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ON CRITICAL LEVEL SETS OF SOME TWO DEGREES OF FREEDOM INTEGRABLE HAMILTONIAN SYSTEMS

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ABSTRACT. We prove that all Liouville's tori generic bifurcations of a large class of two degrees of freedom integrable Hamiltonian systems (the so called Jacobi-Moser-Mumford systems) are nondegenerate in the sense of Bott. Thus, for such systems, Fomenko's theory [4] can be applied (we give the example of Gel'fand-Dikii's system). We also check the Bott property for two interesting systems: the Lagrange top and the geodesic flow on an ellipsoid.

1. **Introduction.** As explained by Mumford in [10] (actually, this idea could be fathered on Jacobi), a two degrees of freedom integrable Hamiltonian system (IHS2) may be associated to a family of polynomials:

$$f_{\xi,\eta}(t) = \sum_{k=0}^{n} a_k(\xi,\eta)t^{n-k} \quad (a_0 = 1).$$

Furthermore, we assume that the functions $(\xi,\eta) \longmapsto a_k(\xi,\eta)$, $k = 1,\ldots,n$, are linear at each variable.

Like Donagi [3], we call it a Jacobi-Moser-Mumford (JMM) system.

Our main result is Theorem 3.1 where we prove that such a system is a Bott system (*i.e.*, its momentum mapping is a Bott map).

The text is organized as follows:

- In Section 2, we define the JMM system (Section 2.1), and its symplectic structure (Section 2.2).
- In Section 3, we give definitions of a Bott function, of a Bott map and of a Bott system (Section 3.1). Then, we state our main theorem (3.1 in Section 3.2): *Two degrees of freedom JMM systems are Bott systems.*
- In Section 4, we give the proof of this theorem. First we study a special case when all the a_k are constant except two of them: $a_i(\xi,\eta) = \xi$ and $a_j(\xi,\eta) = \eta$. Then we consider the general case.
- Section 5 is dedicated to examples and applications. We recall Fomenko's theorem about classification of bifurcations of two-dimensional Liouville's tori. It applies to Gel'fand-Dikii's system, which turns out to be a Bott system (Section 5.1). We also prove that the Lagrange top system is a Bott system (Section 5.2). Finally, we study Jacobi's system found while searching for geodesic flow on an ellipsoid (Section 5.3).

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2. The JMM model of a two degrees of freedom integrable system.

The reader can refer to [10] and [3]. Consider the family of polynomials

$$f_{\xi,\eta}(t) = \sum_{k=0}^{n} a_k(\xi,\eta)t^{n-k} \quad (a_0 = 1).$$

Suppose that the functions $(\xi,\eta) \longmapsto a_k(\xi,\eta)$, $k = 1,\ldots,n$, are linear at each variable. Define the curve: $C = \{(t,s) \in \mathbb{C}^2,\ s^2 = f_{\xi,\eta}(t)\}$. The JMM description of the Jacobian $J(C)$ of C leads naturally to an IHS2.

In this section, we recall the JMM model of such a system. At first, we consider a particular case when $a_1,\ldots,\hat{a}_i,\ldots,\hat{a}_j,\ldots,a_n$ are constant and $a_i = \xi$, $a_j = \eta$. Then we extend the results to the general case when each a_k depends linearly on ξ and η.

2.1. *A special case.* Let us fix i and j such as $1 \le i < j \le n$. In this section, we define the two degrees of freedom JMM system associated with:

$$f_{\xi,\eta}(t) = t^n + a_1t^{n-1} + \cdots + \xi t^{n-i} + \cdots + \eta t^{n-j} + \cdots + a_{n-1}t + a_n.$$

After a short presentation, we give convenient local coordinates and finally, the first integrals of the system.

2.1.1. Presentation. Let V be the set of polynomials (called Jacobi polynomials)

$$u(t) = t^2 + u_1t + u_2, \quad v(t) = v_1t + v_2, \quad w(t) = t^{n-2} + w_1t^{n-3} + \cdots + w_{n-2},$$

satisfying: $f_{\xi,\eta}(t) = u(t)w(t) + v^2(t)$.

A simple identification of the coefficients shows the existence of n functions $H_1,\ldots,H_j$ such as:

$$\begin{aligned} f_{\xi,\eta} = uw + v^2 &\iff \forall t \sum_{k=1}^{n} H_k(u,v,w)t^{n-k} = 0,\\ &\iff H_k(u,v,w) = 0,\ k = 1\ldots n. \end{aligned}$$

Therefore:

$$V = \left\{(u,v,w) \in \mathbb{C}^{n+2}, H_k(u,v,w) = 0, k = 1\ldots n\right\}.$$

PROPOSITION 2.1. *If $f_{\xi,\eta}$ has no double root then V is smooth and biholomorphic to an affine part of the symmetric product $S^2(C)$ of the hyperelliptic curve C.*

PROOF. The reader can refer to [10] for the proof of the proposition. ∎

2.1.2. Local coordinates. Let $M = \{(u, v, w),\ \forall k = 1, \ldots, n,\ k \neq i,\ k \neq j,\ H_k = 0\}$. Consider $(u, v, w) \in M$, then:

$$f_{\xi,\eta}(t) - v^2(t) = u(t)w(t) + H_i t^{n-i} + H_j t^{n-j}. \tag{1}$$

Let x_1 and x_2 be the roots of u. Two cases must be considered depending on whether u has simple roots or a double root.

1. If the roots of u are x_1, x_2, $x_1 \neq x_2$, then we set $y_k = v(x_k)$ that is:

$$u_1 = -x_1 - x_2, \quad u_2 = x_1 x_2, \quad v_1 = \frac{y_1 - y_2}{x_1 - x_2}, \quad v_2 = \frac{x_1 y_2 - x_2 y_1}{x_1 - x_2}.$$

Replacing t by x_k in (1), we get:

$$f_{\xi,\eta}(x_k) - y_k^2 = H_i x_k^{n-i} + H_j x_k^{n-j}, \quad k = 1, 2. \tag{2}$$

This leads to the expressions of the first integrals:

$$H_j = \frac{x_1^{n-i}\left[f_{\xi,\eta}(x_2) - y_2^2\right] - x_2^{n-i}\left[f_{\xi,\eta}(x_1) - y_1^2\right]}{x_1^{n-j} x_2^{n-j}(x_1^{j-i} - x_2^{j-i})},$$
$$H_i = \frac{x_2^{n-j}\left[f_{\xi,\eta}(x_1) - y_1^2\right] - x_1^{n-j}\left[f_{\xi,\eta}(x_2) - y_2^2\right]}{x_1^{n-j} x_2^{n-j}(x_1^{j-i} - x_2^{j-i})}.$$

2. If x_1 is a double root of u then we set $y_1 = v(x_1)$ and $y_2 = v'(x_1)$, that is:

$$u_1 = -2x_1, \quad u_2 = x_1^2, \quad v_1 = y_2, \quad v_2 = y_1 - x_1 y_2.$$

$H_k = 0$, $k \neq i$, $k \neq j$ allow to express w_j, $j = 1, \ldots, n-2$, in terms of u_1, u_2, v_1, v_2 so (u_1, u_2, v_1, v_2) are local coordinates on

$$M = \left\{(u, v, w), \forall k = 1, \ldots, n, k \neq i, k \neq j, H_k = 0\right\}.$$

But we will use the variables (x_1, x_2, y_1, y_2) for most of the following calculations. Obviously, they are not coordinates on M because of the points where the polynomial u has a double root. Nevertheless, they are local coordinates on:

$$M^\star = \left\{(u, v, w)_\star, \forall k = 1, \ldots, n, k \neq i, k \neq j, H_k = 0\right\},$$

where $(u, v, w)_\star$ is such as u has no double root.

From now on, we often infer that $\forall k = 1, \ldots, n$, $k \neq i$, $k \neq j$, $H_k = 0$.

2.1.3. Symplectic structure and first integrals. Consider the canonical Poisson bracket V on $\mathbb{R}^4(x_1, x_2, y_1, y_2)$:

$$V(x_k, x_p) = 0, \quad V(y_k, y_p) = 0, \quad V(x_k, y_p) = \delta_{k,p}, \quad k, p = 1, 2,$$

δ being Kronecker's delta. The corresponding symplectic form is $\Omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. A direct calculation gives $\Omega = dv_2 \wedge du_1 + dv_1 \wedge du_2 + u_1 du_1 \wedge dv_1$.

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REMARK. If we set $\tilde{u_2} = u_2 - \frac{1}{2}u_1^2$, then $\Omega = dv_2 \wedge du_1 + dv_1 \wedge d\tilde{u_2}$. Notice that the change of variables $(x_1, x_2, y_1, y_2) \longmapsto (u_1, \tilde{u_2}, v_2, v_1)$ is a canonical transformation. Its generating function is:

$$F(x_1, x_2, v_1, v_2) = -(x_1 + x_2)v_2 - \frac{1}{2}(x_1^2 + x_2^2)v_1.$$

PROPOSITION 2.2. *Consider the manifold*

$$M = \left\{(u, v, w), \forall k = 1, \ldots, n, k \neq i, k \neq j, H_k = 0\right\},$$

endowed with the Poisson bracket V and the symplectic structure Ω.

- *Since the functions* H_i *and* H_j *are in involution (and independent),* (M, Ω, H_i, H_j) *defines a two degrees of freedom integrable Hamiltonian system.*
- *The system linearizes in variables* (x_1, x_2, y_1, y_2).

PROOF. A direct calculation using the expressions of H_i and H_j in local coordinates (x_1, x_2, y_1, y_2) on $M^\star$ shows that $V(H_i, H_j) = 0$ and that H_i and H_j are independent.

The differential equations expressed in (x_1, x_2, y_1, y_2) are:

$$\left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dy_1}{dt}, \frac{dy_2}{dt}\right) = \Big(V(x_1, H_k), V(x_2, H_k), V(y_1, H_k), V(y_2, H_k)\Big)_{k=i,j}.$$

They yield a linearized system corresponding to each Hamiltonian:

- for H_j:

$$\begin{cases} \frac{x_1^{n-i}dx_1}{y_1} + \frac{x_2^{n-i}dx_2}{y_2} = 0 \\ \frac{x_1^{n-j}dx_1}{y_1} + \frac{x_2^{n-j}dx_2}{y_2} = -2dt. \end{cases}$$

- for H_i:

$$\begin{cases} \frac{x_1^{n-i}dx_1}{y_1} + \frac{x_2^{n-i}dx_2}{y_2} = -2dt \\ \frac{x_1^{n-j}dx_1}{y_1} + \frac{x_2^{n-j}dx_2}{y_2} = 0. \end{cases}$$

Since $M \setminus M^\star$ is of dimension three, whereas M is of dimension four, we can conclude that H_i and H_j are in involution and independent everywhere. ∎

IMPORTANT REMARK. Consider $p \neq i$, $p \neq j$. a_p only appears in the function H_p and in no other H_q, $q \neq p$. ξ (resp η) only appears in H_i (resp in H_j). So ξ and η can be looked upon as the first integrals of the system.

2.2. *General case.* Now, back to the general case, we consider the two degrees of freedom JMM system associated with

$$f_{\xi,\eta}(t) = \sum_{k=0}^{n} a_k(\xi, \eta)t^{n-k} \quad (a_0 = 1).$$

We can also write $f_{\xi,\eta}(t) = c(t) + \xi f_1(t) + \eta f_2(t)$, where c is a monic polynomial of degree n with constant coefficients, f_1 and f_2 are polynomials of degree $n - 1$ with constant coefficients.

We still can write $f(t) - u(t)w(t) - v^2(t) = \sum_{k=1}^n H_k t^{n-k}$ and set up $M = \{(u, v, w), \forall k \neq i, k \neq j, H_k = 0\}$ as we did before, for some fixed i and j.

On M, each $a_k(\xi, \eta)$ is a linear combination of H_i and H_j. Furthermore, there exists two functions H_ξ and H_η containing respectively ξ and η which are linear combinations of H_i and H_j. Indeed,

$$\begin{aligned}
\big[f_1(x_1)f_2(x_2) - f_2(x_1)f_1(x_2)\big]\xi &= H_i\big[x_1^{n-i}f_2(x_2) - x_2^{n-i}f_2(x_1)\big] - H_j\big[x_2^{n-j}f_2(x_1) - x_1^{n-j}f_2(x_2)\big] \\
&\quad + f_2(x_1)\big[c(x_2) - y_2^2\big] - f_2(x_1)\big[c(x_1) - y_1^2\big], \text{ and} \\
\big[f_1(x_1)f_2(x_2) - f_2(x_1)f_1(x_2)\big]\eta &= H_j[x_2^{n-j}f_1(x_2) - x_2^{n-i}f_1(x_1)] \\
&\quad - H_i\big[x_1^{n-i}f_1(x_2) - x_2^{n-i}f_1(x_1)\big].
\end{aligned}$$

By making use of the proposition (2.2), we can state the following:

PROPOSITION 2.3. *Consider the manifold*

$$M = \big\{(u, v, w), \forall k \neq i, k \neq j, H_k = 0\big\}$$

endowed with the symplectic structure Ω.

- *The independent functions* H_ξ *and* H_η *are in involution, so that* $(M, \Omega, H_\xi, H_\eta)$ *defines a two degrees of freedom integrable Hamiltonian system.*
- *The system* $(M, \Omega, H_\xi, H_\eta)$ *linearizes in variables* (x_1, x_2, y_1, y_2).

REMARK. Although it does not appear in our notation, each pair (i, j) defines a two degrees of freedom JMM system.

3. **Bott systems: definitions and main theorem.** In this section, we give the definitions of a Bott function and of a Bott map. Then, we set out our main theorem (the proof will be done in the next section).

3.1. *Definitions.* For the stating of these definitions, we used [5] (p. 6), [4] (p. 55) and [2].

DEFINITION 3.1. The critical submanifold of an analytic function $F\colon \mathbb{C}^m \longrightarrow \mathbb{C}$ is nondegenerate if its hessian d^2F is nondegenerate on normal planes to the submanifold.

DEFINITION 3.2. An analytic function F is a Bott function providing its critical submanifold is nondegenerate.

DEFINITION 3.3. An analytic map $F = (F_1, \ldots, F_n)\colon \mathbb{C}^m \longrightarrow \mathbb{C}^n$ is a Bott map providing the functions $F_k \mid Q\big((c_l)_{l=1\ldots n}\big)$, $k = 1, \ldots, n$, are either constants or Bott functions for any smooth level set:

$$Q\big((c_1, \ldots, c_n)\big) = \Big\{\sum_{k=1}^n c_k H_k = 0\Big\}.$$

DEFINITION 3.4. An integrable Hamiltonian system is a Bott system if its momentum mapping is a Bott map.

3.2. *Main theorem.*

THEOREM 3.1. *Consider*

$$f_{\xi,\eta}(t) = \sum_{k=0}^{n} a_k(\xi,\eta)t^{n-k} \quad (a_0 = 1).$$

Suppose that the functions $(\xi,\eta) \longmapsto a_k(\xi,\eta)$, $k = 1,\dots,n$, *are linear at each variable.*

The two degrees of freedom Jacobi-Moser-Mumford Hamiltonian systems associated to the family of polynomials $f_{\xi,\eta}$ *are Bott systems.*

4. **Proof of the main theorem.** Once again, we begin with the proof of the main theorem in the special case and deduce that the result still holds in the general case.

4.1. *Proof in the special case.* We consider the JMM system associated with:

$$f_{\xi,\eta}(t) = t^n + a_1 t^{n-1} + \cdots + \xi t^{n-i} + \cdots + \eta t^{n-j} + \cdots + a_{n-1}t + a_n.$$

We recall (see (2)) that the first integrals of the system satisfy:

$$f_{\xi,\eta}(x_k) - y_k^2 = H_i x_k^{n-i} + H_j x_k^{n-j}, \quad k = 1, 2.$$

We begin the proof of the theorem with an intermediate step: we study the smoothness of the sets $Q(c_1,c_2) = \{c_1H_i + c_2H_j = 0\}$. Then, we show that the restrictions of the first integrals H_j and H_i to a smooth manifold $Q(c_1,c_2)$ are either constants or Bott functions, considering two different cases: $c_2 \neq 0$ and $c_2 = 0$. Finally, we show that the momentum mapping $H = (H_i,H_j)\colon \mathbb{C}^4 \longrightarrow \mathbb{C}^2$ is a Bott map.

4.1.1. Smoothness of $\{c_1H_i + c_2H_j = 0\}$.

PROPOSITION 4.1. *Suppose that if* $(i,j) \neq (n-1,n)$ *then* $(a_{n-1},a_n) \neq (0,0)$. *Let* $B = \{(\xi,\eta), f_{\xi,\eta}$ *has a double root*$\}$.

Consider (ξ^0,η^0) *in* B *such as* $f_{\xi^0,\eta^0}(t_0) = f'_{\xi^0,\eta^0}(t_0) = 0$.

1. *If* $f''_{\xi^0,\eta^0}(t_0) = 0$ *then any line is tangent to* B *at* (ξ^0,η^0).
2. *If* $f''_{\xi^0,\eta^0}(t_0) \neq 0$ *then the line* $\Delta = \{(\xi,\eta), f_{\xi,\eta}(t_0) = 0\}$ *is tangent to* B *at* (ξ^0,η^0).

PROOF. First, for some $t \neq 0$, let us consider the system:

$$f_{\xi,\eta}(t) = 0, \quad f'_{\xi,\eta}(t) = 0$$

$$\iff \begin{cases} f_{\xi^0,\eta^0}(t) + (\xi-\xi^0)t^{n-i} + (\eta-\eta^0)t^{n-j} = 0 \\ f'_{\xi^0,\eta^0}(t) + (n-i)(\xi-\xi^0)t^{n-i} + (n-j)(\eta-\eta^0)t^{n-j} = 0 \end{cases}$$

$$\iff \begin{cases} \xi = \xi^0 + \frac{(n-j)f_{\xi^0,\eta^0}(t) - t f'_{\xi^0,\eta^0}(t)}{(j-i)t^{n-i}} \\ \eta = \eta^0 - \frac{(n-i)f_{\xi^0,\eta^0}(t) - t f'_{\xi^0,\eta^0}(t)}{(j-i)t^{n-j}}. \end{cases}$$

So a parametrization of B in a neighbourhood of (ξ^0,η^0) is:

$$\varphi\colon t \longmapsto \begin{pmatrix} \xi(t) = \xi^0 + \frac{(n-j)f_{\xi^0,\eta^0}(t) - tf'_{\xi^0,\eta^0}(t)}{(j-i)t^{n-i}} \\ \eta(t) = \eta^0 - \frac{(n-i)f_{\xi^0,\eta^0}(t) - tf'_{\xi^0,\eta^0}(t)}{(j-i)t^{n-j}} \end{pmatrix}.$$

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The gradient of φ at t_0 is:

$$\begin{pmatrix} \frac{d\xi}{dt_1}/t_0 \\ \frac{d\eta}{dt_1}/t_0 \end{pmatrix} = \frac{f''_{\xi^0,\eta^0}(t_0)}{(j-i)t_0^{n-i-1}} \begin{pmatrix} -1 \\ t_0^{j-i} \end{pmatrix}.$$

1. If $f''_{\xi^0,\eta^0}(t_0) \neq 0$ then grad $\varphi(t_0)$ represents the direction of the tangent to B at (ξ^0,η^0) and is colinear to $(-1, t_0^{j-i})$. On the other hand, the line $\Delta = \{(\xi,\eta) f_{\xi,\eta}(t_0) = 0\}$ passes through the point (ξ^0,η^0) and admits $(-1, t_0^{j-i})$ as a directing vector in the plane (ξ,η). Thereby, Δ is tangent to B at (ξ^0,η^0).
2. If $f''_{\xi_0,\eta_0}(t_0) = 0$ then B is singular at (ξ_0,η_0) so any line is tangent to B.

Notice that this process still works if $t_0 = 0$ and $(i,j) = (n-1,n)$. The case $t_0 = 0$ and $(i,j) \neq (n-1,n)$ has been excluded by hypothesis when we said $(a_{n-1},a_n) \neq (0,0)$ if $(i,j) \neq (n-1,n)$. ∎

We decide to adopt the following definition:

DEFINITION 4.1. The pair (c_1,c_2) defining

$$\begin{aligned} Q\big((c_1,c_2)\big) &= \big\{(u,v,w), f = uw + v^2 + H_i t^{n-i} + H_j t^{n-j}, c_1H_i + c_2H_j = 0\big\} \\ &= \{\forall k \neq i, k \neq j, H_k = 0, c_1H_i + c_2H_j = 0\} \end{aligned}$$

is generic if the vector $(-c_2,c_1)$ is never tangent to $B = \{(\xi,\eta), \exists t, f_{\xi,\eta}(t) = f'_{\xi,\eta}(t) = 0\}$.

Henceforward, we suppose that

- the function $f_{\xi,\eta}$ that defines the system satisfies $(a_{n-1},a_n) \neq (0,0)$ whenever $(i,j) \neq (n-1,n)$,
- the pairs (c_1,c_2) are generic.

Let us fix some $c \neq 0$ and define

$$Q\big((c,1)\big) = Q(c) = \big\{(u,v,w), \forall k \neq i, k \neq j, H_k = 0, cH_i + H_j = 0\big\}.$$

By making use of (2), let us define an equivalence $\overset{c}{\approx}$ so that $Q(c) = \{(u,v,w), f \overset{c}{\approx} uw + v^2\}$:

DEFINITION 4.2. Two polynomials P and R are $\overset{c}{\approx}$-equivalent if:

$$\exists d \in \mathbb{C}, P - R = d(t^{n-i} - ct^{n-j}).$$

Let $[P]$ be the $\overset{c}{\approx}$-equivalent class of P.

Likewise, for $Q_i = \{H_i = 0\}$ to be written as $\{f \overset{0}{\sim} uw + v^2\}$, we set out:

DEFINITION 4.3. Two polynomials P and R are $\overset{0}{\sim}$-equivalent if:

$$\exists d \in \mathbb{C}, P - R = dt^{n-i}.$$

Let $[P]_0$ be the $\overset{0}{\sim}$-equivalent class of P.

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Let $(u,v,w)_\star$ stands for a point such as u has no double root.

$$\begin{aligned} Q(c)^\star &= \left\{(u,v,w)_\star, f \overset{c}{\approx} uw+v^2\right\} \\ &= \left\{(x_k,y_k)_{k=1,2}, x_1 \neq x_2, cH_i(x_k,y_k)+H_j(x_k,y_k)=0\right\}. \\ Q_i^\star &= \left\{(u,v,w)_\star, f \overset{0}{\sim} uw+v^2\right\} \\ &= \left\{(x_k,y_k)_{k=1,2}, x_1 \neq x_2, H_i(x_k,y_k)=0\right\}. \end{aligned}$$

PROPOSITION 4.2. *1. The manifold*

$$Q(c) = \{H_1 = \cdots = H_{n-2} = 0, cH_i + H_j = 0\} = \{f \overset{c}{\approx} uw+v^2\}, c \neq 0,$$

is smooth providing the pair $(c,1)$ *is generic.*

2. *The manifold* $Q_i = \{H_i = 0\} = \{f \overset{0}{\sim} uw+v^2\}$ *is smooth providing* $(1,0)$ *is generic.*

PROOF. 1. Consider a manifold $Q(c) = \{cH_i + H_j = 0\}$, $c \neq 0$, such as the pair $(c,1)$ is generic.

Consider $(u,v,w) \in Q(c)$. We look for the dimension of the Zariski tangent space:

$$T_{(u,v,w)}Q(c) = \left\{(\dot u,\dot v,\dot w), u\dot w + w\dot u + 2v\dot v \overset{c}{\approx} 0\right\}.$$

The equation

$$P \overset{c}{\approx} u\dot w + w\dot u + 2v\dot v, \tag{3}$$

always has a solution for any $P(t) = r_1t^{n-1} + \cdots + r_it^{n-i} + \cdots - cr_it^{n-j} + \cdots + r_n$. Indeed, we look for $(\dot u,\dot v,\dot w)$ solution of (3). A problem can appear if there exists x_1 such as:

$$x_1^{j-i} = c,\ u(x_1) = v(x_1) = w(x_1) = 0.$$

For such a $(u,v,w) \in Q(c)$, $f \overset{c}{\approx} uw+v^2$, so that, for some $d \in \mathbb{C}$,

$$f(x_1) = 0, f'(x_1) = -d(j-i)x_1^{n-i-1}.$$

There are three possible cases:

(a) If $f(x_1) = 0$, $f'(x_1) \neq 0$ then (3) can be solved by differentiation.

(b) If $f(x_1) = f'(x_1) = 0$, $f''(x_1) \neq 0$ then

   (i) if $x_1 \neq 0$, we know that $(-1, x_1^{j-i}) = (-1,c)$ is a directing vector of the tangent to B at $\big(\xi(x_1),\eta(x_1)\big)$. This contradicts the genericity of the pair $(c,1)$.

   (ii) if $x_1 = 0$ then 0 is a double root of f which is not allowed.

(c) If $f(x_1) = f'(x_1) = f''(x_1) = 0$ then any line is tangent to B at $\big(\xi(x_1),\eta(x_1)\big)$ which contradicts the genericity of $(c,1)$.

Thereby, if we consider the map $A\colon (\dot u,\dot v,\dot w) \longmapsto [\dot u w + \dot w u + 2\dot v v]$, the dimension of $\operatorname{Im} A$ is the dimension of the set of P-like polynomials, that is $n-1$.

Finally, the dimension of the Zariski tangent space $T_{(u,v,w)}Q(c) = \operatorname{Ker} A$ is $(n+2) - (n-1) = 3$, which shows that $Q(c)$ has no singular points.

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Let us stress upon the meaning of the smoothness of $Q(c)$: all the $\overset{c}{\approx}$-equivalent-to-f polynomials have neither double roots x_1 such as $x_1^{j-i} = c$ nor triple roots.

2. Consider the manifold $Q_i = \{H_i = 0\} = \{f \overset{0}{\sim} uw + v^2\}$, $(1, 0)$ being generic.

The equation

$$P \overset{0}{\sim} u\dot{w} + w\dot{u} + 2v\dot{v} \tag{4}$$

has always a solution for any polynomial $P = r_1t^{n-1} + \cdots + r_{i-1}t^{n-i+1} + r_{i+1}t^{n-i-1} + \cdots + r_n$. Indeed, a problem occurs at a point (u, v, w) such as $u(0) = v(0) = w(0) = 0$. But then:

$$\begin{aligned}(u, v, w) \in Q(c) &\Rightarrow f \overset{0}{\sim} uw + v^2\\ &\Rightarrow \exists d, f = uw + v^2 + dt^{n-i}\\ &\Rightarrow f(0) = 0, f'(0) = 0.\end{aligned}$$

The three possible cases are:

1. If $f(0) = 0, f'(0) \neq 0$ then $P \overset{0}{\sim} u\dot{w} + w\dot{u} + 2v\dot{v}$ can be solved by differentiation.
2. $f(0) = f'(0) = 0, f''(0) \neq 0$ contradicts the genericity of $(1, 0)$.
3. $f(0) = f'(0) = f''(0) = 0$ is not allowed.

Like in the first case, we conclude that the Zariski tangent space $T_{(u,v,w)}Q_i$ has no singular point.

Let us stress upon the meaning of the smoothness of Q_i: all $\overset{0}{\sim}$-equivalent-to-f polynomials have neither a double root at 0 nor triple roots. ∎

4.1.2. Restriction to $Q(c)$. Since H_i and H_j are proportional on the smooth submanifold $Q(c)$, we choose to study H_i. We look for critical points of $H_i \mid Q(c)^\star$.

PROPOSITION 4.3. *Critical points of $H_i \mid Q(c)^\star$ are $(x_1, x_2, 0, y_2) \in Q(c)^\star$ such as x_1 is a double root of a $\overset{c}{\approx}$-equivalent-to-f polynomial (and similar points permuting x_1 and x_2).*

PROOF. From (1) and $cH_i + H_j = 0$, we get the expression of $H_i \mid Q(c)^\star$:

$$H_i \mid Q(c)^\star : (x_1, y_1, x_2) \longmapsto \frac{f_{\xi,\eta}(x_1) - y_1^2}{x_1^{n-j}(x_1^{j-i} - c)}.$$

REMARK. The roots of u are always distinct on $Q(c)^\star$, so if x_1 is such as $x_1^{n-j}(x_1^{j-i} - c) = 0$ then we use the variable x_2 to express $H_i \mid Q(c)^\star$.

$(x_1, x_2, y_1) \in Q(c)^\star$, is a critical point of $H_i / Q(c)^\star$ if:

$$y_1 = 0, x_1 f'(x_1)(x_1^{j-i} - c) - f(x_1)\big[(n-i)x_1^{j-i} + c(n-j)\big] = 0. \tag{5}$$

Now consider the new polynomial

$$f_{\xi + \frac{f(x_1)}{x_1^{n-j}(c - x_1^{j-i})}, \eta - \frac{cf(x_1)}{x_1^{n-j}(c - x_1^{j-i})}}(t) = f_{\xi,\eta}(t) + \frac{f_{\xi,\eta}(x_1)}{x_1^{n-j}(c - x_1^{j-i})}(t^{n-i} - ct^{n-j}).$$

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Clearly, if $(x_1, x_2, y_1) \in Q(c)^\star$ is a singular point of $H_i \mid Q(c)^\star$ then x_1 is a double root of $f_{\xi+\frac{f(x_1)}{x_1^{n-j}(c-x_1^{j-i})},\eta-\frac{cf(x_1)}{x_1^{n-j}(c-x_1^{j-i})}}$ which is $\overset{c}{\approx}$-equivalent to f. ∎

Now we have all the elements to state the following proposition:

PROPOSITION 4.4. *Consider*

$$Q(c) = \{\forall k \neq i, k \neq j, H_k = 0, cH_i + H_j = 0\} \quad (c \neq 0).$$

$H_i \mid Q(c)$ and $H_j \mid Q(c)$ are Bott functions, providing the pair $(c, 1)$ is generic.

PROOF. In this proof, we note h_i the restriction $H_i \mid Q^\star_{n-1}$. Let $S^\star$ be the critical submanifold of h_i. Let N be the plane generated by the derivations $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial y_1}$.

Consider a singular point (x_1, y_1) such as (5) holds. We saw in the former proposition that x_1 is a double root of

$$f_{\xi+\frac{f(x_1)}{x_1^{n-j}(c-x_1^{j-i})},\eta-\frac{cf(x_1)}{x_1^{n-j}(c-x_1^{j-i})}}.$$

If the determinant of the Hessian $d^2(h_i \mid N)(x_1, 0)$ is zero then a direct computation shows that x_1 is a triple root of this new polynomial (indeed, $\det[d^2(h_i \mid N)](x_1, 0) = 0$ iff:

$$x_1^2(x_1^{j-i} - c)f''(x_1) - \left[(n-i)(n-i-1)(x_1^{j-i} - c) - c(i-j)(2n-i-j-1)\right]f(x_1) = 0).$$

This contradicts the genericity of the pair $(c, 1)$ because a $\overset{c}{\approx}$-equivalent-to-f polynomial would have a triple root. Therefore, $\det[d^2(h \mid N)] \neq 0$ and h_i is nondegenerate on the plane N.

The plane N does not contain any tangent vector v to $S^\star$ at the point (x_1, y_1, z) (z stands for the third coordinate in the submanifold $S^\star$). Indeed, h_i is constant on the trajectory of the integral curve of v since $\operatorname{grad} h_i = 0$ on $S^\star$. This can be written $L_v^2 h_i = 0$. But $\langle d^2 h_i v, v\rangle = L_v^2 h_i = 0$ so h_i is degenerate along v. As h_i is never degenerate on N, the tangent vector v does not belong to N. Finally, N is a normal plane to $S^\star$.

Now, let us study the points (u, v, w) such as u has a double root. Call S the critical submanifold of $H_i \mid Q(c)$. First notice that $d^2\big(H_i \mid Q(c)\big)$ is a tensor. It is invariant under any change of coordinates on S (because here $\operatorname{grad} H_i = 0$). Since $Q(c)$ is nondegenerate, $\operatorname{grad} H_j$ is never equal to zero and the flow of H_j has no stationary point on $Q(c)$ or S. Moreover S is invariant by the flow of H_j since $0 = \{H_i, H_j\} = L_{X_j} H_i$. Consequently, the flow of H_i plays the role of a diffeomorphism transforming a point of $S^\star$ into a point of $S \setminus S^\star$. It preserves the rank of the hessian $d^2\big(H_i \mid Q(c)\big)$ which is nondegenerate on normal planes to $S^\star$, so we conclude that $d^2\big(H_i \mid Q(c)\big)$ is nondegenerate everywhere on normal planes to S.

Hence $H_i \mid Q(c)$ is nondegenerate on normal planes to its critical submanifold, that is, $H_i \mid Q(c)$ is a Bott function. As H_j is proportionnal to H_i on $Q(c)$, we draw the same conclusion for $H_j \mid Q(c)$. ∎

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4.1.3. Restriction to Q_i.

PROPOSITION 4.5. *Consider the smooth manifold* $Q_i = \{\forall k \neq j, H_k = 0\}$. *The function* $H_j \mid Q_i$ *is a Bott function.*

PROOF. We proceed for $H_j \mid Q_i$ as we did for $H_i \mid Q(c)$ in the former section but, because of the likenesses of the two cases, we sometimes shorten the explanations.

According to (2), the restriction of the function H_j to the smooth manifold Q_i has the following form:

$$\begin{array}{rccc} G_j : & \mathbb{C}^3 & \longrightarrow & \mathbb{C} \\ & (x_1, y_1, x_2) & \longmapsto & \frac{f(x_1)-y_1^2}{x_1^{n-j}} \end{array}$$

It takes some calculations to show that G_j is Bott:

$$(x_1, y_1, x_2) \in \mathrm{Sing}(G_j) \Longleftrightarrow y_1 = 0, x_1 f'(x_1) + (n-j) f(x_1) = 0. \tag{6}$$

Consider x_1 such as (6) holds. x_1 is a double root of

$$f_{\xi, \eta + \frac{x_1 f'(x_1) - (n-k) f(x_1)}{(i-k) x_1^{n-j}}}(t) = f(t) + \frac{x_1 f'(x_1) - (n-k) f(x_1)}{(i-k) x_1^{n-j}} t^{n-j},$$

for any k, $1 \leq k \leq n$, $k \neq j$. We temporarily call this new polynomial g. Notice that g is $\overset{0}{\sim}$-equivalent to f.

Let us keep the notations of the proof of proposition (4.4).

The determinant of the hessian matrix $d^2(G_j \mid N)(x_1, 0)$ is zero if and only if:

$$x_1 f''(x_1) - (n-j-1) f'(x_1) = 0. \tag{7}$$

We compute easily that: (7) $\Rightarrow g''(x_1) = 0$. But, $g''(x_1) = 0$ contradicts the smoothness of Q_i (because a $\overset{0}{\sim}$-equivalent-to-f polynomial would have a triple root). Thereby, it is impossible to have $\det[d^2(G_j \mid N)] = 0$. $H_j \mid Q_i$ is nondegenerate on normal planes to its critical submanifold, that is $H_j \mid Q_i$ is a Bott function. ■

To sum up, it has been shown that H_i and H_j are Bott functions when they are restricted to smooth manifolds $\{c_1 H_i + c_2 H_j = 0\}$. Thus (H_i, H_j) is a Bott map and the system is a Bott system. The proof of Theorem 3.1 is completed in the special case of

$$f_{\xi,\eta}(t) = t^n + a_1 t^{n-1} + \cdots + \xi t^{n-i} + \cdots + \eta t^{n-j} + \cdots + a_{n-1} t + a_n.$$

4.2. *Proof in the general case.* We recall that the two integrals (H_ξ, H_η) of the JMM system associated with:

$$f_{\xi,\eta}(t) = \sum_{k=0}^{n} a_k(\xi, \eta) t^{n-k} \quad (a_0 = 1),$$

are linear combinations of H_i and H_j.

Thereby, according to the special case that we have been dealing with in (4.1),

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1. $H_\xi \mid \{c_1H_i + c_2H_j = 0\}$ and $H_\eta \mid \{c_1H_i + c_2H_j = 0\}$ are Bott functions for any smooth manifold $\{c_1H_i + c_2H_j = 0\}$,
2. $H_\xi \mid \{c'_1H_\xi + c'_2H_\eta = 0\}$ and $H_\eta \mid \{c'_1H_\xi + c'_2H_\eta = 0\}$ are Bott functions for any smooth manifold $\{c'_1H_\xi + c'_2H_\eta = 0\}$,
3. (H_ξ, H_η) is a Bott map.

This completes the proof of Theorem 3.1 in the general case of $f_{\xi,\eta}(t) = \sum_{k=0}^{n} a_k(\xi,\eta)t^{n-k}$.

Remembering that each $a_k(\xi,\eta)$, $k = 1,\ldots,n$, is a linear combination of H_i and H_j, we also state the following corollary:

COROLLARY 4.1. *The map $(a_1,\ldots,a_n)$ is a Bott map.*

5**. An application and some examples.** Bott functions are often intervening in Hamiltonian systems theory. In this section, we mention Fomenko's theorem (in [4]) about classification of Liouville's tori bifurcations. Then we consider the example of Gel'fand-Dikii's system. In a second part, we prove that the Lagrange top system is a Bott system. Finally, we show that the two degrees of freedom Jacobi's system (or more precisely Neumann's system) is a Bott system.

5.1. *Fomenko's theory about Liouville's tori bifurcations.* So far, we have been considering complex manifolds like $V = \{f = uw + v^2\}$. Now that we study the bifurcations of the systems, we must consider the real part $V_{\mathbb{R}}$ of V. Thus, in this section, we use a "real version" of the propositions and theorems that we have been stating.

The reader can refer to [4] and to [5]. In this section, we just recall Fomenko's theorem (see [4], p. 67) about classification of bifurcations of two-dimensional Liouville's tori. Our Theorem 3.1 proves that it can be applied to the large class of two degrees of freedom JMM systems. For example, we show that Gel'fand-Dikii's system is a Bott system and we study its bifurcations by making use of Fomenko's theorem.

5.1.1. Fomenko's theorem. As kindly explained to us by A. Fomenko, it is possible to classify simple Hamiltonian systems on three dimensional constant energy surfaces. The reader will find more details in [5].

Consider an IHS2 (M^4, H, F, ω) such as the restriction of F to $Q^3 = \{H = \text{const}\}$ is a Bott function. Liouville's foliation of Q^3 is given by the isoenergy surfaces of $f \mid Q^3$. The topological invariant of such a system is a molecule whose atoms are representing the Liouville's foliation singularities. The links between the atoms of a molecule are representing Liouville's tori. Let us consider three simple bifurcations and give their atomic representation:

1. A torus T^2 is contracted to the axial circle of a full torus and then "vanishes". This bifurcation is represented by $-A$.
2. A torus T^2 splits into two tori. This birfurcation is represented by: $-B^{-}_{-}$.
3. A torus T^2 spirals twice round a torus T^2. This bifurcation is represented by: $-A^\star-$.

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Furthermore molecules include numbers indicating the way Liouville's tori are glued (three numbers (r, ε, n) for each glueing).

The (simplified) theorem that we use is the following:

THEOREM 5.1. *1. Any orientable Morse-Bott type bifurcation of Liouville's tori is the direct product of some two dimensional foliated surface (so called "2-atom") on the circle:* $P^2 \times S^1$.

2. *Any non-orientable Morse-Bott type bifurcation of Liouville's tori is the skew product (Seifert product of type (2,1)) of some 2-atom on the circle:* $P^2 \tilde{\times} S^1$.
3. *The classification of all 2-atoms is given.*
4. *The Liouville's type of global Liouville's foliation on the whole isoenergy 3-surface is completely described (up to Liouville's equivalence) by so called "marked-molecule"* $W^\star$ *combined form of 2-atoms and some numerical marks.*

5.1.2. Gel'fand-Dikii's system.

PROPOSITION 5.1. *Consider the family of polynomials* $f_{\xi,\eta}(t) = t^5 + \xi t + \eta$. *The JMM system associated with* $f_{\xi,\eta}$ *is Gel'fand-Dikii's system.*

PROOF. The first integrals of Gel'fand-Dikii's system are:

$$\begin{cases} H = -q_1^4 - q_2^2 + 3q_1^2 q_2 - q_1 p_2^2 - 2p_1 p_2, \\ K = q_1^3 q_2 - 2q_1 q_2^2 + q_1^2 p_2^2 + 2q_1 p_1 p_2 - p_2^2 q_2 + p_1^2. \end{cases}$$

On the other hand, proceeding as in Section 2 with the family of polynomials

$$f_{\xi,\eta}(t) = t^5 + \xi t + \eta,$$

we get a JMM system with first integrals:

$$H_4 = \frac{f(x_1) - y_1^2 - f(x_2) + y_2^2}{x_1 - x_2}, \quad H_5 = \frac{-x_2\left[f(x_1) - y_1^2\right] + x_1\left[f(x_2) - y_2^2\right]}{x_1 - x_2}.$$

By making use of the following canonical transformation:

$$(x_1, x_2, y_1, y_2) \longmapsto \left(q_1 = x_1 + x_2, q_2 = x_1 x_2, p_1 = \frac{x_1 y_1 - x_2 y_2}{x_1 - x_2}, p_2 = \frac{y_2 - y_1}{x_1 - x_2}\right),$$

we can express H and K in terms of $(x_k, y_k)_{i=1,2}$ and see that:

$$H_4 = h - H, \quad H_5 = k - K. \qquad \blacksquare$$

Once again we use Theorem 3.1 and state the following corollary:

COROLLARY 5.1. *Gel'fand-Dikii's system is a Bott system.*

Thus, Fomenko's theorem can be applied to Gel'fand-Dikii's system. The bifurcations of the real part $V_{\mathbb{R}}$ of $V = \{H = \text{const}, K = \text{const}\}$ can be described by observing the behaviour of roots of the polynomial $f(t) = t^5 + \xi t + \eta$ as η varies (ξ being fixed). We sum this up in the following table (α_1 and α_2 are the roots of f'):

ξ	η	roots topological type	domain
$\xi > 0$	$-$	one real root T^2	$D1$
$\xi = 0$	$\eta \neq 0$	one real root T^2	$D1$
$\xi = 0$	$\eta = 0$	0 quintuple root bifurc at 0	$-$
$\xi < 0$	$\eta < -(\alpha_1^5 + \xi\alpha_1)$	one real root T^2	$D1$
$\xi < 0$	$\eta = -(\alpha_1^5 + \xi\alpha_1)$	a double root bifurc at $P_1(\xi, \eta)$	$-$
$\xi < 0$	$-(\alpha_1^5 + \xi\alpha_1) < \eta < -(\alpha_2^5 + \xi\alpha_2)$	three real roots $2T^2$	$D2$
$\xi < 0$	$\eta = -(\alpha_2^5 + \xi\alpha_2)$	a double root bifurc at $P^2(\xi, \eta)$	$-$
$\xi < 0$	$\eta > -(\alpha_2^5 + \xi\alpha_2)$	one real root T^2	$D1$

Let us fix $\xi_0 < 0$. $P_1(\xi_0, -\eta_0)$ and $P_2(\xi_0, \eta_0)$ are the common points of the bifurcation diagram and of the line $\xi = \xi_0$. Going from the domain $D2$ to the domain $D1$ through $P_1(\xi, \eta)$, two tori are transformed into one torus. The bifurcation is ${}^{-}_{-}B-$. Going from the domain $D1$ to the domain $D2$ through $P_2(\xi, \eta)$, it is the "inverse" bifurcation $-B^{-}_{-}$. Going from the domain $D1$ to the domain $D2$ through the origin, we also have the bifurcation $-B^{-}_{-}$.

1. Moving along the line $\xi = \xi_0$, we glue two atoms $-B^{-}_{-}$ and we get the corresponding molecule:

$$-B^{-}_{-}{}^{-}_{-}B- \qquad \textit{i.e.} \quad \_\_ B = B \_\_ \, .$$

2. Now imagine a path going from $D1$ to $D2$ through 0 and going back to $D1$ by passing through P_1. The bifurcations are represented by the same molecule:

$$\_\_ B = B \_\_ \, .$$

Obviously, although their molecules are identical, these bifurcations are different. Indeed, the glueings of the atoms $-B^{-}_{-}$ are different in cases 1. and 2. This shows the importance of the "marks" (numbers) that must be added to the molecules since they reflect these differences. But this is not our intention to compute these numbers here. The reader can refer to [5] to get more information.

REMARK. This subject is treated in a quite different way in [12].

5.2. *The Lagrange top system.* We recall that the Lagrange top is a heavy symmetrical top (see [8]).

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PROPOSITION 5.2. *Fix* $(a_1, a_2) \in \mathbb{C}^2$. *Consider the family of polynomials:*

$$f_{\xi,\eta}(t) = t^4 + a_1 t^3 + a_2 t^2 + \xi t + \eta.$$

The JMM system associated with $f_{\xi,\eta}$ *is a Lagrange top system.*

PROOF. Let us consider the usual Poisson bracket W used for the Lagrange top system:

$W \nearrow$	ω_1	ω_2	ω_3	γ_1	γ_2	γ_3
ω_1	0	$-k\omega_3$	$k^{-1}\omega_2$	0	$-\gamma_3$	γ_2
ω_2	$k\omega_3$	0	$-k^{-1}\omega_1$	γ_3	0	$-\gamma_1$
ω_3	$-k^{-1}\omega_2$	$k^{-1}\omega_1$	0	$-k^{-1}\gamma_2$	$k^{-1}\gamma_1$	0
γ_1	0	$-\gamma_3$	$k^{-1}\gamma_2$	0	0	0
γ_2	γ_3	0	$-k^{-1}\gamma_1$	0	0	0
γ_3	$-\gamma_2$	γ_1	0	0	0	0

The first W-integrals of the system are:

$$F_1 = \omega_3, \quad F_2 = \frac{1}{2}(\omega_1^2 + \omega_2^2 + k\omega_3^2) - \gamma_3.$$

The W-Casimirs are:

$$F_3 = \omega_1\gamma_1 + \omega_2\gamma_2 + k\omega_3\gamma_3, \quad F_4 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2, \quad \text{where } k = 1 + m.$$

Thus, we get the equations describing the motion of the Lagrange top:

$$\begin{cases} \dot{\omega}_1 = -m\omega_2\omega_3 - \gamma_2 \\ \dot{\omega}_2 = m\omega_1\omega_3 + \gamma_1 \\ \dot{\omega}_3 = 0 \\ \dot{\gamma}_1 = \gamma_2\omega_3 - \gamma_3\omega_2 \\ \dot{\gamma}_2 = \gamma_3\omega_1 - \gamma_1\omega_3 \\ \dot{\gamma}_3 = \gamma_1\omega_2 - \gamma_2\omega_1. \end{cases}$$

On the other hand, considering the polynomial:

$$f_{\xi,\eta}(t) = t^4 + a_1 t^2 + a_2 t^2 + \xi t + \eta,$$

and processing like in Section 2, one gets a two degrees of freedom Hamiltonian system with the following first integrals:

$$H_3 = \frac{1}{x_1 - x_2}\left[f(x_1) - y_1^2 - f(x_2) + y_2^2\right],$$
$$H_4 = \frac{1}{x_1 - x_2}\left[x_1\left(f(x_2) - y_2^2\right) - x_2\left(f(x_1) - y_1^2\right)\right].$$

In fact, this JMM system is a Lagrange top system. Indeed, there exists a bi-Hamiltonian structure for the Lagrange top (see [9], Theorem 4.2 and [7]). Briefly, we say that:

1. An appropriate change of variables gives the following relations between the first integrals:
   - $F_1 = \frac{1}{2k}(a_1 - H_1)$,
   - $F_2 = \frac{1}{2}\left[a_2 - H_2 - \frac{m}{4k}(a_1 - H_1)^2\right]$,
   - $F_3 = \frac{1}{2}(H_3 - a_3)$,
   - $F_4 = a_4 - H_4$.
2. Let V be the canonical Poisson bracket given in 2.1.3. Let us express V in variables (ω_i, γ_i):

$V \nearrow$	ω_1	ω_2	ω_3	γ_1	γ_2	γ_3
ω_1	0	0	0	0	-1	0
ω_2	0	0	0	1	0	0
ω_3	0	0	0	0	0	0
γ_1	0	-1	0	0	$k\omega_3$	$-\omega_2$
γ_2	1	0	0	$-k\omega_3$	0	ω_1
γ_3	0	0	0	ω_2	$-\omega_1$	0

3. F_1 and F_2 are V-Casimirs, F_3 and F_4 are V-first integrals of the system of the Lagrange top. Furthermore, $V(.,mF_1F_3 + \frac{1}{2}F_4)$, give the equations of motion of Lagrange top. ■

Hence, Theorem 3.1 holds and the following corollary ensues from it:

COROLLARY 5.2. *The Lagrange top system is a Bott system.*

5.3. *Geodesic flow on an ellipsoid.* First, we briefly recall the statements of R. Donagi, but the reader can refer to [3] for more details.

We consider an ellipsoid:

$$E = \left\{(q_1, q_2, q_3), \frac{q_1^2}{\alpha_1} + \frac{q_2^2}{\alpha_2} + \frac{q_3^2}{\alpha_3} = 1\right\} \subset \mathbb{R}^3.$$

The geodesic flow on the symplectic manifold TE is given by the Hamiltonian function of the squared length. But, the system TE has two drawbacks:

1. TE is not a symplectic space since the induced metrix is degenerate at (complex) points satisfying $(\frac{q_1}{\alpha_1})^2 + (\frac{q_2}{\alpha_2})^2 + (\frac{q_3}{\alpha_3})^2 = 0$,
2. the first integrals are not symmetric.

Nevertheless, the system TE can be recovered as a $\mathbb{C}^\star$-bundle over an hypersurface in the tangent bundle TS of the sphere S:

$$S = \left\{(q_1, q_2, q_3) \in \mathbb{C}^3, q_1^2 + q_2^2 + q_3^2 = 1\right\},$$

$$TS = \left\{(q_k, p_k)_{k=1,2,3} \in \mathbb{C}^6, q_1^2 + q_2^2 + q_3^2 = 1, q_1p_1 + q_2p_2 + q_3p_3 = 0\right\}.$$

The new system TS is symplectic. The three functions:

$$F_k(q,p) = q_k^2 + \sum_{\substack{l=1 \\ l \neq k}}^{3} \frac{(q_kp_l - q_lp_k)^2}{\alpha_k - \alpha_l}, \quad k = 1, 2, 3,$$

subject to the condition $F_1 + F_2 + F_3 = 1$, provide two independent first integrals.

This system is known as Neumann's system.

PROPOSITION 5.3 ([3], P. 28). *Neumann's system is a JMM system (except at points of the circles $q_1^2 + q_2^2 = 1$, $q_2^2 + q_3^2 = 1$ and $q_1^2 + q_3^2 = 1$).*

PROOF. We only give here essential stages of the proof.

· The function that defines the JMM system is:

$$\begin{aligned} f(t) \;=\; & (t-\alpha_1)(t-\alpha_2)^2(t-\alpha_3)^2\xi + (t-\alpha_1)^2(t-\alpha_2)(t-\alpha_3)^2\eta \\ & + (t-\alpha_1)^2(t-\alpha_2)^2(t-\alpha_3)(1-\xi-\eta). \end{aligned}$$

f can be written $c(t) + \xi f_1(t) + \eta f_2(t)$ with

$$\begin{aligned} f_1(t) &= (t-\alpha_1)(t-\alpha_2)^2(t-\alpha_3)^2 - (t-\alpha_1)^2(t-\alpha_2)^2(t-\alpha_3), \\ f_2(t) &= (t-\alpha_1)^2(t-\alpha_2)(t-\alpha_3)^2 - (t-\alpha 1)^2(t-\alpha_2)^2(t-\alpha_3), \\ c(t) &= (t-\alpha_1)^2(t-\alpha_2)^2(t-\alpha_3). \end{aligned}$$

· The Jacobi polynomials are:

$$\begin{aligned} u(t) &= (t-\alpha_2)(t-\alpha_3)q_1^2 + (t-\alpha_1)(t-\alpha_3)q_2^2 + (t-\alpha_1)(t-\alpha_2)q_3^2, \\ v(t) &= \sqrt{-1}\big[(t-\alpha_2)(t-\alpha_3)q_1p_1 + (t-\alpha_1)(t-\alpha_3)q_2p_2 + (t-\alpha_1)(t-\alpha_2)q_3p_3\big], \\ w(t) &= (t-\alpha_2)(t-\alpha_3)p_1^2 + (t-\alpha_1)(t-\alpha_3)p_2^2 + (t-\alpha_1)(t-\alpha_2)p_3^2. \end{aligned}$$

(We recall that $q_1^2 + q_2^2 + q_3^2 = 1$ and $q_1p_1 + q_2p_2 + q_3p_3 = 0$.)

· Thereby, the usual JMM system and Neumann's system are corresponding via the change of variables $\varphi\colon (q_1, q_2, p_1, p_2) \longmapsto (u_1, u_2, v_1, v_2)$ where:

$$\begin{aligned} u_1 &= (\alpha_3-\alpha_1)q_1^2 + (\alpha_3-\alpha_2)q_2^2 + \alpha_1 + \alpha_2, \\ u_2 &= \alpha_2(\alpha_3-\alpha_1)q_1^2 + \alpha_1(\alpha_3-\alpha_2)q_2^2 + \alpha_1\alpha_2, \\ v_1 &= -\sqrt{-1}\big[(\alpha_3-\alpha_1)q_1p_1 + (\alpha_3-\alpha_2)q_2p_2\big], \\ v_2 &= \sqrt{-1}\big[\alpha_2(\alpha_3-\alpha_1)q_1p_1 + \alpha_1(\alpha_3-\alpha_2)q_2p_2\big]. \end{aligned}$$

$\det\big(d\varphi(q_1, q_2, p_1, p_2)\big) = -4(\alpha_3-\alpha_1)^2(\alpha_3-\alpha_2)^2(\alpha_2-\alpha_1)^2q_1^2q_2^2$. If $q_1 \neq 0$ and $q_2 \neq 0$ then φ is a diffeomorphism and Neumann's system is a JMM system. ∎

Now, we state:

COROLLARY 5.3. *The momentum mapping of Neumann's system is a Bott map at any point (q_1, q_2, p_1, p_2) such as $q_1 \neq 0$ and $q_2 \neq 0$.*

PROOF. Neumann's system is a JMM system whenever the change of variables φ (expressed in the proof of proposition 5.3) is a diffeomorphism. So, whenever $q_1 \neq 0$ and $q_2 \neq 0$, we use Theorem 3.1 and say that the momentum mapping of Neumann's system is a Bott map. ∎

REMARK. Nguyen Tien Zung studied Neumann's system in a quite different way (see [11]).

6. **Conclusion.** We have shown that the two degrees of freedom JMM Hamiltonian system associated with:

$$f_{\xi,\eta}(t) = \sum_{k=0}^{n} a_k(\xi,\eta)t^{n-k} \quad (a_0 = 1),$$

(We suppose that the functions $(\xi,\eta) \longmapsto a_k(\xi,\eta)$, $k = 1,\dots,n$, are linear at each variable) is a Bott system.

The logical extension of this paper consists in proving Theorem 3.1 for $k > 2$ degrees of freedom JMM Hamiltonian systems. We believe that the proof is quite similar.

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