcal{N}_{\mathcal{O}}(\ell)$ that do not contain -1 . These three possibilities are distinguished by their orbit signatures as follows:*

- $\mathcal{N}_{\mathcal{O}}(\ell)$ has two nontrivial ℓ -torsion orbits, one of size $\ell - 1$ and one of size $\ell(\ell - 1)$;
- H_1 has three nontrivial ℓ -torsion orbits, one of size $\ell - 1$ and two of size $\ell(\ell - 1)/2$;
- H_2 has three nontrivial ℓ -torsion orbits, two of size $(\ell - 1)/2$ and one of size $\ell(\ell - 1)$.

The cases H_1 and H_2 each arise for a quadratic twist of E that is unique up to F -isomorphism.

Proof. This follows from [LR, Theorem 1.5], [Sut16, Corollary 5.25] and an analysis of the ℓ -torsion orbits of $\mathcal{N}_{\mathcal{O}}(\ell), H_1, H_2$, which are explicitly described in [LR, Theorem 1.5]. \square

In practice, we can quickly determine when we are in the typical case $\rho_{E,\ell}(G_F) = \mathcal{N}_{\mathcal{O}}(\ell)$ of Proposition 12.2 by sampling Frobenius elements at small primes of good reduction, since $\mathcal{N}_{\mathcal{O}}(\ell)$ contains many elements that are not $\text{GL}_2(\ell)$ -conjugate to any element in H_i for $i = 1, 2$. But if we do not find such an element, we cannot rule out the possibility that $\rho_{E,\ell}(G_F) = \mathcal{N}_{\mathcal{O}}(\ell)$, nor can we hope to distinguish H_1 from H_2 by sampling Frobenius elements; since these groups are Gassmann equivalent subgroups of $\text{GL}_2(\ell)$, they contain the same number of elements in each $\text{GL}_2(\ell)$ -conjugacy class.

However, we can use the ℓ -division polynomial of E to compute the orbit-signature of $\rho_{E,\ell}(G_F)$ and thereby uniquely determine which of the three cases in Proposition 12.2 applies. Let $y^2 = f(x)$ be an integral model for E with $f \in \mathcal{O}_F[x]$ a cubic polynomial, and let $h \in \mathcal{O}_F[x]$ be the ℓ -division polynomial of degree $(\ell^2 - 1)/2$, which can be efficiently computed using well known recursive formulas [McK94]; the roots of h are the distinct x -coordinates of the nontrivial points in $E[\ell](\overline{F})$. Each irreducible factor g of h in $F[x]$ corresponds to a Galois orbit of the set $\{x(P) : P \in E[\ell](\overline{F}) \text{ of order } \ell\}$, each of which corresponds to either two Galois orbits of $E[\ell](\overline{F})$ of the same size or a single Galois orbit of $E[\ell](\overline{F})$ of twice the size. These two possibilities can be distinguished by checking whether f is a square in the field $F[x]/(g(x))$. This computation is implemented by the function `TorsionOrbits` in the file `g12.m` in [RSZB21], which can be used to compute the Galois orbit signature of $E[N](F)$ for any positive integer N .

The algorithm sketched above runs in polynomial-time (we can factor polynomials over number fields in polynomial time [Len83, Lan85] and then compute square-roots by Hensel lifting), but it is time-consuming for large values of ℓ , including $\ell = 163$, which is a necessary case to consider for $F = \mathbb{Q}$.

Table 10. Nonmaximal ℓ -adic images of E/\mathbb{Q} with potential CM by \mathcal{O} of discriminant D divisible by a prime $\ell > 3$.

| ℓ | D | H_1 | E_1 | H_2 | E_2 |
|--------|------|-------------------|----------|--------------------|----------|
| 7 | -7 | 7.48.0.3 | 49.a2 | 7.48.0.6 | 49.a4 |
| | -28 | 7.48.0.3 | 49.a1 | 7.48.0.6 | 49.a3 |
| 11 | -11 | 11.120.1.5 | 121.b2 | 11.120.1.10 | 121.b1 |
| 19 | -19 | 19.360.7.9 | 361.a2 | 19.360.7.18 | 361.a1 |
| 43 | -43 | 43.1848.57.21 | 1849.b2 | 43.1848.57.42 | 1849.b1 |
| 67 | -67 | 67.4488.155.31 | 4489.b2 | 67.4488.155.64 | 4489.b1 |
| 163 | -163 | 163.26568.1027.57 | 26569.a2 | 163.26568.1027.138 | 26569.a1 |

A more expedient approach is to determine once and for all the finite set of F -isomorphism classes of elliptic curves E/F with potential CM by an imaginary quadratic order \mathcal{O} that have nonmaximal ℓ -adic image a prime $\ell > 3$ that divides disc \mathcal{O} . For $F = \mathbb{Q}$, this is done by Zywna in [Zyw15a, Proposition 1.14]. Table 10 lists the LMFDB labels of these elliptic curves and the label of their mod- ℓ image H_i . With this table in hand, determining which case in Proposition 12.2 applies is reduced to an isomorphism test.

Proposition 12.3. *Let E be an elliptic curve over \mathbb{Q} with integral model $y^2 = x^3 + a$ and potential CM by $\mathcal{O} = \mathbb{Z}[\zeta_3]$, and let $\ell > 3$ be prime. Then E has maximal ℓ -adic image unless $4a\ell^e$ is an integer cube, where $e \in \{1, 2\}$ satisfies $\ell \equiv \pm 2e \pmod{9}$, in which case $\rho_{E,\ell}(G_{\mathbb{Q}})$ is the unique index-3 subgroup of $\mathcal{N}_{\mathcal{O}}(\ell)$ that does not contain $z_{\mathcal{O}}^2$.*

Proof. This follows from [Zyw15a, Proposition 1.16] and Proposition 12.1 above. □

Corollary 12.4. *Let E/\mathbb{Q} be an elliptic curve with potential CM by $\mathcal{O} = \mathbb{Z}[\zeta_3]$ that has nonmaximal ℓ -adic image with $\ell > 3$. Then $H := \rho_{E,\ell}(G_{\mathbb{Q}})$ has index $i = 2\ell(\ell - (\frac{-3}{\ell}))/3$ in $GL_2(\ell)$ and genus $g = 3g' - 3 + (\ell - (\frac{-1}{\ell}))/4 + 2$, where g' is the genus of $\mathcal{N}_{\mathcal{O}}(\ell)$.*

The subgroup H has label $\ell.i.g.1$ under the labeling system defined in §2.4.

Proof. The index formula follows immediately from Proposition 12.3, since $\mathcal{N}_{\mathcal{O}}(\ell)$ contains an index-2 Cartan subgroup of order $(\ell - 1)(\ell - (\frac{-3}{\ell}))$, and the genus formula follows from a calculation similar to [Bar10, Proposition 7.10].

For $\ell > 37$, the group $GL_2(\ell)$ contains a unique conjugacy class of index i (this follows from Dickson’s classification [Dic58]), which implies that the label for H is as claimed, and a direct computation verifies that this is also true for $3 < \ell \leq 37$. □

12.3. ℓ -adic CM images for $\ell = 2, 3$

With Propositions 12.7, 12.2, 12.3 in hand, the problem of computing $\rho_{E,\ell^\infty}(G_F)$ for an elliptic curve E over $F = \mathbb{Q}(j(E))$ with potential CM is reduced to the case $\ell = 2, 3$, where we must distinguish the subgroups of $GL_2(16)$ and $GL_2(27)$ that can arise for an elliptic curve with potential CM.

Proposition 12.1. *There are 59 subgroups of $GL_2(16)$ that arise as the reduction of a twist of $\mathcal{N}_{\mathcal{O}}$ for some imaginary quadratic order \mathcal{O} , of which 14 arise as maximal 2-adic images (the reduction of some $\mathcal{N}_{\mathcal{O}}$); if one restricts to imaginary quadratic orders \mathcal{O} of class number 1, there are 28, of which 7 are maximal.*

There are 26 subgroups of $GL_2(27)$ that arise as the reduction of a twist of $\mathcal{N}_{\mathcal{O}}$ for some imaginary quadratic order \mathcal{O} , of which 7 are maximal; if one restricts to imaginary quadratic orders \mathcal{O} of class number 1, there are 17, of which 4 are maximal.

Proof. It is clear from the definition of $\mathcal{N}_{\mathcal{O}}$ that $\mathcal{N}_{\mathcal{O}}(16)$ depends only on the residue class of D modulo 64. It thus suffices to enumerate a representative list of 32 imaginary quadratic discriminants

D that realise the residue classes modulo 64 that are squares modulo 4, compute $\mathcal{N}_{\mathcal{O}}(16)$ for each D , and then reduce this collection to a set of nonconjugate subgroups of $\mathrm{GL}_2(16)$. This yields 14 distinct subgroups of $\mathrm{GL}_2(16)$ that arise as maximal twists, and it then suffices to compute the quadratic twists of these subgroups, along with all twists of $\mathcal{N}_{\mathcal{O}}(16)$ for $\mathcal{O} = \mathbb{Z}[i], \mathbb{Z}[\zeta_3]$ to obtain a complete list of 59 nonconjugate subgroups of $\mathrm{GL}_2(16)$. Repeating this process for the 13 discriminants of class number 1 yields the first statement of the theorem. The second statement is proved similarly, but now $\mathcal{N}_{\mathcal{O}}(27)$ depends only on the residue class of D modulo 27. The function `GL2CMTwists` in the file [g12.m](#) in [RSZB21] was used to perform these computations. \square

Remark 12.2. That 28 subgroups of $\mathrm{GL}_2(16)$ arise as reductions of twists of $\mathcal{N}_{\mathcal{O}}$ with \mathcal{O} of class number 1 is also noted in [LR, p. 2].

Corollary 12.3. *Let E be an elliptic curve with potential CM by an imaginary quadratic order \mathcal{O} of discriminant D defined over $F = \mathbb{Q}(j(E))$. If $D \geq -4$, then $\rho_{E,16}(G_F)$ is listed in Table 18 and $\rho_{E,27}(G_F)$ is listed in Table 19. If $D < -4$, then $\rho_{E,16}(G_F)$ is listed in Table 20 or Table 21 and $\rho_{E,27}(G_F)$ is listed in Table 22. Moreover, every group listed in these tables arises for some elliptic curve with potential CM defined over its minimal field of definition.*

As can be seen in Tables 18–22, for any given imaginary quadratic order \mathcal{O} , the 4-orbits and 8-orbits of the possible subgroups $H \leq \mathrm{GL}_2(16)$ that arise as twists of $\mathcal{N}_{\mathcal{O}}(16)$ are sufficient to determine $\rho_{E,16}(G_F)$ except for 11 pairs of subgroups. For each of these 11 pairs, there are many conjugacy classes of $\mathrm{GL}_2(16)$ that arise in one element of the pair and not the other. We can thus compute $\rho_{E,16}(G_F)$ by using division polynomials to compute the 4-torsion and 8-torsion orbits of E as described at the end of the previous subsection and then sampling Frobenius elements to eliminate any remaining ambiguity. Similar comments apply to $\mathcal{N}_{\mathcal{O}}(27)$ using 3-orbits and 9-orbits to distinguish all but three pairs of subgroups that can then be distinguished by sampling Frobenius elements. The script `CMImages.m` in [RSZB21] verifies these claims. This yields the following algorithm, which is implemented in the function `GL2CME11AdicImages` in the file [g12.m](#) in [RSZB21].

Algorithm 12.4. Given an elliptic curve E with potential CM defined over its minimal field of definition $F = \mathbb{Q}(j(E))$, we may compute $\rho_{E,\ell^\infty}(G_F)$ for all primes ℓ as follows:

1. Compute the discriminant D of $\mathcal{O} \simeq \mathrm{End}(E_{\overline{F}})$.⁷
2. Compute $\rho_{E,16}(G_F)$ and $\rho_{E,27}(G_F)$ using Tables 18–22 as described above.
3. Compute $\rho_{E,\ell}(G_F)$ for $\ell > 3$ dividing D via Proposition 12.2 (Table 10 if $F = \mathbb{Q}$).
4. If $j(E) = 0$, compute $\rho_{E,\ell}(G_F)$ for ℓ of bad reduction via Proposition 12.3.
5. $\rho_{E,\ell}(G_F) = \mathcal{N}_{\mathcal{O}}(\ell)$ for all ℓ not addressed in steps (1) to (4), by Proposition 12.7.

Having determined $\rho_{E,\ell^e}(G_F)$ for all primes $\ell = 2, 3, 7, \dots$ with $e = 4, 3, 1, 1, \dots$, $\rho_{E,\ell^\infty}(G_F)$ is the inverse image of $\rho_{E,\ell^e}(G_F)$ under the projection $\mathcal{N}_{\mathcal{O}}(\mathbb{Z}_\ell) \rightarrow \mathcal{N}_{\mathcal{O}}(\ell^e)$, by Proposition 12.4.

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⁷This can be done using the Magma function `HasComplexMultiplication`.

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Appendix A. Jacobians of arbitrary modular curves (with John Voight)

Ribet [Rib04, Section 3] observed that the simple abelian varieties attached to newforms on $\Gamma_1(N)$ are of GL_2 -type over \mathbb{Q} . In this appendix, we generalise this to any modular curve. One may anticipate this result from the fact that $\Gamma(N)$ is $GL_2(\mathbb{Q})$ -conjugate to a subgroup of $\Gamma_1(N^2)$, but the proof requires attention to descent. As a corollary, we show that if A is an isogeny factor of J_H with $r_{an}(A) = 0$, then $r_{alg}(A) = 0$.

By a curve over a perfect field k in this appendix, we mean a smooth, nonsingular scheme of finite type over k , possibly disconnected, but with all components of dimension 1. Isogenies of abelian varieties are by definition defined over the ground field of the objects, but we will sometimes remind the reader of this convention for clarity.

Let A be a simple abelian variety over a number field F , and let $End(A)_{\mathbb{Q}}$ be the \mathbb{Q} -algebra of endomorphisms of A (defined over F). We say A is of GL_2 -type if there exists a subfield $E \subseteq End(A)_{\mathbb{Q}}$ such that $[E : \mathbb{Q}] = \dim A$. Evidently, if A is of GL_2 -type over F , then so is A_K for any field extension $K \supseteq F$ (taking the same subfield E): in fact, A_K is isogenous over K to a power of a simple abelian variety of GL_2 -type over K (called a building block), by Pyle [Pyl04, Proposition 1.3].

Let $N \in \mathbb{Z}_{\geq 3}$. Let $X(N)$ be the modular curve over \mathbb{Q} attached to the trivial subgroup of $GL_2(N)$, parametrising generalised elliptic curves with ‘full level N structure’ in the sense of Katz–Mazur [KM85, 3.1]. (When $N = 1, 2$, in fact $X(N)$ is a stack not a scheme, but its components are of genus 0 and our applications – for example, Proposition A.6 – hold trivially.) Then $X(N)$ is connected over \mathbb{Q} but becomes disconnected over $\mathbb{Q}(\zeta_N)$, with geometrically connected components indexed by $Gal(\mathbb{Q}(\zeta_N) | \mathbb{Q}) \simeq (\mathbb{Z}/N\mathbb{Z})^{\times}$ (see also the proof of Lemma A.2). More precisely, the group $(\mathbb{Z}/N\mathbb{Z})^{\times}$ acts geometrically by diamond operators, letting $d \in (\mathbb{Z}/N\mathbb{Z})^{\times}$ act via $\begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}$, which on moduli problems sends $[(E, P, Q)] \mapsto [(E, P, dQ)]$. Each connected component of $X(N)_{\mathbb{C}}$ is attached to the congruence subgroup $\Gamma(N) \leq SL_2(\mathbb{Z})$. Since these diamond automorphisms are defined on the moduli problem, they are defined over \mathbb{Q} , giving $(\mathbb{Z}/N\mathbb{Z})^{\times} \hookrightarrow Aut(X(N))$.

Let $J(N)$ be the Jacobian of $X(N)$. Let $Pic^0 X(N)$ be the functor of line bundles whose restriction to each geometric component of $X(N)$ has degree 0. Although $X(N)$ is geometrically disconnected, the functor $Pic^0 X(N)$ is representable by an abelian variety $J(N)$ over \mathbb{Q} : see, for example, Kleiman [Kle05, Remark 5.6]. (We note that even though $X(N)$ does not admit a degree one divisor, Pic^0 is representable as long as one sheafifies [Poo17, Warning 5.7.10]). The formation of the Jacobian is functorial, so $J(N)_{\mathbb{Q}(\zeta_N)}$ is isomorphic over $\mathbb{Q}(\zeta_N)$ to the product of the Jacobians of the geometric connected components of $X(N)$.

Next, let $X(N)' := X_{H_0}$, where $H_0 \leq GL_2(\widehat{\mathbb{Z}})$ is the preimage of $\left\{ \begin{bmatrix} * & 0 \\ 0 & 1 \end{bmatrix} \right\} \leq GL_2(\mathbb{Z}/N\mathbb{Z})$. (In Derickx–Sutherland [DS17, Section 2], $X(N)'$ is denoted $X_{0,1}(N, N)$.) We have $X(N)' = X(N)/H_0(N)$, and the moduli problem attached to $X(N)'$ has several equivalent descriptions. Directly from the description as a quotient, we see that $X(N)'$ parametrises triples (E, ϕ, P) , where E is a generalised elliptic curve, P is a point of exact order N and ϕ is a cyclic N -isogeny such that $E[N]$ is generated by P and $\ker \phi$. The curve $X(N)'$ is geometrically connected and attached to the congruence subgroup $\Gamma(N) \leq SL_2(\mathbb{Z})$. The diamond operators above are defined by the same formula and their image $(\mathbb{Z}/N\mathbb{Z})^{\times} \hookrightarrow Aut(X(N)')$ gives a subgroup C .

Remark A.1. See also Poonen–Schaefer–Stoll [PSS07, Sections 4.1–4.2], who describe $X(N)'$ (for $N = 7$ and away from the cusps) as parametrising isomorphism classes of pairs (E, ψ) , where E is a generalised elliptic curve and $\psi : \mu_N \times \mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} E[N]$ is a symplectic isomorphism.

Lemma A.2. *The curve $X(N)$ over \mathbb{Q} is isomorphic to $X(N)'_{\mathbb{Q}(\zeta_N)}$ as a curve over \mathbb{Q} .*

By *as a curve over* \mathbb{Q} , we mean the composition $X(N)'_{\mathbb{Q}(\zeta_N)} \rightarrow \text{Spec } \mathbb{Q}(\zeta_N) \rightarrow \text{Spec } \mathbb{Q}$ as structure morphism: that is, the product $X(N)' \times_{\text{Spec } \mathbb{Q}} \text{Spec } \mathbb{Q}(\zeta_N)$ viewed as a scheme over \mathbb{Q} .

Proof. Let $Y(N) \subseteq X(N)$ and $Y(N)' \subseteq X(N)'$ denote the affine curves obtained by restricting the moduli problem to elliptic curves. In the terminology of Katz–Mazur [KM85, 5.6], for a point $[(E, P, Q)] \in Y(N)(R)$ over a \mathbb{Q} -algebra R , where E is an elliptic curve, we have its determinant $e_N(P, Q)$ (obtained equivalently by the Weil pairing), an element of $R[\zeta_N] := R[x]/(\Phi_N(x))$, where $\Phi_N(x)$ is the N th cyclotomic polynomial. Refining the natural assignment above, we have a map $[(E, P, Q)] \mapsto [(E, \phi, P; e_N(P, Q))]$ on points which induces a map $Y(N) \rightarrow Y(N)'_{\mathbb{Q}[\zeta_N]}$ of nonsingular curves over \mathbb{Q} . Over any geometric point, this map is bijective, so as a map of nonsingular curves it is an isomorphism. Thus we obtain a (unique) map $X(N) \rightarrow X(N)'_{\mathbb{Q}(\zeta_N)}$ of projective, nonsingular curves over \mathbb{Q} . □

For a Dirichlet character $\chi: (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ of modulus N , let $X(N)'_\chi$ be the twist of $X(N)'$ attached to χ (with respect to $C \leq \text{Aut}(X(N)')$) and let $J(N)'_\chi$ be its Jacobian. For a precise description of this twisting operation; see Kida [Kid95, Section 2].

Corollary A.3. *Up to isogeny over \mathbb{Q} , we have*

$$J(N) \sim \text{Res}_{\mathbb{Q}(\zeta_N)|\mathbb{Q}}(J(N)'_{\mathbb{Q}(\zeta_N)}) \sim \prod_\chi J(N)'_\chi,$$

where $\text{Res}_{\mathbb{Q}(\zeta_N)|\mathbb{Q}}$ is a restriction of scalars and χ ranges over Dirichlet characters of modulus N .

Proof. By the universal property of the restriction of scalars, the Jacobian of $X(N)'_{\mathbb{Q}(\zeta_N)}$ as a curve over \mathbb{Q} is the restriction of scalars $\text{Res}_{\mathbb{Q}(\zeta_N)|\mathbb{Q}}(J(N)')$, a line bundle on $X(N)'_{\mathbb{Q}(\zeta_N)}$ as a curve over \mathbb{Q} is just a line bundle on $X(N)'_{\mathbb{Q}(\zeta_N)}$, so the first isogeny follows from Lemma A.2. The second isogeny follows from a result of Kida [Kid95, Theorem, p. 53]. □

Let $\widehat{\mathbb{Q}} := \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$, and let

$$\varpi = \varpi_N := \begin{bmatrix} 0 & 1 \\ N & 0 \end{bmatrix} \in \text{GL}_2(\widehat{\mathbb{Q}}).$$

To prove our main result, we study the effect of conjugating H -level structures by ϖ , when this is defined. Conjugation by ϖ in $\text{GL}_2(\widehat{\mathbb{Q}})$ is given by the formula

$$\varpi \begin{bmatrix} a & b \\ c & d \end{bmatrix} \varpi^{-1} = \begin{bmatrix} d & N^{-1}c \\ Nb & a \end{bmatrix}.$$

Let $H \leq \text{GL}_2(\widehat{\mathbb{Z}})$ be an open subgroup, and suppose that its reduction $H(N) \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ is contained in the Borel subgroup $B(N)$ of upper-triangular matrices. Then the level of H is divisible by N . We define

$$H^\varpi := \varpi H \varpi^{-1} = \left\{ \begin{bmatrix} d & c \\ Nb & a \end{bmatrix} : \begin{bmatrix} a & b \\ Nc & d \end{bmatrix} \in H \right\} \leq \text{GL}_2(\widehat{\mathbb{Z}}).$$

The group H^ϖ is again an open subgroup. If H has level MN , then visibly H^ϖ has level dividing MN^2 , and this bound is in general sharp: for example, if H is the level N preimage of $\left\{ \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \right\} \leq \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$, then $H^\varpi = \left\{ \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \right\} \leq \text{GL}_2(\mathbb{Z}/N^2\mathbb{Z})$, which has level N^2 .

Lemma A.4. *Suppose that $H(N) \leq B(N)$. Then the modular curves X_H and X_{H^ϖ} are isomorphic over \mathbb{Q} .*

Proof. We apply a result of Deligne–Rapoport [DR73, Propositions IV-3.16 and IV-3.19] with $K = H$ and $g = \varpi$: the isomorphism provided is defined on the (open) moduli problem and extends to an isomorphism of curves over \mathbb{Q} . □

The preceding lemma explains in many cases why modular curves are in fact isomorphic over \mathbb{Q} (even though they may look like twists!). The following example illustrates how conjugation by ϖ transforms one moduli problem into an isomorphic moduli problem (over \mathbb{Q}).

Example A.5. Let

$$H_1 := \left\{ \begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix} \right\} \leq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}).$$

Then $\Gamma_{H_1} = \Gamma_1(N)$ (following the conventions in Section 2.2), and the points of X_{H_1} are given by classes $[(E, P)]$, where E is a generalised elliptic curve and $P \in E[N]$ is a point of order N .

We compute that $H_1^{\varpi} = \left\{ \begin{bmatrix} * & * \\ 0 & 1 \end{bmatrix} \right\} \leq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ has level N . (Swapping basis vectors transforms H_1^{ϖ} to the transpose $H_1^T = \left\{ \begin{bmatrix} 1 & 0 \\ * & * \end{bmatrix} \right\} \leq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$.) The points of $X_{H_1^T}$ are classes $[(E, \nu)]$, where E is a generalised elliptic curve and $\nu: \mu_N \hookrightarrow E[N]$ is an inclusion. We have $\Gamma_{H_1^{\varpi}} = \Gamma^1(N)$ (conjugate to $\Gamma_1(N)$ in $\mathrm{SL}_2(\mathbb{Z})$), but of course the moduli interpretation is different for the two cases.

Lemma A.4 shows that in fact we have an isomorphism $X_{H_1} \xrightarrow{\sim} X_{H_1^{\varpi}}$ defined over \mathbb{Q} . In terms of the moduli problems above, this isomorphism is defined as follows: the point $[(E, \nu)]$ over a base ring R maps to the pair $[(E', Q')]$ over R , where $E' := E/\nu(\mu_N)$ and $Q' \in E'[N]$ is the image of any point $Q \in E[N]$ such that $e_N(\nu(\zeta_N), Q) = \zeta_N$ under the Weil pairing e_N . This isomorphism was observed by Poonen–Schaefer–Stoll [PSS07, Remark 5.1] in the case $N = 7$, where they write $X_{\mu}(N)$ for $X_{H_1^T}$.

With this setup, we can prove our first main result.

Proposition A.6. *For all Dirichlet characters χ , every simple isogeny factor of $J(N)_{\chi}'$ over \mathbb{Q} is an isogeny factor of $J_1(N^2)$ and of GL_2 -type over \mathbb{Q} . More precisely, each such factor is isogenous to a modular abelian variety of the form A_f , where f is a Galois orbit of newforms of level dividing N^2 and conductor of modulus N .*

Proof. Let

$$H := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a \equiv 1 \pmod{N}, c \equiv 0 \pmod{N^2} \right\} \leq \mathrm{GL}_2(\widehat{\mathbb{Z}}).$$

Then $\Gamma_H = \Gamma_0(N^2) \cap \Gamma_1(N)$; in particular, there is a natural surjective map $X_1(N^2) \rightarrow X_H$.

By Lemma A.4, we have an isomorphism of modular curves $X_H \xrightarrow{\sim} X_{H^{\varpi}}$. We compute directly that $H^{\varpi} = H_0$ is the preimage of $\left\{ \begin{bmatrix} * & 0 \\ 0 & 1 \end{bmatrix} \right\} \leq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$, so $X_{H^{\varpi}} = X(N)'$.

Ribet [Rib04, Section 3] observed that every isogeny factor of $J_1(N^2)$ is of GL_2 -type over \mathbb{Q} : more precisely, we have a decomposition $J_1(N^2)$ into a product of simple abelian varieties A_f over \mathbb{Q} , possibly with multiplicity, where f is a Galois orbit of newforms of level dividing N^2 . In fact, the natural quotient map $X_1(N^2) \rightarrow X_H$ is defined by the subgroup of the diamond operators $\{\langle d \rangle : d \in (\mathbb{Z}/N^2\mathbb{Z})^{\times} \text{ with } d \equiv 1 \pmod{N}\}$, so the cusp forms on X_H are precisely those with conductor dividing N (so of modulus N). Finally, the isomorphism $X_H \xrightarrow{\sim} X(N)'$ over \mathbb{Q} transports Ribet’s observation to the Jacobian $J(N)'$, as claimed.

To finish, we consider the twist: from the decomposition in the previous paragraph,

$$J(N)_{\chi}' \sim \prod_f A_{f \otimes \chi}^{e_f},$$

where $f \otimes \chi$ is the twist of f by the character χ . Let ψ be the character of f , of modulus N and conductor $\mathrm{cond}(\psi)$. By Shimura [Shi94, Proposition 3.64], the level of $f \otimes \chi$ divides

$$\mathrm{lcm}(N^2, \mathrm{cond}(\chi)^2, \mathrm{cond}(\chi) \mathrm{cond}(\psi)) = N^2,$$

and moreover, $f \otimes \chi$ has character $\psi\chi^2$ still of modulus N . Thus $A_{f \otimes \chi}$ is an isogeny factor over \mathbb{Q} of $J_1(N^2)$, and the result follows as in the previous case. \square

Theorem A.7. *Let $H \leq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ be an open subgroup of level N , let X_H be the modular curve attached to H , and let J_H be its Jacobian. Then each simple factor of J_H is isogenous to a simple factor of $J_1(N^2)$. In particular, every simple factor of J_H is of GL_2 -type.*

Proof. Let A be a simple factor of J_H . We have a map $X(N) \rightarrow X_H$, realising J_H as an isogeny factor of $J(N)$, so A is a simple isogeny factor of $J(N)$. Corollary A.3 implies that A is a simple isogeny factor of $J(N)'_\chi$ for some Dirichlet character χ . By Proposition A.6, every simple factor of $J(N)'_\chi$ is an isogeny factor of $J_1(N^2)$, hence of GL_2 -type. \square

Corollary A.8. *Let A be an isogeny factor of J_H of analytic rank 0. Then $A(\mathbb{Q})$ is finite.*

Proof. Apply Kato [Kat04, Corollary 14.3]. \square

Remark A.9. Assaf [Ass21] has given a method for computing Hecke operators on arbitrary congruence subgroups. According to Theorem A.7, the Galois orbits of cusp forms appearing can be matched with classical modular forms according to the corresponding system of Hecke eigenvalues (forms with level dividing N^2 and character of modulus N), thereby explicitly providing the isotypic decomposition of J_H up to isogeny over \mathbb{Q} .

This method is indeed sensitive to the group H ! For example, the groups 16.24.1.6, 16.24.1.7, 16.24.1.8 (numbered 163, 162, 164 by Rouse–Zureick-Brown [RZB15]) have the same intersection with $\mathrm{SL}_2(\mathbb{Z}/16\mathbb{Z})$, with Cummins–Pauli label 16D¹, as can be seen in Table 17; but the associated modular Jacobians are nonisogenous elliptic curves 128.c2, 128.b2, 128.d2 with j -invariant 128 (so isomorphic over \mathbb{C} but not even isogenous over \mathbb{Q}).

Remark A.10. The same proof works for an arbitrary (possibly geometrically disconnected) Jacobian of a Shimura curve attached to a quaternion algebra over a totally real field with a unique real split place. The step that generalises Ribet’s observation (there are Hecke operators and degeneracy operators) is given by Hida [Hid81, Proposition 4.8].

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Appendix B. Tables

The tables that follow provide data for various subgroups H of ℓ -power level that include:

- the label N.i.g.n of H as defined in Section 2.4;
- a list of generators for H in $\mathrm{GL}_2(N)$ (which we believe is minimal);
- for $g(H) \leq 24$, the label of $H \cap \mathrm{SL}_2(\mathbb{Z})$ assigned by Cummins and Pauli [CP03, CP];
- the number of cusps $\#X_H^\infty(\overline{\mathbb{Q}})$ and rational cusps $\#X_H^\infty(\mathbb{Q})$, as described in Section 2;
- the analytic rank r of $J_H := \mathrm{Jac}(X_H)$, whose computation is described in Section 6;
- the genus g of X_H ;
- the dimensions and LMFDB labels of the Galois orbits of newforms $[f]$ whose modular abelian varieties A_f are isogeny factors of J_H (exponents denote multiplicities).

Table 11. Subgroups H of ℓ -power level that have rational noncuspidal non-CM points but do not arise as $\rho_{E, \ell^\infty}(G_{\mathbb{Q}})$ for any non-CM elliptic curve E/\mathbb{Q} (the noncuspidal non-CM points in $X_H(\mathbb{Q})$ all correspond to elliptic curves E/\mathbb{Q} with $\rho_{E, \ell^\infty}(\mathbb{Q})$ properly contained in H).

| Label | Generators | CP | $\#X_H^\infty(\overline{\mathbb{Q}})$ | $\#X_H^\infty(\mathbb{Q})$ | r | g | Dimensions |
|------------|--|------------------|---------------------------------------|----------------------------|-----|-----|----------------|
| 16.24.1.5 | $\begin{bmatrix} 0 & 5 \\ 1 & 10 \end{bmatrix}, \begin{bmatrix} 4 & 5 \\ 5 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 11 & 10 \end{bmatrix}, \begin{bmatrix} 13 & 10 \\ 8 & 7 \end{bmatrix}$ 128.2.a.a | 16D ¹ | 2 | 2 | 1 | 1 | 1 |
| 16.24.1.10 | $\begin{bmatrix} 14 & 9 \\ 9 & 2 \end{bmatrix}, \begin{bmatrix} 15 & 6 \\ 6 & 15 \end{bmatrix}, \begin{bmatrix} 9 & 2 \\ 10 & 7 \end{bmatrix}, \begin{bmatrix} 12 & 3 \\ 3 & 12 \end{bmatrix}$ 256.2.a.a | 8D ¹ | 3 | 1 | 1 | 1 | 1 |
| 16.24.1.11 | $\begin{bmatrix} 4 & 7 \\ 5 & 12 \end{bmatrix}, \begin{bmatrix} 12 & 3 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 3 & 10 \end{bmatrix}, \begin{bmatrix} 0 & 9 \\ 5 & 0 \end{bmatrix}$ 256.2.a.a | 8D ¹ | 3 | 1 | 1 | 1 | 1 |
| 16.24.1.13 | $\begin{bmatrix} 5 & 4 \\ 8 & 3 \end{bmatrix}, \begin{bmatrix} 10 & 15 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 15 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 12 & 9 \\ 11 & 0 \end{bmatrix}$ 256.2.a.a | 16C ¹ | 3 | 1 | 1 | 1 | 1 |
| 16.24.1.15 | $\begin{bmatrix} 3 & 9 \\ 0 & 13 \end{bmatrix}, \begin{bmatrix} 5 & 14 \\ 6 & 15 \end{bmatrix}, \begin{bmatrix} 3 & 13 \\ 10 & 1 \end{bmatrix}, \begin{bmatrix} 15 & 4 \\ 12 & 3 \end{bmatrix}$ 256.2.a.a | 16C ¹ | 3 | 1 | 1 | 1 | 1 |
| 16.24.1.17 | $\begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 13 & 2 \\ 10 & 9 \end{bmatrix}, \begin{bmatrix} 13 & 14 \\ 6 & 11 \end{bmatrix}$ 256.2.a.b | 16B ¹ | 2 | 2 | 1 | 1 | 1 |
| 16.24.1.19 | $\begin{bmatrix} 5 & 0 \\ 8 & 9 \end{bmatrix}, \begin{bmatrix} 9 & 14 \\ 14 & 3 \end{bmatrix}, \begin{bmatrix} 9 & 14 \\ 6 & 9 \end{bmatrix}, \begin{bmatrix} 0 & 5 \\ 5 & 12 \end{bmatrix}$ 256.2.a.b | 16B ¹ | 2 | 2 | 1 | 1 | 1 |
| 11.12.1.1 | $\begin{bmatrix} 2 & 7 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 3 \\ 0 & 7 \end{bmatrix}$ 11.2.a.a | 11A ¹ | 2 | 2 | 0 | 1 | 1 |
| 11.24.1.1 | $\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 9 & 10 \\ 0 & 7 \end{bmatrix}$ 11.2.a.a | 11A ¹ | 2 | 2 | 0 | 1 | 1 |
| 11.24.1.2 | $\begin{bmatrix} 8 & 1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ 11.2.a.a | 11A ¹ | 2 | 2 | 0 | 1 | 1 |
| 17.18.1.1 | $\begin{bmatrix} 15 & 3 \\ 0 & 11 \end{bmatrix}, \begin{bmatrix} 3 & 14 \\ 0 & 13 \end{bmatrix}$ 17.2.a.a | 17A ¹ | 2 | 2 | 0 | 1 | 1 |
| 17.36.1.1 | $\begin{bmatrix} 13 & 0 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 9 & 1 \\ 0 & 2 \end{bmatrix}$ 17.2.a.a | 17B ¹ | 4 | 2 | 0 | 1 | 1 |
| 17.36.1.2 | $\begin{bmatrix} 8 & 6 \\ 0 & 9 \end{bmatrix}, \begin{bmatrix} 14 & 16 \\ 0 & 8 \end{bmatrix}$ 17.2.a.a | 17B ¹ | 4 | 2 | 0 | 1 | 1 |
| 37.38.2.1 | $\begin{bmatrix} 23 & 33 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 17 & 18 \\ 0 & 15 \end{bmatrix}$ 37.2.a.a, 37.2.a.b | 37A ² | 2 | 2 | 1 | 2 | 1 ² |

Table 12. *Arithmetically maximal subgroups of 3-power level.*

| Label | Generators | CP | $\#X_H^\infty(\overline{\mathbb{Q}})$ | $\#X_H^\infty(\mathbb{Q})$ | r | g | Dimensions |
|-------------|--|-------------------|---------------------------------------|----------------------------|-----|-----|---|
| 9.12.1.1 | $\begin{bmatrix} 5 & 8 \\ 6 & 8 \end{bmatrix}, \begin{bmatrix} 7 & 0 \\ 0 & 8 \end{bmatrix}$ 27.2.a.a | 9A ¹ | 2 | 2 | 0 | 1 | 1 |
| 9.36.2.1 | $\begin{bmatrix} 2 & 3 \\ 6 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 6 \\ 6 & 8 \end{bmatrix}$ 81.2.a.a | 9A ² | 4 | 2 | 0 | 2 | 2 |
| 9.54.1.1 | $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 7 & 0 \\ 0 & 5 \end{bmatrix}$ 27.2.a.a | 9E ¹ | 6 | 1 | 0 | 1 | 1 |
| 9.54.2.2 | $\begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 5 & 3 \\ 3 & 4 \end{bmatrix}$ 81.2.a.a | 9B ² | 6 | 0 | 0 | 2 | 2 |
| 9.81.1.1 | $\begin{bmatrix} 5 & 8 \\ 7 & 5 \end{bmatrix}, \begin{bmatrix} 5 & 8 \\ 2 & 4 \end{bmatrix}$ 27.2.a.a | 9G ¹ | 9 | 0 | 0 | 1 | 1 |
| 9.108.3.1 | $\begin{bmatrix} 4 & 2 \\ 3 & 5 \end{bmatrix}, \begin{bmatrix} 5 & 5 \\ 0 & 4 \end{bmatrix}$ 27.2.a.a, 81.2.a.a | 9A ³ | 12 | 0 | 0 | 3 | 1, 2 |
| 9.108.4.1 | $\begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 8 \\ 5 & 3 \end{bmatrix}$ 81.2.c.b | 9A ⁴ | 12 | 3 | 0 | 4 | 4 |
| 27.36.1.1 | $\begin{bmatrix} 13 & 17 \\ 0 & 20 \end{bmatrix}, \begin{bmatrix} 11 & 16 \\ 0 & 20 \end{bmatrix}$ 27.2.a.a | 27A ¹ | 6 | 2 | 0 | 1 | 1 |
| 27.36.1.2 | $\begin{bmatrix} 13 & 23 \\ 18 & 14 \end{bmatrix}, \begin{bmatrix} 23 & 22 \\ 15 & 23 \end{bmatrix}$ 243.2.a.b | 27B ¹ | 2 | 2 | 0 | 1 | 1 |
| 27.36.2.1 | $\begin{bmatrix} 2 & 17 \\ 18 & 26 \end{bmatrix}, \begin{bmatrix} 17 & 8 \\ 9 & 13 \end{bmatrix}$ 81.2.a.a | 27A ² | 4 | 2 | 0 | 2 | 2 |
| 27.36.2.2 | $\begin{bmatrix} 17 & 26 \\ 12 & 5 \end{bmatrix}, \begin{bmatrix} 5 & 2 \\ 6 & 10 \end{bmatrix}$ 243.2.a.c | 27B ² | 2 | 2 | 0 | 2 | 2 |
| 27.36.3.1 | $\begin{bmatrix} 4 & 1 \\ 21 & 2 \end{bmatrix}, \begin{bmatrix} 11 & 0 \\ 0 & 19 \end{bmatrix}$ 243.2.a.a, 243.2.a.d | 27A ³ | 2 | 2 | 1 | 3 | 1, 2 |
| 27.108.3.1 | $\begin{bmatrix} 5 & 19 \\ 24 & 2 \end{bmatrix}, \begin{bmatrix} 13 & 9 \\ 15 & 5 \end{bmatrix}, \begin{bmatrix} 10 & 23 \\ 21 & 4 \end{bmatrix}$ 27.2.a.a, 81.2.a.a | 27C ³ | 12 | 0 | 0 | 3 | 1, 2 |
| 27.108.4.1 | $\begin{bmatrix} 20 & 10 \\ 6 & 13 \end{bmatrix}, \begin{bmatrix} 10 & 18 \\ 12 & 26 \end{bmatrix}$ 81.2.c.b | 27A ⁴ | 12 | 3 | 0 | 4 | 4 |
| 27.108.4.3 | $\begin{bmatrix} 20 & 22 \\ 18 & 26 \end{bmatrix}, \begin{bmatrix} 1 & 26 \\ 18 & 26 \end{bmatrix}$ 81.2.c.b | 27A ⁴ | 12 | 3 | 0 | 4 | 4 |
| 27.108.4.5 | $\begin{bmatrix} 4 & 25 \\ 6 & 14 \end{bmatrix}, \begin{bmatrix} 8 & 0 \\ 3 & 1 \end{bmatrix}$ 81.2.a.a ² | 27A ⁴ | 12 | 0 | 0 | 4 | 2 ² |
| 27.108.6.1 | $\begin{bmatrix} 14 & 0 \\ 12 & 4 \end{bmatrix}, \begin{bmatrix} 25 & 12 \\ 15 & 2 \end{bmatrix}, \begin{bmatrix} 20 & 23 \\ 21 & 14 \end{bmatrix}$ 243.2.a.e, 243.2.a.f | 27A ⁶ | 6 | 0 | 3 | 6 | 3 ² |
| 27.243.12.1 | $\begin{bmatrix} 20 & 14 \\ 7 & 20 \end{bmatrix}, \begin{bmatrix} 2 & 9 \\ 9 & 25 \end{bmatrix}$ 729.2.a.a, 729.2.a.c | 27A ¹² | 9 | 0 | 12 | 12 | 6 ² |
| 27.729.43.1 | $\begin{bmatrix} 23 & 20 \\ 5 & 9 \end{bmatrix}, \begin{bmatrix} 3 & 13 \\ 14 & 26 \end{bmatrix}$ 27.2.a.a, 243.2.a.b ² , 81.2.a.a, 243.2.a.c, 243.2.a.e, 243.2.a.f, 729.2.a.a ² , 729.2.a.b, 729.2.a.c, 729.2.a.d | none | 27 | 0 | 21 | 43 | 1 ³ , 2 ² , 3 ² , 6 ⁵ |

Table 13. Arithmetically maximal subgroups of 5-power level.

| Label | Generators | CP | $\#X_H^\infty(\overline{\mathbb{Q}})$ | $\#X_H^\infty(\mathbb{Q})$ | r | g | Dimensions |
|-------------|--|-------------------|---------------------------------------|----------------------------|-----|-----|------------------------------------|
| 25.50.2.1 | $\begin{bmatrix} 3 & 12 \\ 11 & 7 \end{bmatrix}, \begin{bmatrix} 11 & 7 \\ 14 & 21 \end{bmatrix}$ 625.2.a.a | 25E ² | 2 | 0 | 2 | 2 | 2 |
| 25.75.2.1 | $\begin{bmatrix} 10 & 3 \\ 23 & 5 \end{bmatrix}, \begin{bmatrix} 12 & 10 \\ 10 & 16 \end{bmatrix}$ 625.2.a.b | 25F ² | 7 | 0 | 2 | 2 | 2 |
| 25.150.4.1 | $\begin{bmatrix} 18 & 15 \\ 15 & 8 \end{bmatrix}, \begin{bmatrix} 7 & 10 \\ 5 & 14 \end{bmatrix}$ 25.2.d.a | 25F ⁴ | 14 | 5 | 0 | 4 | 4 |
| 25.150.4.2 | $\begin{bmatrix} 7 & 20 \\ 20 & 7 \end{bmatrix}, \begin{bmatrix} 22 & 2 \\ 13 & 22 \end{bmatrix}$ 625.2.d.h | 25F ⁴ | 14 | 0 | 0 | 4 | 4 |
| 25.150.4.5 | $\begin{bmatrix} 18 & 15 \\ 0 & 7 \end{bmatrix}, \begin{bmatrix} 18 & 22 \\ 0 & 9 \end{bmatrix}$ 25.2.d.a | 25G ⁴ | 14 | 5 | 0 | 4 | 4 |
| 25.150.4.6 | $\begin{bmatrix} 18 & 15 \\ 0 & 24 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 0 & 18 \end{bmatrix}$ 25.2.d.a | 25G ⁴ | 14 | 5 | 0 | 4 | 4 |
| 25.150.4.7 | $\begin{bmatrix} 24 & 24 \\ 0 & 18 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ 0 & 23 \end{bmatrix}$ 625.2.a.b ² | 25G ⁴ | 14 | 0 | 4 | 4 | 2 ² |
| 25.150.4.8 | $\begin{bmatrix} 8 & 4 \\ 0 & 23 \end{bmatrix}, \begin{bmatrix} 16 & 7 \\ 0 & 8 \end{bmatrix}$ 625.2.d.h | 25G ⁴ | 14 | 0 | 0 | 4 | 4 |
| 25.150.4.9 | $\begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}, \begin{bmatrix} 3 & 18 \\ 0 & 14 \end{bmatrix}$ 625.2.d.h | 25G ⁴ | 14 | 0 | 0 | 4 | 4 |
| 25.150.8.1 | $\begin{bmatrix} 7 & 0 \\ 0 & 23 \end{bmatrix}, \begin{bmatrix} 13 & 15 \\ 0 & 21 \end{bmatrix}$ 125.2.a.a, 125.2.a.b, 125.2.a.c | 25A ⁸ | 10 | 2 | 2 | 8 | 2 ² , 4 |
| 25.250.14.1 | $\begin{bmatrix} 10 & 2 \\ 1 & 15 \end{bmatrix}, \begin{bmatrix} 18 & 7 \\ 6 & 7 \end{bmatrix}$ 625.2.a.a, 625.2.a.b, 625.2.a.c, 625.2.a.e | 25A ¹⁴ | 10 | 0 | 14 | 14 | 2 ³ , 8 |
| 25.375.22.1 | $\begin{bmatrix} 0 & 6 \\ 21 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ 7 & 0 \end{bmatrix}$ 125.2.a.a, 125.2.a.b, 625.2.a.a, 625.2.a.b, 625.2.a.c, 125.2.a.c, 625.2.a.e | 25A ²² | 15 | 1 | 16 | 22 | 2 ⁵ , 4, 8 |
| 25.625.36.1 | $\begin{bmatrix} 17 & 22 \\ 24 & 8 \end{bmatrix}, \begin{bmatrix} 13 & 8 \\ 21 & 13 \end{bmatrix}$ 125.2.a.a, 125.2.a.b, 625.2.a.a ² , 625.2.a.b ² , 625.2.a.c, 625.2.a.d, 125.2.a.c, 625.2.a.e, 625.2.a.f | none | 25 | 0 | 20 | 36 | 2 ⁸ , 4, 8 ² |
| 125.150.8.1 | $\begin{bmatrix} 18 & 82 \\ 0 & 66 \end{bmatrix}, \begin{bmatrix} 2 & 80 \\ 0 & 98 \end{bmatrix}$ 125.2.a.a, 125.2.a.b, 125.2.a.c | 125A ⁸ | 10 | 2 | 2 | 8 | 2 ² , 4 |

Table 14. *Arithmetically maximal subgroups of ℓ -power level for $\ell = 7, 11$.*

| Label | Generators | CP | $\#X_H^\infty(\overline{\mathbb{Q}})$ | $\#X_H^\infty(\mathbb{Q})$ | r | g | Dimensions |
|----------------|--|-------------------|---------------------------------------|----------------------------|-----|-----|---|
| 7.42.1.1 | $\begin{bmatrix} 3 & 5 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 1 \\ 5 & 2 \end{bmatrix}$ 49.2.a.a | 7A ¹ | 6 | 0 | 0 | 1 | 1 |
| 7.56.1.1 | $\begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}$ 49.2.a.a | 7B ¹ | 8 | 2 | 0 | 1 | 1 |
| 7.56.1.2 | $\begin{bmatrix} 0 & 3 \\ 6 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$ 49.2.a.a | 7B ¹ | 8 | 0 | 0 | 1 | 1 |
| 49.56.1.1 | $\begin{bmatrix} 39 & 18 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 47 & 33 \\ 0 & 33 \end{bmatrix}$ 49.2.a.a | 49A ¹ | 8 | 2 | 0 | 1 | 1 |
| 49.147.9.1 | $\begin{bmatrix} 3 & 6 \\ 6 & 32 \end{bmatrix}, \begin{bmatrix} 19 & 42 \\ 34 & 23 \end{bmatrix}$ 2401.2.a.f | 49A ⁹ | 3 | 0 | 9 | 9 | 9 |
| 49.168.12.1 | $\begin{bmatrix} 39 & 6 \\ 36 & 24 \end{bmatrix}, \begin{bmatrix} 11 & 9 \\ 24 & 2 \end{bmatrix}$ 343.2.a.a, 343.2.a.b, 343.2.a.e | 49B ¹² | 6 | 0 | 3 | 12 | 3 ² , 6 |
| 49.168.12.2 | $\begin{bmatrix} 1 & 15 \\ 1 & 27 \end{bmatrix}, \begin{bmatrix} 7 & 44 \\ 4 & 14 \end{bmatrix}$ 343.2.a.c, 343.2.a.d | 49A ¹² | 6 | 0 | 6 | 12 | 6 ² |
| 49.196.9.1 | $\begin{bmatrix} 20 & 46 \\ 25 & 21 \end{bmatrix}, \begin{bmatrix} 40 & 36 \\ 44 & 2 \end{bmatrix}$ 2401.2.a.b, 2401.2.a.c | 49B ⁹ | 10 | 0 | 9 | 9 | 3, 6 |
| 49.1029.69.1 | $\begin{bmatrix} 22 & 10 \\ 1 & 22 \end{bmatrix}, \begin{bmatrix} 29 & 5 \\ 24 & 20 \end{bmatrix}$ 2401.2.a.b, 2401.2.a.c, 2401.2.a.e, 2401.2.a.f, 2401.2.a.g, 2401.2.a.h | none | 21 | 0 | 69 | 69 | 3, 6, 9 ² , 18, 24 |
| 49.1372.94.1 | $\begin{bmatrix} 0 & 12 \\ 43 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 23 & 0 \end{bmatrix}$ 49.2.a.a, 343.2.a.a, 343.2.a.b, 2401.2.a.b, 343.2.a.c, 343.2.a.d, 343.2.a.e, 2401.2.a.c, 2401.2.a.e, 2401.2.a.f, 2401.2.a.g, 2401.2.a.h | none | 28 | 1 | 78 | 94 | 1, 3 ³ , 6 ⁴ , 9 ² , 18, 24 |
| 11.12.1.1 | $\begin{bmatrix} 7 & 2 \\ 0 & 6 \end{bmatrix}, \begin{bmatrix} 6 & 8 \\ 0 & 5 \end{bmatrix}$ 11.2.a.a | 11A ¹ | 2 | 2 | 0 | 1 | 1 |
| 11.55.1.2 | $\begin{bmatrix} 5 & 10 \\ 9 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 7 \\ 10 & 10 \end{bmatrix}$ 121.2.a.a | 11B ¹ | 5 | 0 | 0 | 1 | 1 |
| 11.66.2.1 | $\begin{bmatrix} 6 & 0 \\ 0 & 9 \end{bmatrix}, \begin{bmatrix} 9 & 10 \\ 7 & 0 \end{bmatrix}$ 11.2.a.a, 121.2.a.b | 11A ² | 6 | 1 | 1 | 2 | 1 ² |
| 11.110.4.1 | $\begin{bmatrix} 10 & 9 \\ 4 & 1 \end{bmatrix}, \begin{bmatrix} 7 & 10 \\ 9 & 7 \end{bmatrix}$ 11.2.a.a, 121.2.a.a, 121.2.a.b, 121.2.a.c | 11A ⁴ | 10 | 0 | 1 | 4 | 1 ⁴ |
| 121.605.41.1 | $\begin{bmatrix} 72 & 55 \\ 18 & 93 \end{bmatrix}, \begin{bmatrix} 118 & 83 \\ 17 & 4 \end{bmatrix}$ 121.2.a.b, 14641.2.a.a, 14641.2.a.c | none | 5 | 0 | 41 | 41 | 1, 5, 35 |
| 121.6655.511.1 | $\begin{bmatrix} 1 & 61 \\ 45 & 1 \end{bmatrix}, \begin{bmatrix} 10 & 2 \\ 62 & 111 \end{bmatrix}$ 121.2.a.b, 14641.2.a.a, 14641.2.a.b, 14641.2.a.c, 14641.2.a.d, 14641.2.a.e, 14641.2.a.g, 14641.2.a.i, 14641.2.a.j | none | 55 | 0 | 511 | 511 | 1, 5 ² , 35 ² , 50, 90 ² , 200 |

Table 15. Arithmetically maximal subgroups of ℓ -power level for $\ell = 13, 17, 19$.

| Label | Generators | CP | $\#X_H^\infty(\overline{\mathbb{Q}})$ | $\#X_H^\infty(\mathbb{Q})$ | r | g | Dimensions |
|-------------|---|-------------------|---------------------------------------|----------------------------|-----|-----|--|
| 13.78.3.1 | $\begin{bmatrix} 0 & 11 \\ 8 & 0 \end{bmatrix}, \begin{bmatrix} 11 & 2 \\ 10 & 2 \end{bmatrix}$ 169.2.a.b | 13A ³ | 6 | 0 | 3 | 3 | 3 |
| 13.84.2.1 | $\begin{bmatrix} 1 & 10 \\ 0 & 6 \end{bmatrix}, \begin{bmatrix} 12 & 8 \\ 0 & 11 \end{bmatrix}$ 13.2.e.a | 13A ² | 12 | 6 | 0 | 2 | 2 |
| 13.84.2.2 | $\begin{bmatrix} 3 & 7 \\ 0 & 8 \end{bmatrix}, \begin{bmatrix} 12 & 4 \\ 0 & 12 \end{bmatrix}$ 169.2.e.a | 13A ² | 12 | 0 | 0 | 2 | 2 |
| 13.84.2.3 | $\begin{bmatrix} 9 & 2 \\ 0 & 7 \end{bmatrix}, \begin{bmatrix} 4 & 4 \\ 0 & 7 \end{bmatrix}$ 169.2.b.a | 13A ² | 12 | 0 | 0 | 2 | 2 |
| 13.84.2.4 | $\begin{bmatrix} 8 & 12 \\ 0 & 10 \end{bmatrix}, \begin{bmatrix} 8 & 3 \\ 0 & 9 \end{bmatrix}$ 169.2.e.a | 13A ² | 12 | 0 | 0 | 2 | 2 |
| 13.84.2.5 | $\begin{bmatrix} 2 & 12 \\ 0 & 12 \end{bmatrix}, \begin{bmatrix} 8 & 8 \\ 0 & 1 \end{bmatrix}$ 13.2.e.a | 13A ² | 12 | 6 | 0 | 2 | 2 |
| 13.84.2.6 | $\begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 11 & 3 \\ 0 & 10 \end{bmatrix}$ 169.2.b.a | 13A ² | 12 | 0 | 0 | 2 | 2 |
| 13.91.3.1 | $\begin{bmatrix} 0 & 6 \\ 7 & 0 \end{bmatrix}, \begin{bmatrix} 10 & 0 \\ 0 & 7 \end{bmatrix}$ 169.2.a.b | 13C ³ | 7 | 1 | 3 | 3 | 3 |
| 13.91.3.2 | $\begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 12 & 1 \end{bmatrix}$ 169.2.a.b | 13B ³ | 7 | 0 | 3 | 3 | 3 |
| 169.182.8.1 | $\begin{bmatrix} 25 & 117 \\ 0 & 123 \end{bmatrix}, \begin{bmatrix} 98 & 68 \\ 0 & 75 \end{bmatrix}$ 169.2.a.a, 169.2.a.b, 169.2.a.c | 169A ⁸ | 14 | 2 | 3 | 8 | 2, 3 ² |
| 17.18.1.1 | $\begin{bmatrix} 12 & 4 \\ 0 & 16 \end{bmatrix}, \begin{bmatrix} 12 & 8 \\ 0 & 10 \end{bmatrix}$ 17.2.a.a | 17A ¹ | 2 | 2 | 0 | 1 | 1 |
| 17.136.6.1 | $\begin{bmatrix} 3 & 14 \\ 13 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 6 \\ 9 & 0 \end{bmatrix}$ 289.2.a.a, 289.2.a.b, 289.2.a.d | 17A ⁶ | 8 | 0 | 6 | 6 | 1, 2, 3 |
| 17.153.7.1 | $\begin{bmatrix} 16 & 0 \\ 0 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 9 & 0 \end{bmatrix}$ 17.2.a.a, 289.2.a.a, 289.2.a.b, 289.2.a.d | 17A ⁷ | 9 | 1 | 6 | 7 | 1 ² , 2, 3 |
| 19.20.1.1 | $\begin{bmatrix} 17 & 11 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 9 \end{bmatrix}$ 19.2.a.a | 19A ¹ | 2 | 2 | 0 | 1 | 1 |
| 19.171.8.1 | $\begin{bmatrix} 13 & 1 \\ 5 & 6 \end{bmatrix}, \begin{bmatrix} 18 & 13 \\ 11 & 18 \end{bmatrix}$ 361.2.a.a, 361.2.a.g, 361.2.a.i | 19A ⁸ | 9 | 0 | 8 | 8 | 1, 3, 4 |
| 19.190.9.1 | $\begin{bmatrix} 16 & 0 \\ 0 & 18 \end{bmatrix}, \begin{bmatrix} 0 & 8 \\ 9 & 0 \end{bmatrix}$ 19.2.a.a, 361.2.a.a, 361.2.a.g, 361.2.a.i | 19A ⁹ | 10 | 1 | 8 | 9 | 1 ² , 3, 4 |
| 19.285.14.1 | $\begin{bmatrix} 13 & 17 \\ 8 & 6 \end{bmatrix}, \begin{bmatrix} 16 & 15 \\ 2 & 16 \end{bmatrix}$ 19.2.a.a, 361.2.a.a, 361.2.a.b, 361.2.a.c, 361.2.a.e, 361.2.a.g, 361.2.a.i | 19A ¹⁴ | 15 | 0 | 8 | 14 | 1 ³ , 2 ² , 3, 4 |

Table 16. *Arithmetically maximal subgroups of ℓ -power level for $\ell = 23, 29, 31, 37$.*

| Label | Generators | CP | $\#X_H^\infty(\overline{\mathbb{Q}})$ | $\#X_H^\infty(\mathbb{Q})$ | r | g | Dimensions |
|---------------|--|-------------------|---------------------------------------|----------------------------|-----|-----|---|
| 23.24.2.1 | $\begin{bmatrix} 13 & 13 \\ 0 & 11 \end{bmatrix}, \begin{bmatrix} 20 & 0 \\ 0 & 18 \end{bmatrix}$ 23.2.a.a | 23A ² | 2 | 2 | 0 | 2 | 2 |
| 23.253.13.1 | $\begin{bmatrix} 1 & 19 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 16 & 13 \\ 9 & 7 \end{bmatrix}$ 529.2.a.g, 529.2.a.h, 529.2.a.i | 23A ¹³ | 11 | 0 | 13 | 13 | 4 ² , 5 |
| 23.276.15.1 | $\begin{bmatrix} 0 & 12 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 21 \\ 18 & 0 \end{bmatrix}$ 23.2.a.a, 529.2.a.g, 529.2.a.h, 529.2.a.i | 23A ¹⁵ | 12 | 1 | 13 | 15 | 2, 4 ² , 5 |
| 29.30.2.1 | $\begin{bmatrix} 23 & 26 \\ 0 & 12 \end{bmatrix}, \begin{bmatrix} 27 & 4 \\ 0 & 4 \end{bmatrix}$ 29.2.a.a | 29A ² | 2 | 2 | 0 | 2 | 2 |
| 29.406.24.1 | $\begin{bmatrix} 22 & 27 \\ 28 & 22 \end{bmatrix}, \begin{bmatrix} 2 & 24 \\ 17 & 24 \end{bmatrix}$ 841.2.a.a, 841.2.a.d, 841.2.a.e, 841.2.a.f, 841.2.a.g, 841.2.a.i | 29A ²⁴ | 14 | 0 | 24 | 24 | 2 ² , 3 ² , 6, 8 |
| 29.435.26.1 | $\begin{bmatrix} 0 & 16 \\ 17 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 26 \\ 17 & 0 \end{bmatrix}$ 29.2.a.a, 841.2.a.a, 841.2.a.d, 841.2.a.e, 841.2.a.f, 841.2.a.g, 841.2.a.i | none | 15 | 1 | 24 | 26 | 2 ³ , 3 ² , 6, 8 |
| 29.1015.63.1 | $\begin{bmatrix} 16 & 7 \\ 7 & 20 \end{bmatrix}, \begin{bmatrix} 10 & 21 \\ 15 & 20 \end{bmatrix}$ 29.2.a.a, 841.2.a.a ² , 841.2.a.b, 841.2.a.c, 841.2.a.d, 841.2.a.e ² , 841.2.a.f, 841.2.a.g, 841.2.a.i ² , 841.2.a.j, 841.2.a.k | none | 35 | 0 | 37 | 63 | 2 ⁶ , 3 ³ , 6, 8 ³ , 12 |
| 31.32.2.1 | $\begin{bmatrix} 21 & 12 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 11 & 1 \\ 0 & 13 \end{bmatrix}$ 31.2.a.a | 31A ² | 2 | 2 | 0 | 2 | 2 |
| 31.465.28.1 | $\begin{bmatrix} 0 & 8 \\ 18 & 0 \end{bmatrix}, \begin{bmatrix} 26 & 28 \\ 30 & 26 \end{bmatrix}$ 961.2.a.a, 961.2.a.d, 961.2.a.i, 961.2.a.l | none | 15 | 0 | 28 | 28 | 2 ² , 8, 16 |
| 31.496.30.1 | $\begin{bmatrix} 0 & 26 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 22 \\ 12 & 0 \end{bmatrix}$ 31.2.a.a, 961.2.a.a, 961.2.a.d, 961.2.a.i, 961.2.a.l | none | 16 | 1 | 28 | 30 | 2 ³ , 8, 16 |
| 37.38.2.1 | $\begin{bmatrix} 23 & 20 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 22 & 22 \\ 0 & 36 \end{bmatrix}$ 37.2.a.b, 37.2.a.a | 37A ² | 2 | 2 | 1 | 2 | 1 ² |
| 37.666.43.1 | $\begin{bmatrix} 35 & 20 \\ 33 & 2 \end{bmatrix}, \begin{bmatrix} 28 & 1 \\ 31 & 9 \end{bmatrix}$ 1369.2.a.b, 1369.2.a.c, 1369.2.a.d, 1369.2.a.e, 1369.2.a.i, 1369.2.a.j, 1369.2.a.k, 1369.2.a.l, 1369.2.a.n | none | 18 | 0 | 43 | 43 | 1 ⁴ , 3 ⁴ , 27 |
| 37.703.45.1 | $\begin{bmatrix} 0 & 7 \\ 13 & 0 \end{bmatrix}, \begin{bmatrix} 35 & 0 \\ 0 & 3 \end{bmatrix}$ 37.2.a.a, 37.2.a.b, 1369.2.a.b, 1369.2.a.c, 1369.2.a.d, 1369.2.a.e, 1369.2.a.i, 1369.2.a.j, 1369.2.a.k, 1369.2.a.l, 1369.2.a.n | none | 19 | 1 | 44 | 45 | 1 ⁶ , 3 ⁴ , 27 |
| 37.2109.142.1 | $\begin{bmatrix} 25 & 16 \\ 33 & 13 \end{bmatrix}, \begin{bmatrix} 1 & 16 \\ 4 & 18 \end{bmatrix}$ 37.2.a.a ² , 37.2.a.b ² , 1369.2.a.a ² , 1369.2.a.b ² , 1369.2.a.c ² , 1369.2.a.d ³ , 1369.2.a.e ² , 1369.2.a.f ² , 1369.2.a.g ² , 1369.2.a.h ² , 1369.2.a.i ² , 1369.2.a.j ² , 1369.2.a.k, 1369.2.a.l, 1369.2.a.m, 1369.2.a.n ² , 1369.2.a.o | none | 57 | 0 | 83 | 142 | 1 ¹⁷ , 2 ⁴ , 3 ⁶ , 18, 27 ³ |

Table 17. Arithmetically maximal subgroups of 2-power level.

| Label | Generators | CP | $\#X_H^\infty(\overline{\mathbb{Q}})$ | $\#X_H^\infty(\mathbb{Q})$ | r | g | dims |
|------------|---|---------|---------------------------------------|----------------------------|-----|-----|------|
| 8.12.1.1 | $\begin{bmatrix} 0 & 3 \\ 5 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 5 & 2 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ 5 & 0 \end{bmatrix}$ 32.2.a.a | $8A^1$ | 2 | 2 | 0 | 1 | 1 |
| 8.12.1.2 | $\begin{bmatrix} 5 & 2 \\ 4 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 6 & 1 \\ 7 & 6 \end{bmatrix}$ 64.2.a.a | $8A^1$ | 2 | 2 | 0 | 1 | 1 |
| 8.12.1.3 | $\begin{bmatrix} 0 & 3 \\ 7 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 6 & 1 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 0 & 7 \end{bmatrix}$ 32.2.a.a | $8A^1$ | 2 | 2 | 0 | 1 | 1 |
| 8.12.1.4 | $\begin{bmatrix} 4 & 5 \\ 7 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix}, \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix}$ 64.2.a.a | $8A^1$ | 2 | 2 | 0 | 1 | 1 |
| 8.32.1.2 | $\begin{bmatrix} 1 & 5 \\ 7 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 3 & 7 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 5 & 0 \end{bmatrix}$ 64.2.a.a | $8E^1$ | 4 | 0 | 0 | 1 | 1 |
| 8.32.1.3 | $\begin{bmatrix} 5 & 3 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 7 & 3 \\ 7 & 0 \end{bmatrix}$ 32.2.a.a | $8E^1$ | 4 | 0 | 0 | 1 | 1 |
| 8.48.1.27 | $\begin{bmatrix} 5 & 3 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 4 & 7 \end{bmatrix}, \begin{bmatrix} 7 & 7 \\ 2 & 1 \end{bmatrix}$ 64.2.a.a | $8I^1$ | 6 | 0 | 0 | 1 | 1 |
| 8.48.1.72 | $\begin{bmatrix} 0 & 7 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 6 & 1 \end{bmatrix}$ 32.2.a.a | $8I^1$ | 6 | 0 | 0 | 1 | 1 |
| 8.48.1.91 | $\begin{bmatrix} 0 & 7 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix}, \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$ 32.2.a.a | $8I^1$ | 6 | 2 | 0 | 1 | 1 |
| 8.48.1.92 | $\begin{bmatrix} 0 & 7 \\ 7 & 0 \end{bmatrix}, \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 4 & 5 \end{bmatrix}$ 64.2.a.a | $8I^1$ | 6 | 0 | 0 | 1 | 1 |
| 8.48.2.1 | $\begin{bmatrix} 5 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 5 & 0 \\ 4 & 5 \end{bmatrix}$ 64.2.b.a | $8A^2$ | 6 | 4 | 0 | 2 | 2 |
| 16.24.1.1 | $\begin{bmatrix} 13 & 10 \\ 0 & 13 \end{bmatrix}, \begin{bmatrix} 15 & 1 \\ 0 & 11 \end{bmatrix}, \begin{bmatrix} 15 & 8 \\ 8 & 3 \end{bmatrix}, \begin{bmatrix} 11 & 9 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 5 & 2 \\ 8 & 1 \end{bmatrix}$ 64.2.a.a | $16A^1$ | 4 | 4 | 0 | 1 | 1 |
| 16.24.1.2 | $\begin{bmatrix} 15 & 0 \\ 15 & 5 \end{bmatrix}, \begin{bmatrix} 13 & 0 \\ 6 & 1 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 3 & 3 \end{bmatrix}, \begin{bmatrix} 15 & 0 \\ 10 & 11 \end{bmatrix}, \begin{bmatrix} 7 & 8 \\ 3 & 7 \end{bmatrix}$ 32.2.a.a | $16A^1$ | 4 | 4 | 0 | 1 | 1 |
| 16.24.1.3 | $\begin{bmatrix} 11 & 0 \\ 8 & 7 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 4 & 13 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 8 & 7 \end{bmatrix}, \begin{bmatrix} 7 & 4 \\ 13 & 3 \end{bmatrix}$ 64.2.a.a | $16A^1$ | 4 | 2 | 0 | 1 | 1 |
| 16.24.1.4 | $\begin{bmatrix} 15 & 4 \\ 3 & 15 \end{bmatrix}, \begin{bmatrix} 5 & 12 \\ 3 & 15 \end{bmatrix}, \begin{bmatrix} 5 & 0 \\ 14 & 3 \end{bmatrix}, \begin{bmatrix} 9 & 4 \\ 11 & 7 \end{bmatrix}$ 32.2.a.a | $16A^1$ | 4 | 2 | 0 | 1 | 1 |
| 16.24.1.6 | $\begin{bmatrix} 5 & 8 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 11 & 4 \\ 14 & 7 \end{bmatrix}, \begin{bmatrix} 9 & 14 \\ 13 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 10 \\ 5 & 9 \end{bmatrix}$ 128.2.a.c | $16D^1$ | 2 | 2 | 0 | 1 | 1 |
| 16.24.1.7 | $\begin{bmatrix} 13 & 10 \\ 5 & 13 \end{bmatrix}, \begin{bmatrix} 13 & 10 \\ 3 & 7 \end{bmatrix}, \begin{bmatrix} 7 & 14 \\ 15 & 15 \end{bmatrix}, \begin{bmatrix} 13 & 0 \\ 0 & 9 \end{bmatrix}$ 128.2.a.b | $16D^1$ | 2 | 2 | 0 | 1 | 1 |
| 16.24.1.8 | $\begin{bmatrix} 0 & 7 \\ 13 & 12 \end{bmatrix}, \begin{bmatrix} 8 & 5 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 13 & 14 \\ 4 & 7 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 0 & 13 \end{bmatrix}$ 128.2.a.d | $16D^1$ | 2 | 2 | 0 | 1 | 1 |
| 16.24.1.9 | $\begin{bmatrix} 13 & 8 \\ 4 & 5 \end{bmatrix}, \begin{bmatrix} 13 & 2 \\ 4 & 9 \end{bmatrix}, \begin{bmatrix} 11 & 11 \\ 14 & 1 \end{bmatrix}, \begin{bmatrix} 13 & 10 \\ 14 & 11 \end{bmatrix}$ 256.2.a.d | $8D^1$ | 3 | 1 | 0 | 1 | 1 |
| 16.24.1.12 | $\begin{bmatrix} 0 & 3 \\ 13 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 11 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 8 \\ 8 & 11 \end{bmatrix}, \begin{bmatrix} 10 & 3 \\ 5 & 6 \end{bmatrix}$ 256.2.a.d | $8D^1$ | 3 | 1 | 0 | 1 | 1 |
| 16.24.1.14 | $\begin{bmatrix} 9 & 7 \\ 4 & 15 \end{bmatrix}, \begin{bmatrix} 1 & 7 \\ 0 & 7 \end{bmatrix}, \begin{bmatrix} 5 & 1 \\ 12 & 3 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 10 & 7 \end{bmatrix}$ 256.2.a.d | $16C^1$ | 3 | 1 | 0 | 1 | 1 |
| 16.24.1.16 | $\begin{bmatrix} 3 & 0 \\ 12 & 1 \end{bmatrix}, \begin{bmatrix} 8 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 11 & 12 \\ 8 & 13 \end{bmatrix}, \begin{bmatrix} 6 & 11 \\ 15 & 10 \end{bmatrix}$ 256.2.a.d | $16C^1$ | 3 | 1 | 0 | 1 | 1 |
| 16.24.1.18 | $\begin{bmatrix} 7 & 10 \\ 4 & 15 \end{bmatrix}, \begin{bmatrix} 15 & 14 \\ 2 & 13 \end{bmatrix}, \begin{bmatrix} 5 & 9 \\ 8 & 7 \end{bmatrix}, \begin{bmatrix} 3 & 12 \\ 12 & 7 \end{bmatrix}$ 256.2.a.c | $16B^1$ | 2 | 2 | 0 | 1 | 1 |

(continued)

Table 17. (continued)

| Label | Generators | CP | $\#X_H^\infty(\overline{\mathbb{Q}})$ | $\#X_H^\infty(\mathbb{Q})$ | r | g | dims |
|-------------|--|------------------|---------------------------------------|----------------------------|-----|-----|----------------|
| 16.24.1.20 | $\begin{bmatrix} 5 & 0 \\ 2 & 11 \end{bmatrix}, \begin{bmatrix} 1 & 10 \\ 2 & 7 \end{bmatrix}, \begin{bmatrix} 11 & 13 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 9 & 12 \\ 6 & 11 \end{bmatrix}$ 256.2.a.c | 16B ¹ | 2 | 2 | 0 | 1 | 1 |
| 16.24.2.1 | $\begin{bmatrix} 14 & 11 \\ 11 & 8 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ 11 & 0 \end{bmatrix}, \begin{bmatrix} 8 & 5 \\ 9 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 11 \\ 1 & 6 \end{bmatrix}$ 128.2.b.a | 16A ² | 2 | 2 | 0 | 2 | 2 |
| 16.24.2.2 | $\begin{bmatrix} 11 & 6 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}, \begin{bmatrix} 11 & 7 \\ 10 & 11 \end{bmatrix}, \begin{bmatrix} 13 & 2 \\ 4 & 13 \end{bmatrix}$ 128.2.b.a | 16A ² | 2 | 2 | 0 | 2 | 2 |
| 16.24.2.3 | $\begin{bmatrix} 3 & 4 \\ 12 & 7 \end{bmatrix}, \begin{bmatrix} 10 & 13 \\ 5 & 14 \end{bmatrix}, \begin{bmatrix} 0 & 13 \\ 15 & 0 \end{bmatrix}, \begin{bmatrix} 14 & 3 \\ 13 & 10 \end{bmatrix}$ 256.2.a.e | 16A ² | 2 | 2 | 0 | 2 | 2 |
| 16.24.2.4 | $\begin{bmatrix} 8 & 13 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 11 \\ 15 & 14 \end{bmatrix}, \begin{bmatrix} 9 & 10 \\ 10 & 15 \end{bmatrix}, \begin{bmatrix} 4 & 13 \\ 13 & 8 \end{bmatrix}$ 256.2.a.e | 16A ² | 2 | 2 | 0 | 2 | 2 |
| 16.48.1.115 | $\begin{bmatrix} 1 & 15 \\ 0 & 11 \end{bmatrix}, \begin{bmatrix} 3 & 10 \\ 10 & 13 \end{bmatrix}, \begin{bmatrix} 5 & 3 \\ 0 & 15 \end{bmatrix}, \begin{bmatrix} 1 & 7 \\ 14 & 15 \end{bmatrix}$ 32.2.a.a | 16H ¹ | 6 | 2 | 0 | 1 | 1 |
| 16.48.1.116 | $\begin{bmatrix} 0 & 3 \\ 5 & 0 \end{bmatrix}, \begin{bmatrix} 10 & 1 \\ 7 & 14 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 10 \end{bmatrix}, \begin{bmatrix} 15 & 14 \\ 2 & 1 \end{bmatrix}$ 64.2.a.a | 16H ¹ | 6 | 0 | 0 | 1 | 1 |
| 16.48.1.141 | $\begin{bmatrix} 14 & 13 \\ 15 & 10 \end{bmatrix}, \begin{bmatrix} 10 & 11 \\ 1 & 6 \end{bmatrix}, \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 9 \\ 11 & 8 \end{bmatrix}$ 32.2.a.a | 16H ¹ | 6 | 0 | 0 | 1 | 1 |
| 16.48.1.142 | $\begin{bmatrix} 11 & 0 \\ 4 & 13 \end{bmatrix}, \begin{bmatrix} 13 & 12 \\ 4 & 5 \end{bmatrix}, \begin{bmatrix} 0 & 15 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 15 & 14 \\ 14 & 7 \end{bmatrix}$ 64.2.a.a | 16H ¹ | 6 | 0 | 0 | 1 | 1 |
| 16.48.2.1 | $\begin{bmatrix} 7 & 0 \\ 6 & 1 \end{bmatrix}, \begin{bmatrix} 5 & 0 \\ 6 & 13 \end{bmatrix}, \begin{bmatrix} 7 & 8 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 2 & 15 \end{bmatrix}, \begin{bmatrix} 9 & 8 \\ 6 & 13 \end{bmatrix}$ 64.2.b.a | 16C ² | 6 | 4 | 0 | 2 | 2 |
| 16.48.2.2 | $\begin{bmatrix} 15 & 0 \\ 2 & 11 \end{bmatrix}, \begin{bmatrix} 5 & 0 \\ 0 & 11 \end{bmatrix}, \begin{bmatrix} 9 & 8 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 2 & 11 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 10 & 15 \end{bmatrix}$ 64.2.b.a | 16C ² | 6 | 4 | 0 | 2 | 2 |
| 16.48.2.4 | $\begin{bmatrix} 15 & 4 \\ 14 & 3 \end{bmatrix}, \begin{bmatrix} 5 & 12 \\ 6 & 9 \end{bmatrix}, \begin{bmatrix} 13 & 4 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 11 & 10 \\ 14 & 5 \end{bmatrix}$ 256.2.a.a ² | 8A ² | 6 | 0 | 2 | 2 | 1 ² |
| 16.48.2.7 | $\begin{bmatrix} 8 & 1 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 13 & 10 \end{bmatrix}, \begin{bmatrix} 8 & 9 \\ 3 & 4 \end{bmatrix}$ 256.2.a.a ² | 8A ² | 6 | 0 | 2 | 2 | 1 ² |
| 16.48.2.9 | $\begin{bmatrix} 13 & 0 \\ 14 & 5 \end{bmatrix}, \begin{bmatrix} 5 & 0 \\ 4 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 9 & 13 \end{bmatrix}, \begin{bmatrix} 15 & 0 \\ 0 & 7 \end{bmatrix}, \begin{bmatrix} 15 & 8 \\ 2 & 15 \end{bmatrix}$ 64.2.b.a | 16C ² | 6 | 4 | 0 | 2 | 2 |
| 16.48.2.10 | $\begin{bmatrix} 3 & 15 \\ 12 & 13 \end{bmatrix}, \begin{bmatrix} 15 & 5 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 15 & 4 \\ 0 & 15 \end{bmatrix}, \begin{bmatrix} 9 & 13 \\ 8 & 15 \end{bmatrix}$ 256.2.a.a ² | 16C ² | 6 | 0 | 2 | 2 | 1 ² |
| 16.48.2.16 | $\begin{bmatrix} 0 & 15 \\ 11 & 0 \end{bmatrix}, \begin{bmatrix} 8 & 9 \\ 3 & 14 \end{bmatrix}, \begin{bmatrix} 3 & 6 \\ 2 & 15 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 5 & 0 \end{bmatrix}$ 256.2.a.a ² | 16C ² | 6 | 0 | 2 | 2 | 1 ² |
| 16.48.2.17 | $\begin{bmatrix} 11 & 9 \\ 4 & 13 \end{bmatrix}, \begin{bmatrix} 13 & 5 \\ 4 & 11 \end{bmatrix}, \begin{bmatrix} 1 & 9 \\ 12 & 7 \end{bmatrix}, \begin{bmatrix} 1 & 9 \\ 0 & 5 \end{bmatrix}$ 256.2.a.d ² | 16C ² | 6 | 0 | 0 | 2 | 1 ² |
| 16.48.2.20 | $\begin{bmatrix} 3 & 0 \\ 0 & 11 \end{bmatrix}, \begin{bmatrix} 9 & 14 \\ 1 & 11 \end{bmatrix}, \begin{bmatrix} 11 & 4 \\ 11 & 5 \end{bmatrix}, \begin{bmatrix} 13 & 14 \\ 14 & 11 \end{bmatrix}$ 128.2.a.a, 128.2.a.c | 16E ² | 4 | 0 | 1 | 2 | 1 ² |
| 16.48.2.21 | $\begin{bmatrix} 5 & 0 \\ 0 & 11 \end{bmatrix}, \begin{bmatrix} 6 & 11 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 14 & 1 \\ 15 & 14 \end{bmatrix}, \begin{bmatrix} 8 & 7 \\ 9 & 0 \end{bmatrix}$ 128.2.a.b, 128.2.a.d | 16E ² | 4 | 0 | 0 | 2 | 1 ² |
| 16.48.2.22 | $\begin{bmatrix} 5 & 4 \\ 2 & 11 \end{bmatrix}, \begin{bmatrix} 13 & 2 \\ 8 & 13 \end{bmatrix}, \begin{bmatrix} 7 & 5 \\ 4 & 9 \end{bmatrix}$ 256.2.a.a ² | 16C ² | 6 | 0 | 2 | 2 | 1 ² |
| 16.64.2.1 | $\begin{bmatrix} 1 & 7 \\ 5 & 14 \end{bmatrix}, \begin{bmatrix} 13 & 3 \\ 6 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 9 \\ 5 & 0 \end{bmatrix}$ 256.2.a.a, 256.2.a.b | 16G ² | 4 | 0 | 2 | 2 | 1 ² |
| 16.96.2.15 | $\begin{bmatrix} 9 & 8 \\ 4 & 1 \end{bmatrix}, \begin{bmatrix} 9 & 4 \\ 8 & 9 \end{bmatrix}, \begin{bmatrix} 7 & 8 \\ 8 & 1 \end{bmatrix}, \begin{bmatrix} 15 & 0 \\ 0 & 7 \end{bmatrix}, \begin{bmatrix} 15 & 12 \\ 0 & 5 \end{bmatrix}$ 16.2.e.a | 16I ² | 14 | 6 | 0 | 2 | 2 |
| 16.96.2.16 | $\begin{bmatrix} 7 & 0 \\ 4 & 11 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 15 \end{bmatrix}, \begin{bmatrix} 15 & 4 \\ 12 & 7 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 12 & 9 \end{bmatrix}, \begin{bmatrix} 9 & 12 \\ 0 & 1 \end{bmatrix}$ 64.2.e.a | 16I ² | 14 | 2 | 0 | 2 | 2 |

(continued)

Table 17. (continued)

| Label | Generators | CP | $\#X_H^\infty(\overline{\mathbb{Q}})$ | $\#X_H^\infty(\mathbb{Q})$ | r | g | dims |
|-------------|--|---------|---------------------------------------|----------------------------|-----|-----|------|
| 16.96.2.17 | $\begin{bmatrix} 3 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 12 \\ 4 & 7 \end{bmatrix}, \begin{bmatrix} 15 & 8 \\ 4 & 15 \end{bmatrix}, \begin{bmatrix} 9 & 8 \\ 12 & 1 \end{bmatrix}, \begin{bmatrix} 13 & 8 \\ 0 & 15 \end{bmatrix}$ 128.2.e.a | $16I^2$ | 14 | 2 | 0 | 2 | 2 |
| 16.96.2.18 | $\begin{bmatrix} 7 & 8 \\ 12 & 7 \end{bmatrix}, \begin{bmatrix} 7 & 0 \\ 4 & 13 \end{bmatrix}, \begin{bmatrix} 9 & 4 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 4 & 13 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 8 & 7 \end{bmatrix}$ 128.2.e.b | $16I^2$ | 14 | 2 | 0 | 2 | 2 |
| 16.96.2.35 | $\begin{bmatrix} 11 & 0 \\ 10 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 15 \end{bmatrix}, \begin{bmatrix} 15 & 0 \\ 10 & 7 \end{bmatrix}, \begin{bmatrix} 9 & 8 \\ 14 & 7 \end{bmatrix}, \begin{bmatrix} 7 & 8 \\ 2 & 9 \end{bmatrix}$ 16.2.e.a | $16K^2$ | 14 | 6 | 0 | 2 | 2 |
| 16.96.2.36 | $\begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 14 & 11 \end{bmatrix}, \begin{bmatrix} 13 & 4 \\ 4 & 15 \end{bmatrix}, \begin{bmatrix} 1 & 12 \\ 4 & 15 \end{bmatrix}, \begin{bmatrix} 5 & 0 \\ 8 & 5 \end{bmatrix}$ 64.2.e.a | $16K^2$ | 14 | 2 | 0 | 2 | 2 |
| 16.96.2.37 | $\begin{bmatrix} 7 & 0 \\ 6 & 9 \end{bmatrix}, \begin{bmatrix} 5 & 8 \\ 12 & 1 \end{bmatrix}, \begin{bmatrix} 11 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 13 & 0 \\ 0 & 15 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$ 128.2.e.a | $16K^2$ | 14 | 2 | 0 | 2 | 2 |
| 16.96.2.38 | $\begin{bmatrix} 13 & 2 \\ 0 & 13 \end{bmatrix}, \begin{bmatrix} 13 & 14 \\ 8 & 5 \end{bmatrix}, \begin{bmatrix} 5 & 2 \\ 4 & 3 \end{bmatrix}, \begin{bmatrix} 15 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 8 & 3 \end{bmatrix}$ 128.2.e.b | $16K^2$ | 14 | 2 | 0 | 2 | 2 |
| 16.96.2.49 | $\begin{bmatrix} 15 & 8 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 7 & 0 \\ 8 & 9 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 0 & 11 \end{bmatrix}, \begin{bmatrix} 1 & 8 \\ 8 & 15 \end{bmatrix}, \begin{bmatrix} 7 & 0 \\ 8 & 5 \end{bmatrix}$ 64.2.e.a | $16K^2$ | 14 | 2 | 0 | 2 | 2 |
| 16.96.2.50 | $\begin{bmatrix} 9 & 14 \\ 0 & 7 \end{bmatrix}, \begin{bmatrix} 7 & 12 \\ 0 & 9 \end{bmatrix}, \begin{bmatrix} 9 & 6 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 13 & 2 \\ 0 & 9 \end{bmatrix}, \begin{bmatrix} 15 & 10 \\ 8 & 15 \end{bmatrix}$ 128.2.e.a | $16K^2$ | 14 | 2 | 0 | 2 | 2 |
| 16.96.2.51 | $\begin{bmatrix} 7 & 0 \\ 12 & 11 \end{bmatrix}, \begin{bmatrix} 1 & 8 \\ 12 & 1 \end{bmatrix}, \begin{bmatrix} 7 & 0 \\ 4 & 5 \end{bmatrix}, \begin{bmatrix} 9 & 8 \\ 14 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 8 \\ 2 & 5 \end{bmatrix}$ 128.2.e.b | $16K^2$ | 14 | 2 | 0 | 2 | 2 |
| 16.96.2.52 | $\begin{bmatrix} 9 & 0 \\ 12 & 5 \end{bmatrix}, \begin{bmatrix} 15 & 0 \\ 8 & 11 \end{bmatrix}, \begin{bmatrix} 15 & 12 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 8 \\ 6 & 11 \end{bmatrix}, \begin{bmatrix} 7 & 8 \\ 12 & 15 \end{bmatrix}$ 16.2.e.a | $16K^2$ | 14 | 6 | 0 | 2 | 2 |
| 16.96.2.121 | $\begin{bmatrix} 15 & 1 \\ 0 & 13 \end{bmatrix}, \begin{bmatrix} 15 & 11 \\ 0 & 11 \end{bmatrix}, \begin{bmatrix} 1 & 7 \\ 0 & 9 \end{bmatrix}, \begin{bmatrix} 15 & 11 \\ 0 & 15 \end{bmatrix}$ 16.2.e.a | $16J^2$ | 14 | 6 | 0 | 2 | 2 |
| 16.96.2.122 | $\begin{bmatrix} 9 & 2 \\ 0 & 15 \end{bmatrix}, \begin{bmatrix} 9 & 10 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 7 \\ 0 & 11 \end{bmatrix}, \begin{bmatrix} 7 & 11 \\ 0 & 9 \end{bmatrix}$ 128.2.e.a | $16J^2$ | 14 | 2 | 0 | 2 | 2 |
| 16.96.2.123 | $\begin{bmatrix} 1 & 4 \\ 0 & 13 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 0 & 11 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 15 & 11 \\ 0 & 7 \end{bmatrix}$ 64.2.e.a | $16J^2$ | 14 | 2 | 0 | 2 | 2 |
| 16.96.2.124 | $\begin{bmatrix} 9 & 10 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 9 & 1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 7 & 10 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 15 & 9 \\ 0 & 9 \end{bmatrix}$ 128.2.e.b | $16J^2$ | 14 | 2 | 0 | 2 | 2 |
| 16.96.2.125 | $\begin{bmatrix} 7 & 14 \\ 0 & 15 \end{bmatrix}, \begin{bmatrix} 15 & 8 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 11 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 7 & 3 \\ 0 & 15 \end{bmatrix}$ 16.2.e.a | $16J^2$ | 14 | 6 | 0 | 2 | 2 |
| 16.96.2.126 | $\begin{bmatrix} 11 & 9 \\ 0 & 7 \end{bmatrix}, \begin{bmatrix} 11 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 13 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 5 & 15 \\ 0 & 15 \end{bmatrix}$ 128.2.e.a | $16J^2$ | 14 | 2 | 0 | 2 | 2 |
| 16.96.2.127 | $\begin{bmatrix} 7 & 11 \\ 0 & 15 \end{bmatrix}, \begin{bmatrix} 9 & 6 \\ 0 & 7 \end{bmatrix}, \begin{bmatrix} 5 & 13 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 7 & 6 \\ 0 & 9 \end{bmatrix}$ 64.2.e.a | $16J^2$ | 14 | 2 | 0 | 2 | 2 |
| 16.96.2.128 | $\begin{bmatrix} 9 & 12 \\ 0 & 15 \end{bmatrix}, \begin{bmatrix} 5 & 9 \\ 0 & 7 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 15 \end{bmatrix}, \begin{bmatrix} 7 & 3 \\ 0 & 15 \end{bmatrix}$ 128.2.e.b | $16J^2$ | 14 | 2 | 0 | 2 | 2 |
| 16.96.3.335 | $\begin{bmatrix} 6 & 1 \\ 5 & 0 \end{bmatrix}, \begin{bmatrix} 8 & 5 \\ 3 & 8 \end{bmatrix}, \begin{bmatrix} 6 & 1 \\ 7 & 6 \end{bmatrix}$ 16.2.e.a, 128.2.a.a | $16S^3$ | 8 | 4 | 1 | 3 | 1,2 |
| 16.96.3.338 | $\begin{bmatrix} 15 & 0 \\ 0 & 9 \end{bmatrix}, \begin{bmatrix} 5 & 7 \\ 6 & 1 \end{bmatrix}, \begin{bmatrix} 13 & 15 \\ 6 & 5 \end{bmatrix}$ 128.2.a.a, 128.2.e.b | $16S^3$ | 8 | 0 | 1 | 3 | 1,2 |
| 16.96.3.343 | $\begin{bmatrix} 5 & 12 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 15 \\ 5 & 12 \end{bmatrix}, \begin{bmatrix} 11 & 8 \\ 14 & 9 \end{bmatrix}$ 16.2.e.a, 128.2.a.a | $16S^3$ | 8 | 4 | 1 | 3 | 1,2 |
| 16.96.3.346 | $\begin{bmatrix} 4 & 7 \\ 13 & 0 \end{bmatrix}, \begin{bmatrix} 10 & 13 \\ 15 & 14 \end{bmatrix}, \begin{bmatrix} 5 & 8 \\ 14 & 15 \end{bmatrix}$ 128.2.a.a, 128.2.e.b | $16S^3$ | 8 | 0 | 1 | 3 | 1,2 |
| 32.48.1.1 | $\begin{bmatrix} 29 & 4 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 31 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 29 & 27 \\ 0 & 15 \end{bmatrix}, \begin{bmatrix} 7 & 23 \\ 0 & 25 \end{bmatrix}, \begin{bmatrix} 21 & 5 \\ 0 & 1 \end{bmatrix}$ 32.2.a.a | $32A^1$ | 8 | 4 | 0 | 1 | 1 |

(continued)

Table 17. (continued)

| Label | Generators | CP | $\#X_H^\infty(\overline{\mathbb{Q}})$ | $\#X_H^\infty(\mathbb{Q})$ | r | g | dims |
|------------|--|------------------|---------------------------------------|----------------------------|-----|-----|----------------|
| 32.48.1.2 | $\begin{bmatrix} 3 & 0 \\ 5 & 3 \end{bmatrix}, \begin{bmatrix} 13 & 0 \\ 13 & 15 \end{bmatrix}, \begin{bmatrix} 17 & 0 \\ 7 & 25 \end{bmatrix}, \begin{bmatrix} 19 & 16 \\ 20 & 13 \end{bmatrix}, \begin{bmatrix} 29 & 0 \\ 16 & 5 \end{bmatrix}$ 64.2.a.a | 32A ¹ | 8 | 2 | 0 | 1 | 1 |
| 32.48.1.3 | $\begin{bmatrix} 27 & 16 \\ 2 & 5 \end{bmatrix}, \begin{bmatrix} 13 & 16 \\ 28 & 11 \end{bmatrix}, \begin{bmatrix} 11 & 0 \\ 15 & 9 \end{bmatrix}, \begin{bmatrix} 25 & 0 \\ 3 & 7 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 19 & 19 \end{bmatrix}$ 64.2.a.a | 32A ¹ | 8 | 4 | 0 | 1 | 1 |
| 32.48.1.4 | $\begin{bmatrix} 29 & 0 \\ 20 & 19 \end{bmatrix}, \begin{bmatrix} 29 & 14 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 23 & 11 \\ 0 & 13 \end{bmatrix}, \begin{bmatrix} 9 & 1 \\ 4 & 9 \end{bmatrix}, \begin{bmatrix} 13 & 10 \\ 20 & 31 \end{bmatrix}$ 32.2.a.a | 32A ¹ | 8 | 4 | 0 | 1 | 1 |
| 32.48.2.1 | $\begin{bmatrix} 15 & 16 \\ 7 & 17 \end{bmatrix}, \begin{bmatrix} 25 & 0 \\ 16 & 5 \end{bmatrix}, \begin{bmatrix} 11 & 0 \\ 11 & 25 \end{bmatrix}, \begin{bmatrix} 27 & 16 \\ 28 & 19 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 15 & 11 \end{bmatrix}$ 64.2.b.a | 32A ² | 6 | 4 | 0 | 2 | 2 |
| 32.48.2.2 | $\begin{bmatrix} 5 & 0 \\ 24 & 11 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 6 & 5 \end{bmatrix}, \begin{bmatrix} 5 & 16 \\ 5 & 17 \end{bmatrix}, \begin{bmatrix} 21 & 16 \\ 16 & 23 \end{bmatrix}$ 64.2.b.a | 32A ² | 6 | 4 | 0 | 2 | 2 |
| 32.48.2.3 | $\begin{bmatrix} 21 & 0 \\ 14 & 31 \end{bmatrix}, \begin{bmatrix} 11 & 16 \\ 14 & 21 \end{bmatrix}, \begin{bmatrix} 0 & 15 \\ 15 & 20 \end{bmatrix}, \begin{bmatrix} 11 & 2 \\ 16 & 29 \end{bmatrix}$ 256.2.a.a, 256.2.a.d | 32A ² | 6 | 2 | 1 | 2 | 1 ² |
| 32.48.2.4 | $\begin{bmatrix} 27 & 6 \\ 24 & 31 \end{bmatrix}, \begin{bmatrix} 13 & 10 \\ 0 & 15 \end{bmatrix}, \begin{bmatrix} 23 & 6 \\ 0 & 29 \end{bmatrix}, \begin{bmatrix} 9 & 9 \\ 28 & 11 \end{bmatrix}$ 256.2.a.a, 256.2.a.d | 32A ² | 6 | 2 | 1 | 2 | 1 ² |
| 32.96.2.17 | $\begin{bmatrix} 1 & 3 \\ 0 & 15 \end{bmatrix}, \begin{bmatrix} 15 & 18 \\ 0 & 11 \end{bmatrix}, \begin{bmatrix} 19 & 27 \\ 8 & 13 \end{bmatrix}, \begin{bmatrix} 31 & 1 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 25 & 21 \\ 16 & 11 \end{bmatrix}$ 16.2.e.a | 32B ² | 14 | 6 | 0 | 2 | 2 |
| 32.96.2.18 | $\begin{bmatrix} 15 & 14 \\ 0 & 31 \end{bmatrix}, \begin{bmatrix} 27 & 12 \\ 24 & 27 \end{bmatrix}, \begin{bmatrix} 21 & 29 \\ 24 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 8 \\ 8 & 7 \end{bmatrix}, \begin{bmatrix} 7 & 5 \\ 0 & 29 \end{bmatrix}$ 64.2.e.a | 32B ² | 14 | 2 | 0 | 2 | 2 |
| 32.96.2.19 | $\begin{bmatrix} 31 & 0 \\ 5 & 13 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 17 & 15 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 27 & 19 \end{bmatrix}, \begin{bmatrix} 3 & 24 \\ 14 & 15 \end{bmatrix}, \begin{bmatrix} 19 & 8 \\ 17 & 13 \end{bmatrix}$ 128.2.e.a | 32B ² | 14 | 2 | 0 | 2 | 2 |
| 32.96.2.20 | $\begin{bmatrix} 7 & 16 \\ 5 & 29 \end{bmatrix}, \begin{bmatrix} 15 & 0 \\ 4 & 15 \end{bmatrix}, \begin{bmatrix} 13 & 8 \\ 20 & 29 \end{bmatrix}, \begin{bmatrix} 7 & 0 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 7 & 0 \\ 27 & 5 \end{bmatrix}$ 128.2.e.b | 32B ² | 14 | 2 | 0 | 2 | 2 |
| 32.96.2.21 | $\begin{bmatrix} 7 & 20 \\ 0 & 23 \end{bmatrix}, \begin{bmatrix} 22 & 15 \\ 7 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 12 & 31 \end{bmatrix}$ 1024.2.a.b | 16I ² | 14 | 0 | 2 | 2 | 2 |
| 32.96.2.22 | $\begin{bmatrix} 21 & 0 \\ 20 & 27 \end{bmatrix}, \begin{bmatrix} 13 & 12 \\ 16 & 29 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}$ 1024.2.a.e | 16I ² | 14 | 0 | 0 | 2 | 2 |
| 32.96.2.27 | $\begin{bmatrix} 29 & 21 \\ 8 & 23 \end{bmatrix}, \begin{bmatrix} 25 & 12 \\ 24 & 31 \end{bmatrix}, \begin{bmatrix} 11 & 31 \\ 0 & 19 \end{bmatrix}, \begin{bmatrix} 13 & 23 \\ 8 & 19 \end{bmatrix}, \begin{bmatrix} 15 & 2 \\ 0 & 3 \end{bmatrix}$ 16.2.e.a | 32C ² | 14 | 6 | 0 | 2 | 2 |
| 32.96.2.28 | $\begin{bmatrix} 7 & 17 \\ 0 & 29 \end{bmatrix}, \begin{bmatrix} 7 & 7 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 7 & 28 \\ 0 & 29 \end{bmatrix}, \begin{bmatrix} 23 & 31 \\ 16 & 17 \end{bmatrix}, \begin{bmatrix} 9 & 17 \\ 16 & 1 \end{bmatrix}$ 128.2.e.b | 32C ² | 14 | 2 | 0 | 2 | 2 |
| 32.96.2.29 | $\begin{bmatrix} 9 & 22 \\ 16 & 31 \end{bmatrix}, \begin{bmatrix} 7 & 0 \\ 0 & 27 \end{bmatrix}, \begin{bmatrix} 15 & 25 \\ 0 & 29 \end{bmatrix}, \begin{bmatrix} 31 & 9 \\ 16 & 17 \end{bmatrix}, \begin{bmatrix} 17 & 10 \\ 0 & 31 \end{bmatrix}$ 128.2.e.a | 32C ² | 14 | 2 | 0 | 2 | 2 |
| 32.96.2.30 | $\begin{bmatrix} 17 & 0 \\ 0 & 19 \end{bmatrix}, \begin{bmatrix} 15 & 19 \\ 16 & 23 \end{bmatrix}, \begin{bmatrix} 23 & 17 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 17 & 26 \\ 0 & 15 \end{bmatrix}, \begin{bmatrix} 31 & 30 \\ 0 & 3 \end{bmatrix}$ 64.2.e.a | 32C ² | 14 | 2 | 0 | 2 | 2 |
| 32.96.2.35 | $\begin{bmatrix} 1 & 0 \\ 23 & 15 \end{bmatrix}, \begin{bmatrix} 31 & 16 \\ 20 & 3 \end{bmatrix}, \begin{bmatrix} 31 & 0 \\ 12 & 31 \end{bmatrix}, \begin{bmatrix} 23 & 0 \\ 2 & 25 \end{bmatrix}, \begin{bmatrix} 31 & 0 \\ 31 & 11 \end{bmatrix}$ 128.2.e.b | 32C ² | 14 | 2 | 0 | 2 | 2 |
| 32.96.2.36 | $\begin{bmatrix} 31 & 0 \\ 20 & 29 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 26 & 23 \end{bmatrix}, \begin{bmatrix} 7 & 0 \\ 11 & 25 \end{bmatrix}, \begin{bmatrix} 31 & 0 \\ 0 & 23 \end{bmatrix}, \begin{bmatrix} 15 & 16 \\ 26 & 25 \end{bmatrix}$ 128.2.e.a | 32C ² | 14 | 2 | 0 | 2 | 2 |
| 32.96.2.37 | $\begin{bmatrix} 13 & 6 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 29 & 0 \\ 16 & 25 \end{bmatrix}, \begin{bmatrix} 9 & 1 \\ 0 & 7 \end{bmatrix}, \begin{bmatrix} 21 & 12 \\ 0 & 31 \end{bmatrix}, \begin{bmatrix} 15 & 18 \\ 0 & 31 \end{bmatrix}$ 64.2.e.a | 32C ² | 14 | 2 | 0 | 2 | 2 |
| 32.96.2.38 | $\begin{bmatrix} 19 & 27 \\ 16 & 7 \end{bmatrix}, \begin{bmatrix} 21 & 9 \\ 16 & 25 \end{bmatrix}, \begin{bmatrix} 7 & 13 \\ 0 & 17 \end{bmatrix}, \begin{bmatrix} 3 & 19 \\ 16 & 9 \end{bmatrix}, \begin{bmatrix} 23 & 17 \\ 16 & 23 \end{bmatrix}$ 16.2.e.a | 32C ² | 14 | 6 | 0 | 2 | 2 |
| 32.96.2.63 | $\begin{bmatrix} 19 & 18 \\ 30 & 23 \end{bmatrix}, \begin{bmatrix} 0 & 5 \\ 23 & 8 \end{bmatrix}, \begin{bmatrix} 1 & 8 \\ 0 & 11 \end{bmatrix}$ 1024.2.a.e | 32B ² | 14 | 0 | 0 | 2 | 2 |
| 32.96.2.64 | $\begin{bmatrix} 20 & 23 \\ 7 & 12 \end{bmatrix}, \begin{bmatrix} 31 & 18 \\ 10 & 1 \end{bmatrix}, \begin{bmatrix} 12 & 5 \\ 7 & 12 \end{bmatrix}$ 1024.2.a.b | 32B ² | 14 | 0 | 2 | 2 | 2 |
| 32.96.3.25 | $\begin{bmatrix} 12 & 17 \\ 17 & 20 \end{bmatrix}, \begin{bmatrix} 6 & 25 \\ 15 & 18 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 25 & 0 \end{bmatrix}, \begin{bmatrix} 13 & 4 \\ 28 & 9 \end{bmatrix}$ 256.2.a.a, 1024.2.a.b | 32Q ³ | 10 | 0 | 3 | 3 | 1, 2 |

(continued)

Table 17. (continued)

| Label | Generators | CP | $\#X_H^{\infty}(\overline{\mathbb{Q}})$ | $\#X_H^{\infty}(\mathbb{Q})$ | r | g | dims |
|-------------|---|------------------|---|------------------------------|-----|-----|------|
| 32.96.3.27 | $\begin{bmatrix} 25 & 26 \\ 26 & 31 \end{bmatrix}, \begin{bmatrix} 25 & 30 \\ 6 & 19 \end{bmatrix}, \begin{bmatrix} 7 & 12 \\ 16 & 31 \end{bmatrix}, \begin{bmatrix} 11 & 13 \\ 0 & 5 \end{bmatrix}$ 256.2.a.b, 1024.2.a.c | 32L ³ | 4 | 0 | 3 | 3 | 1, 2 |
| 32.96.3.82 | $\begin{bmatrix} 9 & 12 \\ 20 & 11 \end{bmatrix}, \begin{bmatrix} 24 & 23 \\ 29 & 0 \end{bmatrix}, \begin{bmatrix} 13 & 0 \\ 24 & 31 \end{bmatrix}, \begin{bmatrix} 4 & 25 \\ 23 & 12 \end{bmatrix}$ 256.2.a.a, 1024.2.a.b | 32P ³ | 10 | 2 | 3 | 3 | 1, 2 |
| 32.96.3.83 | $\begin{bmatrix} 8 & 27 \\ 1 & 8 \end{bmatrix}, \begin{bmatrix} 0 & 19 \\ 21 & 0 \end{bmatrix}, \begin{bmatrix} 8 & 11 \\ 3 & 8 \end{bmatrix}, \begin{bmatrix} 28 & 27 \\ 23 & 4 \end{bmatrix}$ 256.2.a.a, 1024.2.a.e | 32P ³ | 10 | 2 | 1 | 3 | 1, 2 |
| 32.96.3.84 | $\begin{bmatrix} 28 & 5 \\ 15 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 5 \\ 27 & 0 \end{bmatrix}, \begin{bmatrix} 11 & 8 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 16 & 15 \\ 5 & 0 \end{bmatrix}$ 256.2.a.b, 1024.2.a.c | 32L ³ | 4 | 2 | 3 | 3 | 1, 2 |
| 32.96.3.229 | $\begin{bmatrix} 13 & 6 \\ 10 & 19 \end{bmatrix}, \begin{bmatrix} 7 & 10 \\ 2 & 9 \end{bmatrix}, \begin{bmatrix} 25 & 6 \\ 31 & 15 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 11 & 27 \end{bmatrix}$ 256.2.a.a, 1024.2.a.e | 16P ³ | 10 | 2 | 1 | 3 | 1, 2 |
| 32.96.3.230 | $\begin{bmatrix} 11 & 0 \\ 12 & 21 \end{bmatrix}, \begin{bmatrix} 18 & 11 \\ 1 & 6 \end{bmatrix}, \begin{bmatrix} 6 & 31 \\ 29 & 10 \end{bmatrix}, \begin{bmatrix} 28 & 29 \\ 1 & 0 \end{bmatrix}$ 256.2.a.a, 1024.2.a.b | 16P ³ | 10 | 2 | 3 | 3 | 1, 2 |
| 32.96.3.231 | $\begin{bmatrix} 6 & 9 \\ 21 & 26 \end{bmatrix}, \begin{bmatrix} 18 & 25 \\ 19 & 14 \end{bmatrix}, \begin{bmatrix} 26 & 17 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 8 & 9 \\ 11 & 8 \end{bmatrix}$ 256.2.a.b, 1024.2.a.c | 32L ³ | 4 | 2 | 3 | 3 | 1, 2 |
| 32.96.3.234 | $\begin{bmatrix} 0 & 5 \\ 29 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 12 \\ 20 & 13 \end{bmatrix}, \begin{bmatrix} 25 & 30 \\ 2 & 25 \end{bmatrix}, \begin{bmatrix} 28 & 31 \\ 7 & 28 \end{bmatrix}$ 256.2.a.a, 1024.2.a.e | 16P ³ | 10 | 0 | 1 | 3 | 1, 2 |
| 32.96.3.235 | $\begin{bmatrix} 11 & 18 \\ 6 & 11 \end{bmatrix}, \begin{bmatrix} 21 & 6 \\ 22 & 27 \end{bmatrix}, \begin{bmatrix} 16 & 3 \\ 29 & 0 \end{bmatrix}, \begin{bmatrix} 23 & 30 \\ 18 & 7 \end{bmatrix}$ 256.2.a.a, 1024.2.a.b | 16P ³ | 10 | 0 | 3 | 3 | 1, 2 |
| 32.96.3.236 | $\begin{bmatrix} 31 & 10 \\ 22 & 7 \end{bmatrix}, \begin{bmatrix} 0 & 21 \\ 19 & 24 \end{bmatrix}, \begin{bmatrix} 25 & 18 \\ 26 & 23 \end{bmatrix}, \begin{bmatrix} 20 & 19 \\ 27 & 12 \end{bmatrix}$ 256.2.a.a, 1024.2.a.b | 32P ³ | 10 | 0 | 3 | 3 | 1, 2 |
| 32.96.3.237 | $\begin{bmatrix} 0 & 1 \\ 7 & 0 \end{bmatrix}, \begin{bmatrix} 9 & 2 \\ 14 & 9 \end{bmatrix}, \begin{bmatrix} 20 & 7 \\ 7 & 28 \end{bmatrix}, \begin{bmatrix} 24 & 27 \\ 19 & 24 \end{bmatrix}$ 256.2.a.a, 1024.2.a.e | 32P ³ | 10 | 0 | 1 | 3 | 1, 2 |
| 32.96.3.241 | $\begin{bmatrix} 0 & 29 \\ 27 & 0 \end{bmatrix}, \begin{bmatrix} 20 & 23 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 7 & 30 \\ 14 & 15 \end{bmatrix}, \begin{bmatrix} 10 & 13 \\ 5 & 14 \end{bmatrix}$ 256.2.a.a, 1024.2.a.b | 16Q ³ | 10 | 0 | 3 | 3 | 1, 2 |
| 32.96.3.242 | $\begin{bmatrix} 3 & 20 \\ 18 & 5 \end{bmatrix}, \begin{bmatrix} 29 & 6 \\ 0 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 30 & 15 \end{bmatrix}, \begin{bmatrix} 23 & 15 \\ 6 & 25 \end{bmatrix}$ 256.2.a.b, 1024.2.a.c | 32L ³ | 4 | 0 | 3 | 3 | 1, 2 |
| 32.96.5.9 | $\begin{bmatrix} 24 & 9 \\ 13 & 8 \end{bmatrix}, \begin{bmatrix} 1 & 6 \\ 26 & 15 \end{bmatrix}, \begin{bmatrix} 31 & 14 \\ 18 & 13 \end{bmatrix}, \begin{bmatrix} 16 & 9 \\ 17 & 0 \end{bmatrix}$ 256.2.a.a, 1024.2.a.g | 32F ⁵ | 6 | 0 | 5 | 5 | 1, 4 |
| 32.96.5.10 | $\begin{bmatrix} 19 & 0 \\ 8 & 15 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 17 & 24 \end{bmatrix}, \begin{bmatrix} 7 & 20 \\ 12 & 23 \end{bmatrix}, \begin{bmatrix} 27 & 30 \\ 30 & 1 \end{bmatrix}$ 256.2.a.a, 1024.2.a.j | 32I ⁵ | 6 | 0 | 1 | 5 | 1, 4 |
| 32.96.5.11 | $\begin{bmatrix} 0 & 21 \\ 19 & 16 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 16 & 21 \end{bmatrix}, \begin{bmatrix} 25 & 18 \\ 26 & 19 \end{bmatrix}, \begin{bmatrix} 4 & 21 \\ 3 & 28 \end{bmatrix}$ 256.2.a.b, 1024.2.a.h | 32F ⁵ | 4 | 0 | 5 | 5 | 1, 4 |
| 32.96.5.12 | $\begin{bmatrix} 16 & 15 \\ 29 & 0 \end{bmatrix}, \begin{bmatrix} 19 & 22 \\ 6 & 5 \end{bmatrix}, \begin{bmatrix} 28 & 1 \\ 11 & 20 \end{bmatrix}, \begin{bmatrix} 11 & 20 \\ 20 & 7 \end{bmatrix}$ 256.2.a.b, 1024.2.a.i | 32F ⁵ | 4 | 0 | 1 | 5 | 1, 4 |
| 32.96.5.19 | $\begin{bmatrix} 21 & 12 \\ 4 & 1 \end{bmatrix}, \begin{bmatrix} 24 & 31 \\ 21 & 0 \end{bmatrix}, \begin{bmatrix} 24 & 5 \\ 25 & 8 \end{bmatrix}, \begin{bmatrix} 0 & 21 \\ 27 & 8 \end{bmatrix}$ 256.2.a.a, 1024.2.a.g | 32H ⁵ | 6 | 2 | 5 | 5 | 1, 4 |
| 32.96.5.20 | $\begin{bmatrix} 15 & 28 \\ 20 & 15 \end{bmatrix}, \begin{bmatrix} 21 & 20 \\ 20 & 27 \end{bmatrix}, \begin{bmatrix} 28 & 17 \\ 9 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 17 \\ 27 & 0 \end{bmatrix}$ 256.2.a.a, 1024.2.a.j | 32H ⁵ | 6 | 2 | 1 | 5 | 1, 4 |
| 32.96.5.21 | $\begin{bmatrix} 4 & 13 \\ 7 & 28 \end{bmatrix}, \begin{bmatrix} 31 & 0 \\ 24 & 21 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 25 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 28 \\ 12 & 31 \end{bmatrix}$ 256.2.a.b, 1024.2.a.h | 32F ⁵ | 4 | 2 | 5 | 5 | 1, 4 |
| 32.96.5.22 | $\begin{bmatrix} 9 & 20 \\ 4 & 9 \end{bmatrix}, \begin{bmatrix} 20 & 9 \\ 7 & 12 \end{bmatrix}, \begin{bmatrix} 21 & 0 \\ 8 & 19 \end{bmatrix}, \begin{bmatrix} 24 & 23 \\ 27 & 0 \end{bmatrix}$ 256.2.a.b, 1024.2.a.i | 32F ⁵ | 4 | 2 | 1 | 5 | 1, 4 |
| 32.96.5.69 | $\begin{bmatrix} 13 & 13 \\ 0 & 27 \end{bmatrix}, \begin{bmatrix} 1 & 22 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 23 & 25 \\ 18 & 17 \end{bmatrix}, \begin{bmatrix} 25 & 11 \\ 8 & 3 \end{bmatrix}$ 256.2.a.a, 1024.2.a.g | 16J ⁵ | 6 | 2 | 5 | 5 | 1, 4 |
| 32.96.5.70 | $\begin{bmatrix} 2 & 5 \\ 25 & 22 \end{bmatrix}, \begin{bmatrix} 28 & 27 \\ 7 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 27 \\ 25 & 28 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 27 & 8 \end{bmatrix}$ 256.2.a.a, 1024.2.a.j | 16J ⁵ | 6 | 2 | 1 | 5 | 1, 4 |

(continued)

Table 17. (continued)

| Label | Generators | CP | $\#X_H^{\infty}(\overline{\mathbb{Q}})$ | $\#X_H^{\infty}(\mathbb{Q})$ | r | g | dims |
|------------|---|------------------|---|------------------------------|-----|-----|-------------------|
| 32.96.5.71 | $\begin{bmatrix} 24 & 7 \\ 21 & 8 \end{bmatrix}, \begin{bmatrix} 17 & 8 \\ 20 & 7 \end{bmatrix}, \begin{bmatrix} 5 & 30 \\ 6 & 11 \end{bmatrix}, \begin{bmatrix} 29 & 0 \\ 12 & 19 \end{bmatrix}$ 256.2.a.b, 1024.2.a.h | 32E ⁵ | 4 | 2 | 5 | 5 | 1, 4 |
| 32.96.5.72 | $\begin{bmatrix} 21 & 0 \\ 0 & 27 \end{bmatrix}, \begin{bmatrix} 30 & 27 \\ 21 & 18 \end{bmatrix}, \begin{bmatrix} 16 & 25 \\ 31 & 16 \end{bmatrix}, \begin{bmatrix} 16 & 23 \\ 23 & 0 \end{bmatrix}$ 256.2.a.b, 1024.2.a.i | 32E ⁵ | 4 | 2 | 1 | 5 | 1, 4 |
| 32.96.5.73 | $\begin{bmatrix} 27 & 5 \\ 30 & 9 \end{bmatrix}, \begin{bmatrix} 23 & 3 \\ 0 & 17 \end{bmatrix}, \begin{bmatrix} 13 & 20 \\ 0 & 21 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix}$ 256.2.a.a, 1024.2.a.g | 16I ⁵ | 6 | 0 | 5 | 5 | 1, 4 |
| 32.96.5.74 | $\begin{bmatrix} 0 & 15 \\ 7 & 16 \end{bmatrix}, \begin{bmatrix} 22 & 15 \\ 9 & 30 \end{bmatrix}, \begin{bmatrix} 30 & 11 \\ 13 & 30 \end{bmatrix}, \begin{bmatrix} 0 & 19 \\ 5 & 8 \end{bmatrix}$ 256.2.a.a, 1024.2.a.j | 16I ⁵ | 6 | 0 | 1 | 5 | 1, 4 |
| 32.96.5.75 | $\begin{bmatrix} 15 & 26 \\ 18 & 1 \end{bmatrix}, \begin{bmatrix} 19 & 26 \\ 28 & 23 \end{bmatrix}, \begin{bmatrix} 23 & 0 \\ 22 & 1 \end{bmatrix}, \begin{bmatrix} 9 & 13 \\ 20 & 23 \end{bmatrix}$ 256.2.a.a, 1024.2.a.g | 32I ⁵ | 6 | 0 | 5 | 5 | 1, 4 |
| 32.96.5.76 | $\begin{bmatrix} 13 & 11 \\ 24 & 19 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 14 & 5 \end{bmatrix}, \begin{bmatrix} 7 & 7 \\ 4 & 9 \end{bmatrix}, \begin{bmatrix} 5 & 12 \\ 0 & 21 \end{bmatrix}$ 256.2.a.a, 1024.2.a.j | 32I ⁵ | 6 | 0 | 1 | 5 | 1, 4 |
| 32.96.5.77 | $\begin{bmatrix} 23 & 22 \\ 30 & 23 \end{bmatrix}, \begin{bmatrix} 24 & 11 \\ 25 & 24 \end{bmatrix}, \begin{bmatrix} 16 & 27 \\ 25 & 8 \end{bmatrix}, \begin{bmatrix} 0 & 5 \\ 9 & 0 \end{bmatrix}$ 256.2.a.a, 1024.2.a.g | 16I ⁵ | 6 | 0 | 5 | 5 | 1, 4 |
| 32.96.5.78 | $\begin{bmatrix} 3 & 8 \\ 20 & 29 \end{bmatrix}, \begin{bmatrix} 0 & 31 \\ 9 & 24 \end{bmatrix}, \begin{bmatrix} 16 & 7 \\ 17 & 0 \end{bmatrix}, \begin{bmatrix} 19 & 22 \\ 6 & 27 \end{bmatrix}$ 256.2.a.a, 1024.2.a.j | 16I ⁵ | 6 | 0 | 1 | 5 | 1, 4 |
| 32.96.5.79 | $\begin{bmatrix} 0 & 1 \\ 7 & 0 \end{bmatrix}, \begin{bmatrix} 23 & 28 \\ 12 & 31 \end{bmatrix}, \begin{bmatrix} 1 & 18 \\ 2 & 9 \end{bmatrix}, \begin{bmatrix} 8 & 7 \\ 23 & 4 \end{bmatrix}$ 256.2.a.b, 1024.2.a.h | 32E ⁵ | 4 | 0 | 5 | 5 | 1, 4 |
| 32.96.5.80 | $\begin{bmatrix} 18 & 17 \\ 19 & 14 \end{bmatrix}, \begin{bmatrix} 1 & 18 \\ 22 & 31 \end{bmatrix}, \begin{bmatrix} 24 & 1 \\ 11 & 24 \end{bmatrix}, \begin{bmatrix} 24 & 23 \\ 21 & 24 \end{bmatrix}$ 256.2.a.b, 1024.2.a.i | 32E ⁵ | 4 | 0 | 1 | 5 | 1, 4 |
| 32.96.7.1 | $\begin{bmatrix} 10 & 27 \\ 27 & 14 \end{bmatrix}, \begin{bmatrix} 6 & 17 \\ 13 & 26 \end{bmatrix}, \begin{bmatrix} 16 & 3 \\ 5 & 0 \end{bmatrix}, \begin{bmatrix} 12 & 9 \\ 11 & 4 \end{bmatrix}$ 256.2.a.b, 1024.2.a.a, 1024.2.a.d, 1024.2.a.f | 32A ⁷ | 4 | 0 | 1 | 7 | 1, 2 ³ |
| 32.96.7.2 | $\begin{bmatrix} 1 & 4 \\ 18 & 27 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 12 & 31 \end{bmatrix}, \begin{bmatrix} 23 & 0 \\ 26 & 9 \end{bmatrix}, \begin{bmatrix} 17 & 13 \\ 12 & 31 \end{bmatrix}$ 256.2.a.b, 1024.2.a.a, 1024.2.a.d, 1024.2.a.f | 32A ⁷ | 4 | 2 | 1 | 7 | 1, 2 ³ |
| 32.96.7.13 | $\begin{bmatrix} 22 & 7 \\ 17 & 26 \end{bmatrix}, \begin{bmatrix} 17 & 12 \\ 20 & 31 \end{bmatrix}, \begin{bmatrix} 0 & 13 \\ 29 & 16 \end{bmatrix}, \begin{bmatrix} 17 & 30 \\ 18 & 31 \end{bmatrix}$ 256.2.a.b, 1024.2.a.a, 1024.2.a.d, 1024.2.a.f | 32A ⁷ | 4 | 2 | 1 | 7 | 1, 2 ³ |
| 32.96.7.14 | $\begin{bmatrix} 3 & 24 \\ 24 & 27 \end{bmatrix}, \begin{bmatrix} 3 & 22 \\ 6 & 11 \end{bmatrix}, \begin{bmatrix} 6 & 13 \\ 27 & 26 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 20 & 31 \end{bmatrix}$ 256.2.a.b, 1024.2.a.a, 1024.2.a.d, 1024.2.a.f | 32A ⁷ | 4 | 0 | 1 | 7 | 1, 2 ³ |

Table 18. Twists H of $\mathcal{N}_{\mathcal{O}}(16)$ for $\mathcal{O} = \mathbb{Z}[i], \mathbb{Z}[\zeta_3]$ and E/\mathbb{Q} with potential CM by \mathcal{O} and $\rho_{E,2^{\infty}}(G_{\mathbb{Q}}) = H$.

| Subgroup | 4-orbits | 8-orbits | D | E/\mathbb{Q} |
|--------------|------------------------------------|--------------------------------------|-----|----------------|
| 16.64.2.1 | 12 | 48 | -3 | 27.a3 |
| 16.96.4.13 | 4, 8 | 16, 32 | -4 | 288.a1 |
| 16.192.9.83 | 4, 8 | 16, 32 | -3 | 36.a3 |
| 16.192.9.143 | 4 ³ | 16 ³ | -4 | 288.d3 |
| 16.192.9.145 | 4, 8 | 8 ² , 32 | -4 | 2304.a2 |
| 16.192.9.178 | 2 ² , 8 | 8 ² , 32 | -4 | 288.d4 |
| 16.192.9.189 | 4 ³ | 16 ³ | -4 | 2304.a1 |
| 16.384.9.462 | 2 ² , 4 ² | 16, 8 ⁴ | -4 | 64.a3 |
| 16.384.9.568 | 2 ⁴ , 4 | 16, 8 ⁴ | -4 | 32.a3 |
| 16.384.9.600 | 2 ² , 4 ² | 4 ² , 16 ² , 8 | -4 | 64.a4 |
| 16.384.9.633 | 1 ² , 2, 4 ² | 4 ² , 16 ² , 8 | -4 | 32.a |
| 16.384.9.817 | 4, 8 | 16 ² , 8 ² | -4 | 256.c2 |
| 16.384.9.819 | 4, 8 | 16 ² , 8 ² | -4 | 256.b2 |
| 16.384.9.829 | 4 ³ | 16 ² , 8 ² | -4 | 256.c1 |
| 16.384.9.831 | 4 ³ | 16 ² , 8 ² | -4 | 256.b1 |

Table 19. Twists H of $\mathcal{N}_{\mathcal{O}}(27)$ for $\mathcal{O} = \mathbb{Z}[\zeta_3]$ and E/\mathbb{Q} with potential CM by \mathcal{O} and $\rho_{E,3^\infty}(G_{\mathbb{Q}}) = H$.

| Subgroup | 3-orbits | 9-orbits | D | E/\mathbb{Q} |
|---------------|---------------------------------|---|-----|----------------|
| 27.324.18.1 | 2, 6 | 18, 54 | -3 | 144.a3 |
| 27.648.18.1 | 1 ² , 6 | 9 ² , 54 | -3 | 36.a4 |
| 27.648.18.4 | 2, 3 ² | 18, 27 ² | -3 | 36.a3 |
| 27.972.55.16 | 2 ² , 4 | 6, 12, 18, 36 | -3 | 432.e3 |
| 27.972.55.22 | 2, 6 | 18 ² , 36 | -3 | 3888.h1 |
| 27.972.55.23 | 2, 6 | 18 ² , 36 | -3 | 3888.n1 |
| 27.1944.55.31 | 1 ² , 2 ³ | 3 ² , 6 ² , 18 ³ | -3 | 27.a4 |
| 27.1944.55.37 | 1 ² , 3 ² | 6 ³ , 9 ² , 18 ² | -3 | 27.a3 |
| 27.1944.55.43 | 1 ² , 6 | 9 ² , 18 ³ | -3 | 243.b2 |
| 27.1944.55.44 | 1 ² , 6 | 9 ² , 18 ³ | -3 | 243.a2 |
| 27.1944.55.49 | 2, 3 ² | 9 ² , 18 ³ | -3 | 243.b1 |
| 27.1944.55.50 | 2, 3 ² | 9 ² , 18 ³ | -3 | 243.a1 |

Table 20. Subgroups $\mathcal{N}_{\mathcal{O}}(16)$ of determinant index 1 that arise for imaginary quadratic orders \mathcal{O} of discriminant D and their quadratic twists (noted by asterisks), along with elliptic curves that realise them, identified by LMFDB label or an element $a \in \mathbb{Q}(j(E))$ to twist by, where $a^4 - 2a^3 - 13a^2 + 14a + 19 = 0$.

| Subgroup | 4-orbits | 8-orbits | D -constraint | Example | | |
|---------------|---------------------------------|--------------------------------------|-----------------|---------|--|----------------------------------|
| | | | | D | $E/\mathbb{Q}(j(E))$ | $\mathbb{Q}(j(E))$ |
| 16.64.2.1 | 12 | 48 | 5 mod 8 | -11 | 121.b2 | \mathbb{Q} |
| 16.96.3.323 | 2 ² , 8 | 8 ² , 32 | 16 mod 64 | -48 | 36.1-a3 | $\mathbb{Q}(\sqrt{3})$ |
| *16.192.3.539 | 1 ² , 2, 8 | 4 ⁴ , 32 | | | *16.1-a4 | |
| *16.192.3.546 | 1 ² , 2, 8 | 4 ² , 8, 32 | | | *256.1-c4 | |
| *16.192.3.553 | 2 ² , 4 ² | 4 ² , 16 ² , 8 | | | *16.1-a3 | |
| *16.192.3.564 | 2 ² , 4 ² | 8 ² , 16 ² | | | *256.1-c3 | |
| 16.96.3.325 | 2 ² , 8 | 8 ² , 32 | 48 mod 64 | -16 | 288.d1 | \mathbb{Q} |
| *16.192.3.540 | 1 ² , 2, 8 | 4 ⁴ , 32 | | | *32.a2 | |
| *16.192.3.545 | 1 ² , 2, 8 | 4 ² , 8, 32 | | | *64.a2 | |
| *16.192.3.554 | 2 ² , 4 ² | 4 ² , 16 ² , 8 | | | *32.a1 | |
| *16.192.3.563 | 2 ² , 4 ² | 16 ² , 8 ² | | | *64.a1 | |
| 16.96.3.353 | 4, 8 | 8 ² , 32 | 4 mod 32 | -28 | 49.a1 | \mathbb{Q} |
| 16.96.3.366 | 4, 8 | 16, 32 | 20 mod 32 | -12 | 36.a1 | \mathbb{Q} |
| 16.96.4.13 | 4, 8 | 16, 32 | 28 mod 32 | -4 | 288.a1 | \mathbb{Q} |
| 16.96.5.135 | 4, 8 | 16, 32 | 56 mod 64 | -8 | 2304.h1 | \mathbb{Q} |
| *16.192.5.602 | 4, 8 | 8 ² , 32 | | | *256.d1 | |
| *16.192.5.607 | 4, 8 | 8 ² , 32 | | | *256.a2 | |
| *16.192.5.617 | 4, 8 | 16 ³ | | | *256.d2 | |
| *16.192.5.624 | 4, 8 | 16 ³ | | | *256.a1 | |
| 16.96.5.136 | 4, 8 | 16, 32 | 24 mod 64 | -40 | 1.1-a1 | $\mathbb{Q}(\sqrt{5}, \sqrt{6})$ |
| *16.192.5.604 | 4, 8 | 8 ² , 32 | | | *(-a ³ +11a ² -29a)/19 | |
| *16.192.5.605 | 4, 8 | 8 ² , 32 | | | *(-a ³ -8a ² -10a+19)/19 | |
| *16.192.5.622 | 4, 8 | 16 ³ | | | *(a ³ +8a ² +10a-19)/19 | |
| *16.192.5.619 | 4, 8 | 16 ³ | | | *(a ³ -11a ² +29a)/19/19 | |
| 16.96.5.148 | 4, 8 | 16, 32 | 12 mod 32 | -20 | 4096.1-k1 | $\mathbb{Q}(\sqrt{5})$ |
| 16.192.9.205 | 4 ³ | 16 ² , 8 ² | 1 mod 8 | -7 | 49.a4 | \mathbb{Q} |

Table 21. Subgroups $\mathcal{N}_{\mathcal{O}}(16)$ of determinant index $i > 1$ for imaginary quadratic orders \mathcal{O} of discriminant D and their quadratic twists (noted by asterisks), along with elliptic curves that realise them, identified by LMFDB label except for the curve $E_1 : y^2 + axy + y = x^3 + \frac{a^2}{2}x^2 + \frac{48a^3+45a^2-250a-246}{2}x + (144a^3 + 121a^2 - 752a - 638)$, where $a^4 - 6a^2 + 4 = 0$.

| Subgroup | 4-orbits | 8-orbits | Constraint | Example | | |
|--------------------|-------------|-----------------|------------|---------|----------------------|----------------------------------|
| | | | | D | $E/\mathbb{Q}(j(E))$ | $\mathbb{Q}(j(E))$ |
| 8.2.2-16.48.0.297 | $2^2, 8$ | $4^2, 8, 32$ | 0 mod 64 | -64 | 1568.2-i1 | $\mathbb{Q}(\sqrt{2})$ |
| *8.2.2-16.96.0.280 | $1^2, 2, 8$ | $2^4, 8, 32$ | | | *32.1-a2 | |
| *8.2.2-16.96.0.292 | $1^2, 2, 8$ | $4^4, 32$ | | | *1024.1-k2 | |
| *8.2.2-16.96.0.304 | $2^2, 4^2$ | $4^4, 16^2$ | | | *32.1-a1 | |
| *8.2.2-16.96.0.316 | $2^2, 4^2$ | $4^2, 8, 16^2$ | | | *1024.1-k1 | |
| *8.2.2-16.96.0.372 | $2^2, 8$ | $2^2, 4, 8, 32$ | | | *1024.1-f1 | |
| *8.2.2-16.96.0.377 | $2^2, 8$ | $2^2, 4, 8, 32$ | | | *256.1-a1 | |
| *8.2.2-16.96.0.397 | $2^2, 8$ | $4^2, 8, 16^2$ | | | *256.1-a2 | |
| *8.2.2-16.96.0.400 | $2^2, 8$ | $4^2, 8, 16^2$ | | | *1024.1-f2 | |
| 8.2.2-16.48.1.216 | $2^2, 8$ | $4^2, 8, 32$ | 32 mod 64 | -32 | 3136.2-a1 | $\mathbb{Q}(\sqrt{2})$ |
| *8.2.2-16.96.1.353 | $2^2, 8$ | $2^4, 8, 32$ | | | *64.1-a2 | |
| *8.2.2-16.96.1.366 | $1^2, 2, 8$ | $4^4, 32$ | | | *256.1-c2 | |
| *8.2.2-16.96.1.386 | $2^2, 4^2$ | $4^4, 16^2$ | | | *64.1-a1 | |
| *8.2.2-16.96.1.397 | $2^2, 4^2$ | $4^2, 16^2, 8$ | | | *256.1-c1 | |
| *8.2.2-16.96.1.473 | $2^2, 8$ | $2^2, 4, 8, 32$ | | | *1024.1-d2 | |
| *8.2.2-16.96.1.476 | $2^2, 8$ | $2^2, 4, 8, 32$ | | | *1024.1-m1 | |
| *8.2.2-16.96.1.497 | $2^2, 8$ | $4^2, 16^2, 8$ | | | *1024.1-d1 | |
| *8.2.2-16.96.1.500 | $2^2, 8$ | $4^2, 16^2, 8$ | | | *1024.1-m2 | |
| 8.2.2-16.48.1.265 | 4, 8 | 16, 32 | 40 mod 64 | -24 | 81.1-b1 | $\mathbb{Q}(\sqrt{2})$ |
| 8.2.2-16.48.1.266 | 4, 8 | 16, 32 | 8 mod 64 | -120 | E_1 | $\mathbb{Q}(\sqrt{2}, \sqrt{5})$ |

Table 22. Subgroups $\mathcal{N}_{\mathcal{O}}(27)$ that arise for imaginary quadratic orders \mathcal{O} of discriminant D and their quadratic twists (noted by asterisks), along with elliptic curves that realise them, identified by LMFDB label.

| Subgroup | 3-orbits | 9-orbits | D -constraint | Example | | |
|---------------|----------|----------------|-----------------|---------|----------------------|------------------------|
| | | | | D | $E/\mathbb{Q}(j(E))$ | $\mathbb{Q}(j(E))$ |
| 27.243.12.1 | 8 | 72 | 2 mod 3 | -7 | 49.a4 | \mathbb{Q} |
| 27.324.13.16 | 2, 6 | 6, 12, 54 | 0 mod 27 | -27 | 432.e2 | \mathbb{Q} |
| *27.648.13.25 | $1^2, 6$ | $3^2, 6^2, 54$ | | | *27.a2 | |
| *27.648.13.34 | $2, 3^2$ | $6^3, 27^2$ | | | *27.a1 | |
| 27.324.16.26 | 2, 6 | 6, 12, 54 | 18 mod 27 | -36 | 144.1-b1 | $\mathbb{Q}(\sqrt{3})$ |
| *27.648.16.35 | $1^2, 6$ | $3^2, 6^2, 54$ | | | *9.1-a2 | |
| *27.648.16.53 | $2, 3^2$ | $6^3, 27^2$ | | | *9.1-a1 | |
| 27.324.16.29 | 2, 6 | 6, 12, 54 | 9 mod 27 | -72 | 16.1-a3 | $\mathbb{Q}(\sqrt{6})$ |
| *27.648.16.38 | $1^2, 6$ | $3^2, 6^2, 54$ | | | *1.1-a4 | |
| *27.648.16.56 | $2, 3^2$ | $6^3, 27^2$ | | | *1.1-a3 | |
| 27.324.18.1 | 2, 6 | 18, 54 | 6 mod 9 | -12 | 144.a1 | \mathbb{Q} |
| *27.648.18.1 | $1^2, 6$ | $9^2, 54$ | | | *36.a2 | |
| *27.648.18.4 | $2, 3^2$ | $18, 27^2$ | | | *36.a1 | |
| 27.324.19.10 | 2, 6 | 18, 54 | 3 mod 9 | -15 | 2025.1-e1 | $\mathbb{Q}(\sqrt{5})$ |
| *27.648.19.19 | $1^2, 6$ | $9^2, 54$ | | | *81.1-a1 | |
| *27.648.19.22 | $2, 3^2$ | $18, 27^2$ | | | *81.1-a2 | |
| 27.486.28.7 | 4^2 | 12, 24, 36 | 1 mod 3 | -11 | 121.b2 | \mathbb{Q} |

Conflicts of Interest. None.

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References

- [Ass21] Eran Assaf, *Computing classical modular forms for arbitrary congruence subgroups*, Arithmetic Geometry, Number Theory, and Computation, 2021, pp. 43–104.
- [BDM+19] Jennifer S. Balakrishnan, Netan Dogra, J. Steffen Müller, Jan Tuitman, and Jan Vonk, *Explicit Chabauty-Kim for the split Cartan modular curve of level 13*, Ann. of Math. (2) **189** (2019), no. 3, 885–944. MR3961086
- [BDM+21] Jennifer S. Balakrishnan, Netan Dogra, J. Steffen Müller, Jan Tuitman, and Jan Vonk, *Quadratic Chabauty for modular curves: Algorithms and examples*, arXiv:2021.01862v2 (2021).
- [BHK+16] Jennifer S. Balakrishnan, Wei Ho, Nathan Kaplan, Simon Spicer, William Stein, and James Weigandt, *Databases of elliptic curves ordered by height and distributions of Selmer groups and ranks*, LMS J. Comput. Math. **19** (2016), no. suppl. A, 351–370. MR3540965
- [BC14] Barinder S. Banwait and John E. Cremona, *Tetrahedral elliptic curves and the local-global principle for isogenies*, Algebra Number Theory **8** (2014), no. 5, 1201–1229. MR3263141
- [Bar10] Burcu Baran, *Normalizers of non-split Cartan subgroups, modular curves, and the class number one problem*, J. Number Theory **130** (2010), no. 12, 2753–2772. MR2684496
- [BS22] Razvan Barbulescu and Sudarshan Shinde, *A classification of ECM-friendly families of elliptic curves using modular curves*, Math. Comp. **91** (2022), 1405–1436. MR4405500
- [BC18] Karim Belabas and Henri Cohen, *Modular forms in Pari/GP*, Res. Math. Sci. **5** (2018), no. 3, Paper No. 37, 19. MR3846373
- [BBB+21] Alex Best, Jonathan Bober, Andrew R. Booker, Edgar Costa, John Cremona, Derickx Maarten, Min Lee, David Lowry-Duda, David Roe, Andrew V. Sutherland, and John Voight, *Computing classical modular forms*, Arithmetic Geometry, Number Theory, and Computation, 2021, pp. 123–213.
- [Bis11] Gaetan Bisson, *Computing endomorphism rings of elliptic curves under the GRH*, J. Math. Cryptol. **5** (2011), no. 2, 101–113. MR2838371
- [BS11] Gaetan Bisson and Andrew V. Sutherland, *Computing the endomorphism ring of an ordinary elliptic curve over a finite field*, J. Number Theory **131** (2011), no. 5, 815–831. MR2772473
- [BCP97] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3–4, 235–265. MR1484478
- [BP11] Yuri Bilu and Pierre Parent, *Serre’s uniformity problem in the split Cartan case*, Ann. of Math. (2) **173** (2011), no. 1, 569–584. MR2753610
- [BPR13] Yuri Bilu, Pierre Parent, and Marusia Rebolledo, *Rational points on $X_0^+(p^r)$* , Ann. Inst. Fourier (Grenoble) **63** (2013), no. 3, 957–984.
- [BC20] Abbey Bourdon and Pete L. Clark, *Torsion points and Galois representations on CM elliptic curves*, Pacific J. Math. **305** (2020), no. 1, 43–88.
- [BEL+19] Abbey Bourdon, Özlem Ejder, Yuan Liu, Frances Odumodu, and Bianca Viray, *On the level of modular curves that give rise to isolated j -invariants*, Adv. Math. **357** (2019), 106824, 33. MR4016915
- [BGRW20] Abbey Bourdon, David R. Gill, Jeremy Rouse, and Lori D. Watson, *Odd degree isolated points on $X_1(N)$ with rational j -invariant*, arXiv:2006.14966v1 (2020).
- [BJ16] Julio Brau and Nathan Jones, *Elliptic curves with 2-torsion contained in the 3-torsion field*, Proc. Amer. Math. Soc. **144** (2016), no. 3, 925–936. MR3447646
- [BLS12] Reinier Bröker, Kristin Lauter, and Andrew V. Sutherland, *Modular polynomials via isogeny volcanoes*, Math. Comp. **81** (2012), no. 278, 1201–1231. MR2869057
- [BS10] Nils Bruin and Michael Stoll, *The Mordell–Weil sieve: proving non-existence of rational points on curves*, LMS Journal of Computation and Mathematics **13** (2010), 272–306. MR2685127
- [CP] Francesco Campagna and Riccardo Pengo, *Entanglement in the family of division fields of elliptic curves with complex multiplication*, Pacific J. Math. **317** (2022), 21–66. <https://msp.org/pjm/2022/317-1/p02.xhtml>
- [Cen16] Tommaso Giorgio Centeleghe, *Integral Tate modules and splitting of primes in torsion fields of elliptic curves*, Int. J. Number Theory **12** (2016), no. 1, 237–248. MR3455277
- [CR21] Michael Cerchia and Jeremy Rouse, *Uniform bounds on the image of the arboreal Galois representations attached to non-CM elliptic curves*, Proc. Amer. Math. Soc. **149** (2021), no. 2, 583–589. MR4198067
- [Che04] Imin Chen, *Jacobians of modular curves associated to normalizers of Cartan subgroups of level p^n* , C. R. Math. Acad. Sci. Paris **339** (2004), no. 3, 187–192. MR2078072
- [Che98] Imin Chen, *The Jacobians of non-split Cartan modular curves*, Proc. London Math. Soc. (3) **77** (1998), no. 1, 1–38. MR1625491

- [CC04] Imin Chen and Chris Cummins, *Elliptic curves with nonsplit mod 11 representations*, *Math. Comp.* **73** (2004), no. 246, 869–880. MR2031412
- [CLR21] Garen Chiloyan and Álvaro Lozano-Robledo, *A classification of isogeny-torsion graphs of \mathbb{Q} -isogeny classes of elliptic curves*, *Trans. London Math. Soc.* **8** (2021), no. 1, 1–34.
- [CFS19] Edgar Costa, Francesc Fité, and Andrew V. Sutherland, *Arithmetic invariants from Sato-Tate moments*, *C. R. Math. Acad. Sci. Paris* **357** (2019), no. 11–12, 823–826. MR4038255
- [Cre95] J. E. Cremona, *Computing the degree of the modular parametrization of a modular elliptic curve*, *Math. Comp.* **64** (1995), no. 211, 1235–1250. MR1297466
- [CP03] Chris Cummins and Sebastian Pauli, *Congruence subgroups of $PSL(2, \mathbb{Z})$ of genus less than or equal to 24*, *Experiment. Math.* **12** (2003), no. 2, 243–245. MR2016709
- [CP] Chris Cummins and Sebastian Pauli, *Congruence subgroups of $PSL(2, \mathbb{Z})$* . Online database, accessed June 2021.
- [DGJ] Harris B. Daniels and Enrique González-Jiménez, *Serre’s constant of elliptic curves over the rationals*, *Exp. Math.*, published online December 2, 2019, to appear in print, 1–19.
- [DLR] Harris B. Daniels and Álvaro Lozano-Robledo, *Coincidences of division fields*, *Ann. Inst. Fourier*, to appear ([arXiv:1912.05618v2](https://arxiv.org/abs/1912.05618v2)).
- [DLRM21] Harris B. Daniels, Álvaro Lozano-Robledo, and Jackson S Morrow, *Towards a classification of entanglements of Galois representations attached to elliptic curves*, [arXiv:2105.02060v1](https://arxiv.org/abs/2105.02060v1) (2021).
- [DLRNS18] Harris B. Daniels, Álvaro Lozano-Robledo, Filip Najman, and Andrew V. Sutherland, *Torsion subgroups of rational elliptic curves over the compositum of all cubic fields*, *Math. Comp.* **87** (2018), no. 309, 425–458. MR3716201
- [DM] Harris B. Daniels and Jackson S. Morrow, *A group theoretic perspective on entanglements of division fields*, *Trans. Amer. Math. Soc.*, to appear ([arXiv:2008.09886v2](https://arxiv.org/abs/2008.09886v2)).
- [DR73] P. Deligne and M. Rapoport, *Les schémas de modules de courbes elliptiques*, *Modular functions of one variable, II* (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), 1973, pp. 143–316. *Lecture Notes in Math.*, Vol. **349**. MR0337993
- [DEvH+] Maarten Derickx, Anastassia Etropolski, Mark van Hoeij, Jackson S. Morrow, and David Zureick-Brown, *Sporadic cubic torsion*, *Algebra Number Theory* **15**, no. 7, 1837–1864. MR4333666
- [DS17] Maarten Derickx and Andrew V. Sutherland, *Torsion subgroups of elliptic curves over quintic and sextic number fields*, *Proc. Amer. Math. Soc.* **145** (2017), no. 10, 4233–4245. MR3690609
- [DS05] Fred Diamond and Jerry Shurman, *A first course in modular forms*, *Graduate Texts in Mathematics*, vol. **228**, Springer-Verlag, New York, 2005. MR2112196
- [Dic58] Leonard Eugene Dickson, *Linear groups: With an exposition of the Galois field theory*, Dover Publications, Inc., New York, 1958. With an introduction by W. Magnus. MR0104735
- [DD13] Tim Dokchitser and Vladimir Dokchitser, *Identifying Frobenius elements in Galois groups*, *Algebra Number Theory* **7** (2013), no. 6, 1325–1352. MR3107565
- [DT02] W. Duke and Á. Tóth, *The splitting of primes in division fields of elliptic curves*, *Experiment. Math.* **11** (2002), no. 4, 555–565 (2003). MR1969646
- [Elk98] Noam D. Elkies, *Elliptic and modular curves over finite fields and related computational issues*, *Computational perspectives on number theory*, 1998, pp. 21–76. MR1486831
- [Elk06] Noam D. Elkies, *Elliptic curves with 3-adic Galois representation surjective mod 3 but not mod 9*, [arXiv:0612734](https://arxiv.org/abs/0612734) (2006). Preprint.
- [ES10] Andreas Enge and Andrew V. Sutherland, *Class invariants by the CRT method*, *Algorithmic Number Theory Symposium (ANTS IX)*, 2010, pp. 142–156. MR2721418
- [GJG03] Enrique González-Jiménez and Josep González, *Modular curves of genus 2*, *Math. Comp.* **72** (2003), no. 241, 397–418 (electronic). MR1933828
- [GJLR16] Enrique González-Jiménez and Álvaro Lozano-Robledo, *Elliptic curves with abelian division fields*, *Math. Z.* **283** (2016), no. 3–4, 835–859. MR3519984
- [GJLR17] Enrique González-Jiménez and Álvaro Lozano-Robledo, *On the minimal degree of definition of p -primary torsion subgroups of elliptic curves*, *Math. Res. Lett.* **24** (2017), no. 4, 1067–1096. MR3723804
- [GJLR18] Enrique González-Jiménez and Álvaro Lozano-Robledo, *On the torsion of rational elliptic curves over quartic fields*, *Math. Comp.* **87** (2018), no. 311, 1457–1478. MR3766394
- [GJN20] Enrique González-Jiménez and Filip Najman, *Growth of torsion groups of elliptic curves upon base change*, *Math. Comp.* **89** (2020), no. 323, 1457–1485. MR4063324
- [GL85] Mark Green and Robert Lazarsfeld, *A simple proof of Petri’s theorem on canonical curves*, *Geometry today (Rome, 1984)*, 1985, pp. 129–142. MR0895152
- [Gre12] Ralph Greenberg, *The image of Galois representations attached to elliptic curves with an isogeny*, *Amer. J. Math.* **134** (2012), no. 5, 1167–1196. MR2975233
- [GRSS14] Ralph Greenberg, Karl Rubin, Alice Silverberg, and Michael Stoll, *On elliptic curves with an isogeny of degree 7*, *Amer. J. Math.* **136** (2014), no. 1, 77–109. MR3163354
- [Guž21] Tomislav Gužvić, *Torsion of elliptic curves with rational j -invariant defined over number fields of prime degree*, *Proc. Amer. Math. Soc.* **149** (2021), 3261–3275. MR4273133

- [HS16] David Harvey and Andrew V. Sutherland, *Computing Hasse-Witt matrices of hyperelliptic curves in average polynomial time, II*, Frobenius distributions: Lang-Trotter and Sato-Tate conjectures, 2016, pp. 127–147. MR3502941
- [Hid81] Haruzo Hida, *On abelian varieties with complex multiplication as factors of the Jacobians of Shimura curves*, Amer. J. Math. **103** (1981), no. 4, 727–776. MR0623136
- [JM22] Nathan Jones and Ken McMurdy, *Elliptic curves with non-abelian entanglements* **28** (2022), 182–229. MR4374148
- [JR10] Rafe Jones and Jeremy Rouse, *Galois theory of iterated endomorphisms*, Proc. Lond. Math. Soc. (3) **100** (2010), no. 3, 763–794. Appendix A by Jeffrey D. Achter. MR2640290
- [KP01] Jerzy Kaczorowski and Alberto Perelli, *Strong multiplicity one for the Selberg class*, C. R. Acad. Sci. Paris Sér. I Math. **332** (2001), no. 11, 963–968. MR1838120
- [Kat04] Kazuya Kato, *p -adic Hodge theory and values of zeta functions of modular forms*, 2004, pp. ix, 117–290. Cohomologies p -adiques et applications arithmétiques. III. MR2104361
- [Kat81] Nicholas M. Katz, *Galois properties of torsion points on abelian varieties*, Invent. Math. **62** (1981), no. 3, 481–502. MR0604840
- [KM85] Nicholas M. Katz and Barry Mazur, *Arithmetic moduli of elliptic curves*, Annals of Mathematics Studies, vol. **108**, Princeton University Press, Princeton, NJ, 1985. MR0772569
- [Ken80] M. A. Kenku, *The modular curve $X_0(169)$ and rational isogeny*, J. London Math. Soc. (2) **22** (1980), no. 2, 239–244. MR0588271
- [Ken81] M. A. Kenku, *On the modular curves $X_0(125)$, $X_1(25)$ and $X_1(49)$* , J. London Math. Soc. (2) **23** (1981), no. 3, 415–427. MR0616546
- [Kid95] Masanari Kida, *Galois descent and twists of an abelian variety*, Acta Arith. **73** (1995), no. 1, 51–57. MR1358187
- [Kle05] Steven L. Kleiman, *The Picard scheme*, Fundamental algebraic geometry, 2005, pp. 235–321. MR3287693
- [Koh96] David Kohel, *Endomorphism rings of elliptic curves over finite fields*, 1996. Thesis (Ph.D.) University of California, Berkeley. MR2695524
- [Lan85] Susan Landau, *Factoring polynomials over algebraic number fields*, SIAM J. Comput. **14** (1985), no. 1, 184–195. MR774938
- [LFL21] Samuel Le Fourn and Pedro Lemos, *Residual Galois representations of elliptic curves with image contained in the normaliser of a nonsplit Cartan*, Algebra Number Theory **15** (2021), no. 3, 747–771. MR4261100
- [Len83] A. K. Lenstra, *Factoring polynomials over algebraic number fields*, Computer algebra (London, 1983), 1983, pp. 245–254, DOI [10.1007/3-540-12868-9_108](https://doi.org/10.1007/3-540-12868-9_108). MR774816
- [LJP92] Hendrik W. Lenstra Jr. and Carl Pomerance, *A rigorous time bound for factoring integers*, J. Amer. Math. Soc. **5** (1992), no. 3, 483–516. MR1137100
- [Lig75] Gérard Ligozat, *Courbes modulaires de genre 1*, Société Mathématique de France, Paris, 1975. Bull. Soc. Math. France, Mém. 43, Supplément au Bull. Soc. Math. France Tome 103, no. 3. MR0417060
- [Lig77] Gérard Ligozat, *Courbes modulaires de niveau 11*, Modular functions of one variable, V (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976), 1977, pp. 149–237. Lecture Notes in Math., Vol. **601**. MR0463118
- [LMF] The LMFDB Collaboration, *The L -functions and modular forms database*. Online database, accessed June 2021.
- [Lom17] Davide Lombardo, *Galois representations attached to abelian varieties of CM type*, Bull. Soc. Math. France **145** (2017), no. 3, 469–501. MR3766118
- [LT] Davide Lombardo and Sebastiano Tronto, *Explicit Kummer theory for elliptic curves*, Int. Math. Res. Not. IMRN, published online August 18, 2021, to appear in print.
- [LT21] Davide Lombardo and Sebastiano Tronto, *Some uniform bounds for elliptic curves over \mathbb{Q}* , [arXiv:2106.09950v1](https://arxiv.org/abs/2106.09950v1) (2021).
- [LR] Álvaro Lozano-Robledo, *Galois representations attached to elliptic curves with complex multiplication*, Algebra Number Theory, to appear ([arXiv:1809.02584v3](https://arxiv.org/abs/1809.02584v3)).
- [Maz77a] B. Mazur, *Rational points on modular curves*, Modular functions of one variable, V (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976), 1977, pp. 107–148. Lecture Notes in Math., Vol. **601**. MR0450283
- [Maz77b] B. Mazur, *Modular curves and the Eisenstein ideal*, Inst. Hautes Études Sci. Publ. Math. **47** (1977), 33–186 (1978). MR0488287
- [Maz78] B. Mazur, *Rational isogenies of prime degree (with an appendix by D. Goldfeld)*, Invent. Math. **44** (1978), no. 2, 129–162. MR0482230
- [McK94] James McKee, *Computing division polynomials*, Math. Comp. **63** (1994), no. 208, 767–771. MR1248973
- [MS20] Pietro Mercuri and René Schoof, *Modular forms invariant under non-split Cartan subgroups*, Math. Comp. **99** (2020), no. 324, 1969–1991. MR4081925
- [Mir95] Rick Miranda, *Algebraic curves and Riemann surfaces*, Graduate Studies in Mathematics, vol. **5**, American Mathematical Society, Providence, RI, 1995. MR1326604
- [Mom84] Fumiyuki Momose, *Rational points on the modular curves $X_{\text{split}}(p)$* , Compositio Math. **52** (1984), no. 1, 115–137. MR742701
- [Mom86] Fumiyuki Momose, *Rational points on the modular curves $X_0^+(p^r)$* , J. Fac. Sci. Univ. Tokyo Sect. IA Math. **33** (1986), no. 3, 441–466. MR866046

- [MS02] Fumiuyuki Momose and Mahoro Shimura, *Lifting of supersingular points on $X_0(p^r)$ and lower bound of ramification index*, Nagoya Math. J. **165** (2002), 159–178. MR1892103
- [Mor19] Jackson S. Morrow, *Composite images of Galois for elliptic curves over \mathbb{Q} and entanglement fields*, Math. Comp. **88** (2019), no. 319, 2389–2421. MR3957898
- [OS19] Ekin Ozman and Samir Siksek, *Quadratic points on modular curves*, Math. Comp. **88** (2019), no. 319, 2461–2484. MR3957901
- [Pet23] K. Petri, *Über die invariante Darstellung algebraischer Funktionen einer Veränderlichen*, Mathematische Annalen **88** (1923), 242–289 (ger). MR151213
- [Poo01] Bjorn Poonen, *Computing torsion points on curves*, Experiment. Math. **10** (2001), no. 3, 449–465. MR1917430
- [Poo17] Bjorn Poonen, *Rational points on varieties*, Graduate Studies in Mathematics, vol. **186**, American Mathematical Society, Providence, RI, 2017. MR3729254
- [PSS07] Bjorn Poonen, Edward F. Schaefer, and Michael Stoll, *Twists of $X(7)$ and primitive solutions to $x^2 + y^3 = z^7$* , Duke Math. J. **137** (2007), no. 1, 103–158. MR2309145
- [Pyl04] Elisabeth E. Pyle, *Abelian varieties over \mathbb{Q} with large endomorphism algebras and their simple components over $\overline{\mathbb{Q}}$* , Modular curves and abelian varieties, 2004, pp. 189–239. MR2058652
- [Rak21] Rakvi, *A classification of genus 0 modular curves with rational points*, arXiv:2105.14623v1 (2021).
- [Rei21] Tomer Reiter, *Isogenies of elliptic curves and arithmetical structures on graphs*, Ph.D. Thesis, 2021. Available at <https://etd.library.emory.edu/concern/etds/kk91fm75v>.
- [Rib04] Kenneth A. Ribet, *Abelian varieties over \mathbb{Q} and modular forms*, Modular curves and abelian varieties, 2004, pp. 241–261. MR2058653
- [RSZB21] Jeremy Rouse, Andrew V. Sutherland, and David Zureick-Brown, *ell-adic-galois-images*, 2021. GitHub: [AndrewV-Sutherland/ell-adic-galois-images](https://github.com/AndrewV-Sutherland/ell-adic-galois-images).
- [RZB15] Jeremy Rouse and David Zureick-Brown, *Elliptic curves over \mathbb{Q} and 2-adic images of Galois*, Res. Number Theory **1** (2015), Art. 12, 34. MR3500996
- [Sch85] René Schoof, *Elliptic curves over finite fields and the computation of square roots mod p* , Math. Comp. **44** (1985), no. 170, 483–494. MR777280
- [Sch95] René Schoof, *Counting points on elliptic curves over finite fields*, J. Théor. Nombres Bordeaux **7** (1995), no. 1, 219–254. MR1413578
- [Sch87] René Schoof, *Nonsingular plane cubic curves over finite fields*, J. Combin. Theory Ser. A **46** (1987), no. 2, 183–211. MR0914657
- [Ser72] Jean-Pierre Serre, *Propriétés Galoisiennes des points d'ordre fini des courbes elliptiques*, Invent. Math. **15** (1972), no. 4, 259–331. MR387283
- [Ser81] Jean-Pierre Serre, *Quelques applications du théorème de densité de Chebotarev*, Inst. Hautes Études Sci. Publ. Math. **54** (1981), 323–401. MR0644559
- [Ser98] Jean-Pierre Serre, *Abelian- ℓ -adic representations and elliptic curves*, Research Notes in Mathematics, vol. **7**, A K Peters, Ltd., Wellesley, MA, 1998. With the collaboration of Willem Kuyk and John Labute, Revised reprint of the 1968 original. MR1484415
- [ST68] Jean-Pierre Serre and John Tate, *Good reduction of abelian varieties*, Ann. of Math. **88** (1968), 492–517, DOI [10.2307/1970722](https://doi.org/10.2307/1970722). MR236190
- [Shi94] Goro Shimura, *Introduction to the arithmetic theory of automorphic functions*, Publications of the Mathematical Society of Japan, vol. **11**, Princeton University Press, Princeton, NJ, 1994. Reprint of the 1971 original, Kanô Memorial Lectures, 1. MR0314766
- [SS15] Igor E. Shparlinski and Andrew V. Sutherland, *On the distribution of Atkin and Elkies primes for reductions of elliptic curves on average*, LMS J. Comput. Math. **18** (2015), no. 1, 308–322. MR3349320
- [Sou04] Kannan Soundararajan, *Strong multiplicity one for the Selberg class*, Canad. Math. Bull. **47** (2004), no. 3, 468–474. MR2073408
- [SW02] William A. Stein and Mark Watkins, *A database of elliptic curves—first report*, Algorithmic Number Theory Symposium (ANTS I), 2002, pp. 267–275. MR2041090
- [Sut11] Andrew V. Sutherland, *Computing Hilbert class polynomials with the Chinese remainder theorem*, Math. Comp. **80** (2011), no. 273, 501–538. MR2728992
- [Sut12a] Andrew V. Sutherland, *A local-global principle for rational isogenies of prime degree*, J. Théor. Nombres Bordeaux **24** (2012), no. 2, 475–485. MR2950703
- [Sut12b] Andrew V. Sutherland, *Identifying supersingular elliptic curves*, LMS J. Comput. Math. **15** (2012), 317–325. MR2988819
- [Sut13] Andrew V. Sutherland, *Isogeny volcanoes*, ANTS X—Proceedings of the Tenth Algorithmic Number Theory Symposium, 2013, pp. 507–530. MR3207429
- [Sut16] Andrew V. Sutherland, *Computing images of Galois representations attached to elliptic curves*, Forum Math. Sigma **4** (2016), Paper No. e4, 79. MR3482279
- [SZ17] Andrew V. Sutherland and David Zywina, *Modular curves of prime-power level with infinitely many rational points*, Algebra Number Theory **11** (2017), no. 5, 1199–1229. MR3671434

- [vHN05] Mark van Hoeij and Andrew Novocin, *A reduction algorithm for algebraic function fields* (2005). Unpublished, available at <https://www.math.fsu.edu/~hoeij/papers/HoeijNovocin.pdf>.
- [VZB22] John Voight and David Zureick-Brown, *The canonical ring of a stacky curve*, Mem. Amer. Math. Soc. **277** (2022), no. 1362, 1–144. MR4403928
- [Wat69] William C. Waterhouse, *Abelian varieties over finite fields*, Ann. Sci. École Norm. Sup. (4) **2** (1969), 521–560. MR0265369
- [Zyw15a] David Zywina, *On the possible images of the mod ℓ representations associated to elliptic curves over \mathbb{Q}* , [arXiv:1508.07660v1](https://arxiv.org/abs/1508.07660v1) (2015).
- [Zyw15b] David Zywina, *Possible indices for the Galois image of elliptic curves over \mathbb{Q}* , [arXiv:1508.07663v1](https://arxiv.org/abs/1508.07663v1) (2015).
- [Zyw21] David Zywina, *Computing actions on cusp forms*, [arXiv:2001.07270v2](https://arxiv.org/abs/2001.07270v2) (2021).
- [Zyw22] David Zywina, *On the surjectivity of mod ℓ representations associated to elliptic curves*, Bull. London. Math. Soc., published online 30 June 2022, to appear in print. <https://doi.org/10.1112/blms.12701>