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Nagoya Math. J.
Vol. 47 (1972), 91–99

PRIME ENTIRE FUNCTIONS WITH PRESCRIBED NEVANLINNA DEFICIENCY

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1. Introduction.

According to [4] a meromorphic function $h(z) = f(g)(z)$ is said to have $f(z)$ and $g(z)$ as left and right factors respectively, provided that $f(z)$ is non-linear and meromorphic and $g(z)$ is non-linear and entire (g may be meromorphic when $f(z)$ is rational). $h(z)$ is said to be E-prime (E-pseudo prime) if every factorization of the above form into entire factors implies that one of the functions f, or g is linear (polynomial). $h(z)$ is said to be prime (pseudo-prime) if every factorization of the above form, where the factors may be meromorphic, implies that one of f or g is linear (a polynomial or f is rational).

Recently the following result was proved by Goldstein [3].

THEOREM 1. *Let $F(z)$ be an entire function of finite order such that $\delta(a, F) = 1$ for some $a \neq \infty$, where $\delta(a, F)$ denotes the Nevanlinna deficiency. Then $F(z)$ is E-pseudo prime.*

The above theorem might suggest that for an entire function of finite order the existence of Nevanlinna deficiency and the primeness of a function are closely related to each other. The purpose of this note is to show that it is not the case in general. More precisely, we shall show the following:

THEOREM 2. *Given any integer $k > 0$, and constant c, $0 \leq c \leq 1$, one can construct a prime function f of order k with $\delta(0, f) = c$.*

Remark. By a well-known result of Nevanlinna [7] one sees immediately why the above result cannot hold for an arbitrary real positive k.

Received September 6, 1971.

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The proof of Theorem 2 also yields the following result.

THEOREM 3. *Given any* $0 \leq c \leq 1$, *there exist real constants* λ_1 *and* λ_2 *such that the function* $F = ze^{\lambda_1 e^z}(e^{\lambda_2 e^z} + 1)$ *satisfies* $\delta(0, F) = c$.

Theorem 3 gives us an example of functions of infinite order which are not pseudo-prime and which have a prescribed deficiency. The analogous problem for functions of finite order remains open.

2. Definitions and preliminary lemmas.

We shall say that a polynomial in z with complex coefficients has property R if (i) $p(z)$ is monic, (ii) $p(0) = 0$, and (iii) for some sequence of points (a_r) tending to ∞ each root of $p(z) - a_r = 0$ lies on one of a finite number of fixed rays $r_1, \cdots, r_l$ out from $z = 0$, for some positive integer l. If $z \in C, z \neq 0$, and $z = |z|e^{i\theta}$ where $-\pi \leq \theta < \pi$ we define $\arg(z)$ to be θ.

LEMMA I. (i) *The polynominal* $p(z)$ *has property* R *if and only if* $p(z) = z^{\frac{1}{2}k}(z^{\frac{1}{2}k} + b)$ *for some* $b \in C$ *and positive integer* k. (ii) *If* $b \neq 0$ *all but at most a finite number of the* a_r *lie on the ray defined by* $\arg(z) \equiv 2(\arg(b))$ modulo 2π, *while if* $b = 0$ *the* a_r's *lie on any finite collection of rays out from* $z = 0$.

Proof. We shall first show the "if" part of (i). If $b = 0$ this is trivial. If $b \neq 0$ set $b = |b|\varepsilon$. Choose the a_r to all be of the form $a_r = |a_r|\varepsilon^2$. Then we may write our equations as $(z^{\frac{1}{2}k}\varepsilon^{-1})^2 + |b|(z^{\frac{1}{2}k}\varepsilon^{-1}) = |a_r|$. Since $|b|^2 + 4|a_r| > 0$ each z which is a root must be such that $z^{\frac{1}{2}k}\varepsilon^{-1}$ is real. Thus the roots must lie on a finite number of rays out from $z = 0$.

The greater part of this proof will be spent establishing the "only if" part of (i). In doing so we shall show, also, that if $p(z)$ has property R then there exists a subsequence of the a_r consisting only of points a_r with each $\arg(a_r) = \alpha$ for some $-\pi \leq \alpha < \pi$. We shall now use this last assertion to help prove (ii) and shall then return to the proof of (i). If $b = 0$ in (ii) there is nothing to prove. If $b \neq 0$ pass to a subsequence of the (a_r) where each $\arg(a_r) \not\equiv 2(\arg(b))$ modulo 2π. (If this is not possible we are through.) We shall now obtain a contradiction. Note that as $|a_r|$ goes to ∞ the absolute values of the roots of $p(z) - a_r = 0$ go to ∞ also. Now $\arg(a_r) = \arg(p(z_{1,r}))$ where $p(z_{1,r}) = a_r$ and each $z_{1,r}$ belongs to the ray r_1, say. Thus

$$\arg(a_r) = \arg((z_{1,r}^{\frac{1}{2}k})(z_{1,r}^{\frac{1}{2}k} + b)) \tag{1}$$
$$\equiv (k(\arg(z_{1,r})) + \arg(1 + bz_{1,r}^{-\frac{1}{2}k})) \text{ modulo } 2\pi\ .$$

Since $\arg(a_r)$ and $\arg(z_{1,r})$ are constants then so is $\arg(1 + bz_{1,r}^{-\frac{1}{2}k})$. As $|a_r|$ goes to infinity $|z_{1,r}|$ goes to infinity and $\arg(1 + bz_{1,r}^{-\frac{1}{2}k})$ goes to zero. Thus each $\arg(1 + bz_{1,r}^{-\frac{1}{2}k}) = 0$ so every $bz_{1,r}^{-\frac{1}{2}k}$ is real and each $b^2 z_{1,r}^{-k}$ is positive. Also, from (1) we have now that

$$\arg(a_r) \equiv k(\arg(z_{1,r})) \text{ modulo } 2\pi$$

so, since $b^2 z_{1,r}^{-k}$ is positive,

$$\arg(a_r) \equiv 2(\arg(b)) \text{ modulo } 2\pi\ .$$

This contradiction proves (ii) subject to our (as yet) unproven assertion.

We next begin the proof of the "only if" part of (i). Let us look at the k different algebraic functions

$$z_j(a) = \rho^j a^{k^{-1}} + b_0 + b_{-1}\rho^{-j} a^{-k^{-1}} + \cdots$$

for $(1 \leq j \leq k)$ which are roots of $p(z) = a$, where $\rho = \exp(2\pi i k^{-1})$ and the expressions are valid for all sufficiently large $|a|$. Let us now pass to a subsequence of the (a_r) such that each series for $z_j(a_r)$ converges and each $\arg(z_j(a_r))$ is constant (recall that there are only a finite number of values possible). Define $-\pi \leq \varepsilon_j(\gamma) < \pi$ by

$$\arg(z_j(a_r)) \equiv (k^{-1}\arg(a_r) + jk^{-1}(2\pi) + \varepsilon_j(\gamma)) \text{ modulo } 2\pi\ , \tag{2}$$

for each $1 \leq j \leq k$. Note that for each $1 \leq j_1, j_2 \leq k$

$$\varepsilon_{j_1}(\gamma) - \varepsilon_{j_2}(\gamma) \equiv (\arg(z_{j_1}(a_r)) - \arg(z_{j_2}(a_r)) \tag{3}$$
$$- (j_1 - j_2)k^{-1}2\pi) \text{ modulo } 2\pi\ ,$$

and the right hand side above is a constant. Also each $\lim_{r\to\infty} \varepsilon_j(\gamma) = 0$ since $\rho^j(a_r)^{k^{-1}}$ is the dominant term of the expansion for $z_j(a_r)$ about infinity. Thus each $\lim_{r\to\infty}(\varepsilon_{j_1}(\gamma) - \varepsilon_{j_2}(\gamma)) = 0 - 0 = 0$, so every $\varepsilon_{j_1}(\gamma) - \varepsilon_{j_2}(\gamma) \equiv 0$ modulo 2π.

We now require that each $|a_r|$ be sufficiently large to guarantee that every $|\varepsilon_j(\gamma)| < k^{-1}\pi/2$. Then every $\varepsilon_{j_1}(\gamma) - \varepsilon_{j_2}(\gamma) = 0$. Set $\varepsilon(\gamma) = \varepsilon_1(\gamma) = \cdots = \varepsilon_k(\gamma)$. Since $\pm a_r = \prod_{j=1}^{k} z_j(a_r)$ we have

$$\arg(a_r) \equiv \arg(a_r) + (k-1)\pi + k\varepsilon(\gamma) \text{ modulo } \pi\ ,$$

so $k\varepsilon(\gamma) \equiv 0$ modulo π. Thus $\varepsilon(\gamma) = 0$ for all sufficiently large γ. Then by (2) with each $\varepsilon_j(\gamma) = 0$ we see that $\arg(a_\gamma)$ is a constant on our subsequence. (This proves the statement needed in the proof of (ii).) From now on we assume that $k > 2$, since there is nothing to prove if $k = 2$. Also we have from (2) that, for each $1 \leq j \leq k$,

$$\arg(z_j(a_\gamma)) \equiv (k^{-1}(\arg(a_\gamma)) + jk^{-1}(2\pi)) \text{ modulo } 2\pi \,. \tag{4}$$

Equation (4) says that each $z_j(a_\gamma)$ has an argument equal to the argument of the dominant term in its expansion about $a_\gamma = \infty$. We shall next show by induction that for all non-negative integers n, $b_{-n} = 0$ *unless* k divides $2(n + 1)$. Further if $b_{-n} \neq 0$, then, for sufficiently large γ, $\arg(b_{-n}(\rho^j a_\gamma^{k-1})^{-n}) \equiv \arg(\rho^j a_\gamma^{k-1})$ modulo π. (Actually, we are only interested in proving the first statement but the second statement is needed in order to make the induction go through.) Since $k > 2$ we must show that $b_0 = 0$. Suppose $b_0 \neq 0$, then for sufficiently large γ we see that $z_j(a_\gamma) - \rho^j(a_\gamma)^{k-1}$ does not vanish so

$$\begin{aligned} k^{-1}(\arg(a_\gamma) + j(2\pi)) &\equiv \lim_{\gamma\to\infty} (\arg(z_j(a_\gamma)) - \rho^j(a_\gamma)^{k-1}) \\ &\equiv \arg(b_0) \text{ modulo } \pi \,, \end{aligned}$$

for *each* $0 \leq j \leq k - 1$. Since $k > 2$ this is impossible. Thus $b_0 = 0$. Now assume the induction assumption for all $0 \leq l \leq n - 1$ and that $b_{-n} \neq 0$. If γ is sufficiently large $z_j(a_\gamma) - \sum_{l=0}^{n-1} b_{-l}(\rho^j a_j^{k-1})^{-l} \neq 0$ so that we have

$$\begin{aligned} k^{-1}(\arg(a_\gamma) + j(2\pi)) &\equiv \arg(z_j(a_\gamma)) \text{ modulo } 2\pi \\ &\equiv \arg\Big(z_j(a_\gamma) - \sum_{l=0}^{n-1} b_{-l}(\rho^j a_\gamma^{k-1})^{-l}\Big) \text{ modulo } \pi \\ &\equiv \arg(b_{-n}(\rho^j a_\gamma^{k-1})^{-n}) \,. \end{aligned} \tag{5}$$

This proves the second statement in our induction assumption. Also we see from (5) that

$$k^{-1}((\arg(a_\gamma))(n + 1) + j(2\pi)(n + 1)) \equiv \arg(b_{-n}) \text{ modulo } \pi \,.$$

Setting $j = 1, 0$ and subtracting we see that $k^{-1}2(n + 1)(\pi) \equiv 0$ modulo π. Therefore k divides $2(n + 1)$ if $b_{-n} \neq 0$. This completes the proof by induction.

We know that $p(z) - a = \prod_{j=1}^{k} (z - z_j(a))$ and that each

$$z_j(a) = \rho^j a^{k-1} + b_{-(\frac{1}{2}k-1)}(\rho^j a^{k-1})^{-(\frac{1}{2}k-1)} + b_{-(k-1)}(\rho^j a^{k-1})^{-(k-1)} \\ + O((a^{k-1})^{-(\frac{3}{2}k-1)}) \, ,$$

where the last term indicates an infinite number of terms of order $(a^{k-1})^{-(\frac{3}{2}k-1)}$ and lower. Since the coefficients of $p(z)$ are independent of a, if we put in the different series for the $z_j(a)$ in $\prod_{j=1}^{k}(z - z_j(a))$ and find the total coefficient of $a^0 z^l = z^l$, for $0 < l < k-1$, we will have the coefficient of z^l in $p(z)$. Our statement which must be demonstrated is that this coefficient vanishes if above $l \neq \frac{1}{2}k$. We shall show that it is impossible to find a term in the product above which equals a coefficient times $a^0 z^l$, if $0 < l < k-1$ and $l \neq \frac{1}{2}k$. It is clearly impossible to obtain such a term if we choose any factor from $O((a^{k-1})^{-(\frac{3}{2}k-1)})$. Also choosing a factor of $(\rho^j a^{k-1})^{-(k-1)}$, for any $1 \leq j \leq k$, forces us to choose $k-1$ factors of the form $(\rho^{j_1} a^{k-1})$ and forces l to be zero. Thus the problem reduces to showing that one cannot find two non-negative integers h_1 and h_2 such that $0 < h_1 + h_2 < k$ and $(a^{k-1})^{h_1}(a^{k-1})^{-h_2(\frac{1}{2}k-1)}) = a^0 = 1$ unless $h_1 + h_2 = \frac{1}{2}k$. Since $k > 2, h_2$ can equal only either 1 or 2. If $h_2 = 1$, then $h_1 = \frac{1}{2}k - 1$, so $h_1 + h_2 = \frac{1}{2}k$. If $h_2 = 2$ then $h_1 = k-2$ so $h_1 + h_2 = k$, contrary to our assumption. This proves Lemma I.

LEMMA II. *If α, β, γ are complex constants with $\beta\gamma \neq 0$ and n is a positive integer then $y = \gamma z(e^{\alpha z^n} + e^{\beta z^n})$ takes on all values.*

Proof. Suppose the statement is false. Then, by a result of Borel [1], one will obtain a contradiction. We leave the details to the reader.

LEMMA III. *The function $y = \gamma z(e^{\alpha z^n} + e^{\beta z^n})$ cannot be written in the form $p(g)$ where g is entire, p is any nonzero, nonlinear polynomial, n is a positive integer, $\beta\gamma \neq 0$, and $\alpha\beta^{-1}$ is real.*

Proof. We shall assume that $y = p(g)$ where y, g, and $p = p(w)$ are as above. This will lead us to the conclusion that y takes on at least one value infinitely often with *multiplicity larger than one*; however, this latter conclusion will subsequently be shown to be false. Since $p(w)$ is nonlinear, $p'(w) = 0$ has at least one solution, w_0. Thus when $g(z) = w_0$ we have that $y(z) = p(w_0)$ and has multiplicity greater than one. If $p'(w) = 0$ has two or more solutions g cannot omit *both* roots, hence y must take on the value of $p(w_0)$ infinitely often with multiplicity greater than one, for some w_0 such that $p'(w_0) = 0$. If w_0 is the only root of

$p'(w) = 0$ then $p(w) = p^{(k)}(w_0)(k!)^{-1}(w - w_0)^k + p(w_0)$ for some positive integer $k \geq 2$. Then either g takes on the value w_0 infinitely often (so that y takes on the value $p(w_0)$ infinitely often with multiplicity greater than one) or g takes on the value w_0 only finitely often (so y takes on the value $p(w_0)$ only finitely often, since $p(w) - p(w_0)$ has only one zero). By Lemma II y does not omit any values, therefore y does assume some value $a = p(w_0)$ infinitely often with multiplicity greater than one. We shall next show that this is impossible.

It is necessary first to dispose of the special cases when $\alpha = 0$ or $\alpha = \beta$. Suppose $\alpha = 0$. Then replacing z by $\sqrt[n]{\beta}\, z$ and then $p(w)$ by $(\gamma(\sqrt[n]{\beta})^{-1})^{-1}p(w)$ we may assume, without loss of generality, that $y = z(e^{z^n} + 1)$. (Similarly, if $\alpha\beta^{-1} = 1$, we may assume that $y = ze^{z^n}$.) Notice that $a \neq 0$, since if $z \neq 0$, $z(e^{z^n} + 1) = 0$, and $nz^n e^{z^n} + (e^{z^n} + 1) = 0$ we would have that $e^{z^n} = 0$. If $a \neq 0$ then, for all nonzero z, if $y(z) = a$ and $y'(z) = 0$ we have $0 = (y'(z))(y(z))^{-1} = z^{-1} + nz^{n-1}e^{z^n}(e^{z^n} + 1)^{-1} = z^{-1} + nz^n e^{z^n} a^{-1} = z^{-1} + nz^{n-1}(a - z)a^{-1} = z^{-1} + nz^{n-1} - na^{-1}z^n$. For fixed $a \neq 0$ this equation has at most $n + 1$ distinct solutions. Suppose that $ze^{z^n} = a$, $e^{z^n} + nz^n e^{z^n} = 0$, and $z \neq 0$. Then $z^{-1}a + nz^{n-1}a = 0$. Since $z \neq 0$ we see that $a \neq 0$. Thus we have $z^{-1} + nz^{n-1} = 0$ which can have at most n distinct solutions.

If $\alpha\beta\gamma \neq 0$ and $\alpha\beta^{-1} \neq 1$, then without loss of generality we may take y to be of the form $y = z(e^{\lambda z^n} + e^{z^n})$ where $\lambda < 1$ but $\lambda \neq 0$. Suppose $a = 0$. Then requiring that $z \neq 0$, the equations $z(e^{\lambda z^n} + e^{z^n}) = 0$ and $(e^{\lambda z^n} + e^{z^n}) + nz^n(\lambda e^{\lambda z^n} + e^{z^n}) = 0$ imply that $e^{\lambda z^n} = e^{z^n} = 0$. This contradiction shows that $a \neq 0$. Now assuming that $a \neq 0$ and $z \neq 0$ we have $0 = z^{-1} + nz^n(\lambda e^{\lambda z^n} + e^{z^n})(z(e^{\lambda z^n} + e^{z^n}))^{-1} = z^{-1} + nz^{n-1}a^{-1}(a + z(\lambda - 1)e^{\lambda z^n})$. Then $e^{\lambda z^n} = a(1 + nz^n)(nz^{n+1}(1 - \lambda))^{-1}$, so substituting back in $z(e^{\lambda z^n} + e^{z^n}) = a$ we have

$$z(a(1 + nz^n)(nz^{n+1}(1 - \lambda))^{-1}) + z(a(1 + nz^n)(nz^{n+1}(1 - \lambda))^{-1})^{\lambda^{-1}} = a\,,$$

for an appropriate choice of the λ-th root above. Regardless of this choice, however, we see upon taking absolute values that $\infty > |a| \geq |z|\cdot|a(1 + nz^n)(nz^{n+1}(1 - \lambda))^{-1}|^{\lambda^{-1}} - |z|\cdot|a(1 + nz^n)(nz^{n+1}(1 - \lambda))^{-1}|$. As $|z|$ goes to infinity the first term on the right hand side above goes to $+\infty$ while the second term remains bounded. This contradiction proves Lemma III.

The following lemma is essentially an observation out of Goldstein's proof of Theorem 1.

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LEMMA IV. *Let* $F(z) = ze^{z^k}(e^{az^k} + 1)$, *where* k *is a positive integer and* a *is a positive real number. Then* F *is E-pseudo prime.*

Sketch of the proof. Set

$$K(z) = (e^{az^k} + 1) .$$

Then $\delta(-1, K) = 1$, and so by virtue of a result of Edrei and Fuchs [2, pp. 281–283] the estimate [2, p. 281] holds for K along a sequence of arcs and segments. Now we note along *those arcs* and segments e^{z^k} is bounded. Hence the mentioned estimate holds not only for K but also for $F(z)$. Then following Goldstein's argument we will arrive at the conclusion.

3. Proof of Theorem2.

First of all, it is easy to verify that for any non-zero constants λ_1 and λ_2 and any positive integer $k, F(z) = ze^{\lambda_1 z^k}(e^{\lambda_2 z^k} + 1)$ cannot be periodic. Thus by virtue of a result of the first author [5], we need only to show that F is E-prime.

When $c = 0$ or $c = 1$ we choose $F = z(e^{z^k} + 1)$ or $F = ze^{z^k}$, respectively, and it is easy to verify that they are all prime functions of order k. Therefore, we restrict ourselves to the case $0 < c < 1$.

Let us choose

$$F(z) = ze^{\lambda_1 z^k}(e^{\lambda_2 z^k} + 1) , \tag{6}$$

where $\lambda_1 > 0$ and $\lambda_2 > 0$ are chosen such that $\dfrac{\lambda_1}{\lambda_1 + \lambda_2} = c$. We claim that $f(z)$ is E-prime with $\delta(0, F) = c$. We first show that F is E-prime. F is E-pseudo prime by virtue of Lemma IV. By Lemma III, F also cannot assume the form $F = p(g)$ with p a polynomial and g transcendental entire. Thus we only need to consider the possibility that F can be factorized as

$$F(z) = g(p(z)) , \tag{7}$$

where g is transcendental, and p is a nonlinear polynomial. We may assume without loss of generality that $p(0) = 0$ and that the leading coefficient of p is one.

Now, according to Lemma 1,

$$p(z) = z^{n/2}(z^{n/2} + b) . \tag{8}$$

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where n is an integer and b a constant. We claim that $n = 1$. Suppose that $n \geq 2$. Then from (7) and (8) we have

$$F(z) = ze^{\lambda_1 z^k}(e^{\lambda_2 z^k} + 1) \equiv g(z^{n/2}(z^{n/2} + b)) . \tag{9}$$

Now if $b \neq 0$, then n has to be even. Let us substitute z by ζz into identity (9) where ζ is a $(n/2)$-th root of unity other than one when $n > 2$, and substitute z by $-z - 6$ when $n = 2$. Then by Borel's result mentioned earlier one will obtain a contradiction. If $b = 0$, then n can be even or odd. We again substitute z by ζz into identity (9) and obtain a contradiction unless $n = 1$ which means $p(z)$ is linear. Thus we have also excluded the possibility (7). Hence F is E-prime, therefore is also prime.

Now we proceed to show that $\delta(0, F) = c$. Let us choose a non-negative number λ such that $\lambda + \lambda_1 = n\lambda_2$, n a positive integer.

Multiplying F by $e^{\lambda z^k}$ we have

$$H(z) = e^{\lambda z^k}F = ze^{n\lambda_2 z^k}(e^{\lambda_2 z^k} + 1) , \tag{10}$$

or

$$H(z) = zf^n(z)(f(z) + 1) , \tag{11}$$

where $f(z) = e^{\lambda_2 z^k}$.

According to a result of Hayman [6, p. 7]

$$\begin{aligned} T(r, H) &= T(r, zf^n(z)(f(z) + 1)) \sim T\{r, f^n(z)(f(z) + 1)\} \\ &\sim (n + 1)T(r, f) \sim \frac{(n + 1)\lambda_2}{\pi} r^k , \qquad \text{as } r \to \infty . \end{aligned} \tag{12}$$

Now we have by Nevanlinna's first fundamental theorem and equation (10) that

$$\begin{aligned} T(r, F) &= F(r, He^{-\lambda z^k}) \\ &\geq T(r, H) - T(r, e^{-\lambda z^k}) + O(1) \\ &\geq \frac{(n + 1)\lambda_2}{\pi} r^k - \frac{\lambda}{\pi} r^k + O(1) \\ &= \frac{(n\lambda_2 + \lambda_2 - \lambda)}{\pi} r^k + O(1) \\ &= \frac{\lambda_1 + \lambda_2}{\pi} r^k + O(1) . \end{aligned} \tag{14}$$

On the other hand

$$
\begin{aligned}
T(r, F) &= T(r, ze^{\lambda_1 z^k}(e^{\lambda_2 z^k} + 1)) \\
&\le T(r, e^{\lambda_1 z^k}) + T(r, e^{\lambda_2 z^k}) + O(\log r) \\
&\sim \frac{\lambda_1}{\pi} r^k + \frac{\lambda_2}{\pi} r^k + O(\log r) \qquad (14) \\
&= \frac{\lambda_1 + \lambda_2}{\pi} r^k + O(\log r) \, .
\end{aligned}
$$

Thus from (13), (14), and noticing the fact that F is transcendental, we conclude

$$
T(r, F) \sim (1 + o(1)) \frac{(\lambda_1 + \lambda_2)}{\pi} r^k \qquad \text{as } r \to \infty \, . \tag{15}
$$

Now the counting function $N\left(r, \frac{1}{F}\right)$ is equal to $N\left(r, \frac{1}{e^{\lambda_2 z^k} + 1}\right)$ which is asymptotic to $T(r, e^{\lambda_2 z^k})$ by Nevanlinna's second fundamental theorem.

Thus from this and (15) we have

$$
\begin{aligned}
\delta(0, F) &= 1 - \varlimsup_{r \to \infty} \frac{N(r, 1/F)}{T(r, F)} \qquad (16) \\
&= 1 - \varlimsup_{r \to \infty} \frac{(\lambda_2/\pi) r^k}{((\lambda_1 + \lambda_2)/\pi) r^k} = \frac{\lambda_1}{\lambda_1 + \lambda_2} = c \, .
\end{aligned}
$$

The theorem is thus proved.

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