A TENSOR BOUNDARY VALUE PROBLEM OF MIXED TYPE

G. F. D. DUFF

The boundary value problems of generalized potential theory on finite Riemannian manifolds may be regarded as extensions of the Dirichlet and Neumann problems for harmonic functions. In the tensor theory there is, in fact, a greater variety of such problems; that is to say, these generalizations from classical potential theory can be made in various ways. We here introduce yet another pair of boundary value problems for the tensor equation of Laplace.

Two boundary value problems for harmonic p-tensors, which, for $p = 0$ or $p = N$, reduce to the classical Dirichlet and Neumann problems, were discussed in **(1.b)** by means of the Poincaré-Fredholm integral equation technique. In one, values of components are assigned on the boundary, while in the other, values of components of the derivative and co-derivative are specified. As in the scalar problems, these are related, inasmuch as the system of integral equations appropriate to the one, when transposed, leads to the other. The whole formulation is invariant: that is, only tensor quantities and operators defined invariantly on the boundary surface appear in the statements and proofs.

In this paper we shall discuss a second invariant generalization of the Dirichlet and Neumann problems. This type of problem is mixed, in the sense that values both of components and of their first derivatives are assigned at each boundary point. Although there are at first sight two problems of this kind, again related by transposition, the second mixed problem for p-tensors is equivalent to the first mixed problem for $(N-p)$-tensors. The eigentensors of these problems are harmonic fields whose tangential or normal components vanish on the boundary, while the dimension of the eigenspace is a relative or absolute Betti number of the manifold. By specializing the boundary values, we obtain theorems for closed or co-closed harmonic forms, and also for harmonic fields, thus bringing these hitherto separate theories together with that of the tensor Laplace equation.

1. Formulation of the problem. We consider orientable Riemannian manifolds of dimension N and differentiability class C^∞. M will denote a compact manifold with boundary B of dimension $N - 1$ and class C^∞, while F will denote a closed and compact manifold which is the double of M. A positive definite metric tensor g_{ik} of class C^∞ is supposed given on M, and can be extended to F.

Received August 18, 1953.

On F we consider skew-symmetric covariant tensors

$$\phi_{i_1 \dots i_p}$$

with which are associated exterior differential forms ϕ of degree p. We have the differential operator d, the dual operator $*$, the co-differential

$$\delta = (-1)^{Np+N+1} * d *,$$

and the Laplacian $\Delta = \delta d + d\delta$. Precise definitions of these are given in **(1.b)** or **(5)** to either of which we refer for brevity. The scalar product $(\phi, \psi)_F = \int_F \phi \wedge *\psi$ defines a Hilbert space of p-forms on F (or on M, if the integration is extended over M) since $N(\phi) = (\phi, \phi)$ is positive unless $\phi \equiv 0$.

As in **(1.c)** we shall make use of double skew-symmetric p-tensor fields $A_{i_1 \dots i_p, j_1 \dots j_p}$ which are symmetric in the two groups of indices $i_1 \dots i_p$ and $j_1 \dots j_p$.

We shall approach the study of the Laplace equation for harmonic forms

$$\Delta\phi = 0 \tag{1.1}$$

via an equation of type

$$L\phi = \Delta\phi + A\phi = 0, \tag{1.2}$$

where $A\phi$ is the differential form corresponding to the tensor

$$A_{i_1 \dots i_p, (j_1 \dots j_p)} \ \phi^{(j_1 \dots j_p)}$$

and where the matrix of independent components of A will be taken as positive definite. Green's formulae for (1.2) are

$$(d\phi, d\psi) + (\delta\phi, \delta\psi) + (\phi, A\psi) - (\phi, L\psi) = \int_B (\phi \wedge *d\psi - \delta\psi \wedge *\phi), \tag{1.3}$$

and

$$(\psi, L\phi) - (\phi, L\psi) = \int_B (\phi \wedge *d\psi - \delta\psi \wedge *\phi - \psi \wedge *d\phi + \delta\phi \wedge *\psi), \tag{1.4}$$

where the integrals on the right are taken over the boundary surface of the domain of integration indicated by the round brackets. From (1.3) we see that the Dirichlet integral for (1.2) is

$$E(\phi, \psi) = (d\phi, d\psi) + (\delta\phi, \delta\psi) + (\phi, A\psi) = E(\psi, \phi). \tag{1.5}$$

When A is positive definite, $E(\phi, \phi) > 0$ unless $\phi \equiv 0$. The formulae for Laplace's equation are found by setting $A \equiv 0$ in the above. Then $E(\phi, \phi) = 0$ implies only that $d\phi = 0$ and $\delta\phi = 0$.

In **(1.c)** it was shown that, for A positive definite, the equation (1.2) has a fundamental singularity in the large $g_A(x, y)$ in any compact closed space F satisfying our conditions. As in the paper referred to, we use this singularity for the construction of single and double layer potentials.

The boundary operators t and n, satisfying $*t = n*$, $*n = t*$, are defined as in **(1.b)**. Thus $t\phi$ is the induced p-form on the boundary B.

From **(1.b)** we recall that in the first (or Dirichlet) boundary value problem for harmonic forms, the quantities $t\phi$ and $n\phi$ are assigned on B, and that the eigenspace O for this problem is finite-dimensional, depending on the topological structure of the manifold, or possibly zero. In the second or Neumann problem we assign $nd\phi$ and $t\delta\phi$, subject to a certain orthogonality condition. The associated eigenspace F is the space of harmonic fields ($d\phi = 0, \delta\phi = 0$) on M, and is known to be infinite-dimensional if $1 \leqslant p \leqslant N - 1$. The corresponding problems for (1.2) with A positive definite have unique solutions.

The two boundary value problems which we now introduce shall be known as the K and M problems. In the K problem we assign $t\phi$ and $t\delta\phi$; in the M problem, $n\phi$ and $nd\phi$. It is easily verified that the number of conditions so prescribed is, in each case, equal to the number $\binom{N}{p}$ of independent components of ϕ. These boundary conditions are also self-adjoint, in the sense that if ϕ and ψ both satisfy one of these homogeneous conditions, the right hand side of (1.4) vanishes.

The mixed boundary condition of Robin's type which was discussed in **(1.b**, §6**)** can be made to yield the Dirichlet (D), Neumann (N) or the K and M problems as limiting cases. For example, to obtain formally the K boundary condition, from (6.4) of **(1.b)**, let $A_p \to \infty$ and $A_{N-p} \to 0$. It is clear that these limiting cases must all be treated separately since the hypotheses of **(1.b**, §6**)** are not then satisfied.

The eigensolutions of these problems may be characterized with the help of (1.2). We see that $L\phi = 0$, $t\phi = 0$, $t\delta\phi = 0$ imply $E(\phi, \phi) = 0$, the integral being taken over M. Thus $\phi \equiv 0$ in M, whenever A is positive definite, and if A is zero identically, we still have $d\phi = 0$, $\delta\phi = 0$ but not necessarily $\phi = 0$. Similarly, if $L\phi = 0$, $n\phi = 0$, $nd\phi = 0$, we have $E(\phi, \phi) = 0$ with $\phi = 0$ if A is positive definite, and $d\phi = 0$, $\delta\phi = 0$ if $A \equiv 0$.

Thus the independent conditions satisfied by a solution of the homogeneous K-problem are $d\phi = 0$, $\delta\phi = 0$ and $t\phi = 0$. In **(2)** it was shown that if the relative periods of ϕ on $R_p(M, B)$ independent relative p-cycles are assigned, ϕ is uniquely determined. Indeed, if these relative periods are given to be zero, then, according to **(1.a)**, ϕ is the derivative $d\chi$ of a $(p - 1)$-form χ whose tangential part vanishes on B. Thus, by the shorter form of Green's formula,

$$N(\phi) = (\phi, d\chi) = (\delta\phi, \chi) + \int_B \chi \wedge *\phi = 0,$$

so that ϕ vanishes identically. Therefore the dimension of the eigenspace K of the K problem is $R_p(M, B) = R_{N-p}(M)$, by the Lefschetz duality theorem.

Reasoning exactly dual to this shows that the eigenspace M of the M problem has dimension $R_p(M) = R_{N-p}(M, B)$. Indeed, if ϕ satisfies a boundary condition of the K type, its dual $*\phi$ is an $N - p$ form satisfying a corresponding boundary condition of the M type.

We remark that the intersection of the eigenspace K and the eigenspace M is the eigenspace O of the Dirichlet problem in which both $t\phi$ and $n\phi$ vanish.

The letters K and M will also be used to denote projection operators (in the L^2 norm) on the K and M eigenspaces. In this connection we have the operator relations $dK=0$, $\delta K=0$, $dM=0$, $\delta M=0$. If $\rho\in K$, and ψ is arbitrary, then, since

$$(\rho, \delta\psi) = (d\rho, \psi) - \int_B \rho\wedge *\psi = 0$$

we have $K\delta = 0$. Similarly $Md = 0$. In fact, this last follows also from the evident formulae $*M = K*$, $*K = M*$.

2. Potentials. Let $g = g_A(x, y)$ denote the fundamental solution of $\Delta\phi + A\phi = 0$ (A positive definite) in the double F. Based on this double form we have the potentials

$$\mu = \int_B (\rho\wedge *dg - \delta\rho\wedge *g) = \int_B (\rho\wedge *dg + *g\wedge *d*\rho) \tag{2.1}$$

and

$$\nu = \int_B (g\wedge *d\sigma - \delta g\wedge *\sigma) = \int_B (g\wedge *d\sigma + *\sigma\wedge *d*g), \tag{2.2}$$

each of which contains both single and double layer terms. The layer densities $t\rho$, $t\delta\rho$, $nd\sigma$ and $n\sigma$ are assumed here to be Hölder continuous on B. We shall calculate the discontinuities of these quantities as the argument point crosses B from M to the complementary part CM in F, and for this purpose we use the formulae of §3 of **(1.b)**, noting that the singularity of the de Rham kernel $g(x, y)$ is asymptotically equal to that of $g_A(x, y)$.

From (3.5) of **(1.b)** we see that the first term of $t\mu$ increases by $t\rho$: the second term of $t\mu$ is clearly continuous. From the same formulae we see that the first term of $t*\mu$ is continuous, and so also is the second. To calculate the discontinuity of $t*d\mu$, we have

$$t*d\mu = t*d \int_B (\rho \wedge *dg - \delta\rho \wedge *g)$$

and from (3.6) of **(1.b)** we see that the second term is continuous across B. If we take the dual of (4.12) of **(1.b)** we see that the first term is also continuous. Lastly, we must examine $t*\, d*\mu$. According to (3.5) of **(1.b)** the second term of $t*\, d*\mu$ decreases by $t*\, d*\rho$, while from (3.6) of the same paper, the first term is continuous. Analogous results for the dual potential ν may be found by taking the duals of the above results with p replaced by $N - p$; or directly. Collecting together the results so found, we see that $t\mu$, $t*\, d*\mu$, $t*\nu$, and $t*\, d\nu$ have the respective discontinuities $t\rho$, $-t*\, d*\rho$, $-t*\sigma$, and $t*\, d\sigma$; while $t*\mu$, $t*\, d\mu$, $t\nu$, and $t*\, d*\nu$ are continuous across B.

We conclude that on B, we have as limits from M,

$$\begin{aligned} t_-\mu &= \tfrac{1}{2}t\rho + t\int_B (\rho \wedge *dg + *g \wedge *d*\rho), \\ t_-*d*\mu &= -\tfrac{1}{2}t*d*\rho + t*d*\int_B (\rho \wedge *dg + *g \wedge *d*\rho), \\ t_-*d\nu &= -\tfrac{1}{2}t*d\sigma + t*d\int_B (g \wedge *d\sigma + *\sigma \wedge *d*g), \\ t_-*\nu &= \tfrac{1}{2}t*\sigma + t*\int_B (g \wedge *d\sigma + *\sigma \wedge *d*g). \end{aligned} \tag{2.3}$$

https://doi.org/10.4153/CJM-1954-043-x Published online by Cambridge University Press

Here the integrals on the right are understood to be evaluated on the boundary. The singularity of $g_A(x, y)$ for $x = y$ is such that principal values of these integrals must be taken. The $-$ sign appended to the operator t on the left of each equation indicates a limiting value from M; a $+$ sign will be used to indicate limits from CM.

The reasoning of the Poincaré-Fredholm method now shows that the solution of the K problem with assigned data $t\phi$, $t\delta\phi$ is to be sought by solving the system of singular integral equations

$$t_{-}\mu = t\phi, \quad t_{-}*d*\mu = t*d*\phi. \tag{2.4}$$

Similarly, the solution of the M problem will be found by solving the system

$$t_{-}*d\nu = t*d\phi, \quad t_{-}*\nu = t*\phi \tag{2.5}$$

with given data $t*\phi$, $t*d\phi$.

The kernel of the system (2.4) is

$$\begin{pmatrix} t_x\, t_y *_y d_y\, g_A(x, y), & t_x\, t_y *_y g_A(x, y) \\ t_x\, t_y *_x d_x *_x *_y d_y\, g_A(x, y), & t_x\, t_y *_x d_x *_y g_A(x, y) \end{pmatrix},$$

while the transpose of this kernel, namely

$$\begin{pmatrix} t_x\, t_y *_x d_x\, g_A(x, y), & t_x\, t_y *_x d_x *_y d_y *_y g_A(x, y) \\ t_x\, t_y *_x g_A(x, y), & t_x\, t_y *_x *_y d_y *_y g_A(x, y) \end{pmatrix},$$

is the kernel of the system (2.5). Thus the analogy with the case considered in **(1.b)** is complete.

3. Solution of the integral equations. The condition for the compatibility of (2.3) or (2.4) is, that the non-homogeneous terms should be orthogonal, over the domain of integration B, to every solution of the homogeneous transposed equations **(3)**. In each case the homogeneous transposed equation arises when we try to solve the boundary value problem of the dual type for the domain CM.

For the K problem we will show that any H-continuous solution of the homogeneous transposed equation

$$\begin{aligned} 0 &= \tfrac{1}{2}t*d\sigma + t*d \int_B (g \wedge *d\sigma + *\sigma \wedge *d*g), \\ 0 &= -\tfrac{1}{2}t*\sigma + t* \int_B (g \wedge *d\sigma + *\sigma \wedge *d*g), \end{aligned} \tag{3.1}$$

is identically zero. The potential ν of (2.2) corresponding to any such solution σ satisfies on B,

$$t_{+} *d\nu = 0, \quad t_{+} *\nu = 0, \tag{3.2}$$

and also is a solution of $\Delta\nu + A\nu = 0$ in CM. Thus the Dirichlet integral over CM of ν is zero, and hence ν vanishes identically in CM. Passing through the boundary B to M, we see that

https://doi.org/10.4153/CJM-1954-043-x Published online by Cambridge University Press

$$(3.3) \qquad t\nu = 0, \quad t\_*\nu = -t\_*\sigma, \quad t\_*d\nu = t*d\sigma, \quad t\_*d*\nu = 0.$$

From the first and last of these relations, it follows that ν has a zero Dirichlet integral in M, and so vanishes identically. Thus finally $t*\sigma = 0$ and $t*d\sigma = 0$.

That is, according to (3), the equations (2.4) possess a solution for arbitrary continuous data. Reasoning along the lines of this proof, we could easily show that this solution of the integral equations is also unique. The proof of existence of a solution of (2.5) parallels that just given and so will be omitted.

THEOREM I. *If A is positive definite in M, the differential equation $\Delta\phi + A\phi = 0$ has unique solutions with either $t\phi$, $t\delta\phi$ or $n\phi$, $nd\phi$ having assigned H-continuous boundary values.*

In the usual way we can now assert the existence of Green's forms corresponding to the K and M problems on M. These are obtained by subtracting from the fundamental singularity $g_A(x, y)$ solutions of the differential equation, the appropriate boundary values of which agree with those of $g_A(x, y)$. For the K problem we find a domain functional $K_p(x, y)$ which satisfies

$$(3.4) \quad \Delta_x K + AK = 0 \ (x \neq y); \quad t_x K = 0, \ t_x\delta_x K = 0; \qquad K(x, y) \sim g(x, y), \ (x \sim y).$$

The symmetry property $K(x, y) = K(y, x)$ is easily established. For the M problem we construct $M_p(x, y)$ satisfying

$$(3.5) \quad \Delta_x M + AM = 0 \ (x \neq y); \quad n_x M = 0, \ n_x d_x M = 0; \qquad M(x, y) \sim g(x, y), \ (x \sim y).$$

Finally, we see that $*_x *_y K_p(x, y) = M_{N-p}(x, y)$.

4. Laplace's Equation. The above method requires modification when applied to Laplace's equation, for which the matrix A is zero. We shall construct a modified Green's function in F, which will appear in the kernel of the integral equations corresponding to (2.4). For this purpose let A_0 be a sufficiently differentiable matrix tensor which is positive definite in $\tilde{M}$ and vanishes in M **(1, c)**. Then to the differential equation

$$(4.1) \qquad \Delta\phi + A_0\phi = 0$$

there corresponds the Dirichlet integral

$$(4.2) \qquad D_0(\phi, \phi) = N(d\phi) + N(\delta\phi) + (\phi, A_0\phi)\tilde{M}.$$

Any form which is harmonic in M and vanishes in $\tilde{M}$ annuls this integral and is an eigenform of (4.1) in F. That is, the eigenspace of (4.1) is $O = M \cap K$ as defined in § 1.

The method of **(5)** for constructing a Green's form applies to (4.1) as in **(1, c)**, and it is a straightforward matter to verify in the present case the existence of a Green's form $g_0(x, y)$ for (4.1) which satisfies the following equation:

https://doi.org/10.4153/CJM-1954-043-x Published online by Cambridge University Press

(4.3) $$\Delta_x g_0(x, y) = -\alpha_0(x, y), \qquad x \neq y,$$

where

$$\alpha_0 = \sum \omega_i(x)\, \omega_i(y)$$

is the reproducing kernel in M of the orthonormalized forms $\omega_i(x)$ of the eigenspace O. Thus $\alpha_0(x, y)$ is a harmonic field with vanishing boundary values on B. Hence also

$$d\delta\, dg_0 = 0, \quad \delta d\, \delta g_0 = 0.$$

This kernel is symmetric as usual. With $g_0(x, y)$ we may construct surface layers of the types (2.1) and (2.4), and for the remainder of this section μ_0 and ν_0 will denote these modified potentials.

If ρ is an eigenform of the K problem, that is, if $d\rho = 0$, $\delta\rho = 0$, $t\rho = 0$, and if $\Delta\phi = 0$ in M, then from Green's formula we have

$$(\rho, \Delta\phi) + D(\rho, \phi) = \int_B (\rho \wedge *d\phi - \delta\phi \wedge *\rho).$$

The left-hand side vanishes, as does the first term on the right. We then find

(4.4) $$\int_B \delta\phi \wedge *\rho = 0$$

as a necessary condition for the solution of the K problem by harmonic forms.

Writing

(4.5) $$\mu_0 = \int_B (\rho \wedge *dg_0 - \delta\rho \wedge *g_0)$$

we see that

$$\Delta\mu_0 = \int_B \delta\rho \wedge *\alpha_0 = 0$$

since $t\alpha_0 = 0$. That is, a surface layer (4.5) is a harmonic form. The solution of the K problem will then be attained if there exists a surface layer (4.5) satisfying the boundary conditions.

The integral equations of the problem again take the form

(4.6) $$t_-\mu_0 = t\phi, \quad t_-\delta\mu_0 = t\delta\phi.$$

A solution $(t\rho, t\delta\rho)$ exists if and only if the nonhomogeneous terms $(t\phi, t\delta\phi)$ are orthogonal to every solution of the homogeneous transposed equations. As in §3, to such an eigensolution $(t{*}d\sigma, t{*}\sigma)$ corresponds a potential ν_0, satisfying (4.1), with

(4.7) $$t_+{*}d\nu_0 = 0, \quad t_+{*}\nu_0 = 0 \qquad \text{on } B.$$

Since A_0 is positive definite in $\tilde{M}$, we find

(4.8) $$\nu_0 \equiv 0 \qquad \text{in } \tilde{M}.$$

The discontinuity conditions then yield

(4.9) $$t_-\nu_0 = 0, \quad t_-{*}\nu_0 = -t{*}\sigma, \quad t_-{*}d\nu_0 = t{*}d\sigma, \; t_-{*}d{*}\nu_0 = 0.$$

https://doi.org/10.4153/CJM-1954-043-x Published online by Cambridge University Press

Since $g_0(x, y)$ satisfies (4.3), we find

$$\Delta\nu_0 = \sum c r\, \zeta_r \tag{4.10}$$

where the ζ_r form a basis for the eigenspace O. Then in M, we see from (4.9),

$$\nu_0 = \int_B (g_0 \wedge *d\sigma - \delta g_0 \wedge *\sigma) = \int_B (g_0 \wedge *d\nu_0 + \delta g_0 \wedge *\nu_0).$$

Therefore

$$\Delta\nu_0 = - \int_B (\alpha_0 \wedge *d\nu_0 + \delta\alpha_0 \wedge *\nu_0) = 0, \tag{4.11}$$

since $\delta\alpha = 0$ and $t\alpha_0 = 0$. Thus ν_0 is a harmonic form in M. Next we calculate

$$\begin{aligned} D_M(\nu_0) &= (d\nu_0, d\nu_0)_M + (\delta\nu_0, \delta\nu_0)_M \\ &= (\nu_0, \Delta\nu_0) + \int_B (\nu_0 \wedge *d\nu_0 - \delta\nu_0 \wedge *\nu_0) = 0 \end{aligned}$$

since $\Delta\nu_0 = 0$ and $t\_\nu_0 = 0$, $t\_*d*\nu_0 = 0$. Therefore ν_0 is in fact a harmonic field in M, and since $t\_\nu_0 = 0$, ν_0 is a member of the eigenspace K.

The orthogonality condition sufficient for the existence of a solution of the integral equations (4.6) now becomes

$$\begin{aligned} 0 &= \int_B (\phi \wedge *d\sigma - \delta\phi \wedge *\sigma) = - \int_B \delta\phi \wedge *\sigma \\ &= \int_B \delta\phi \wedge *\nu_0, \end{aligned}$$

in view of the second of (4.9). That is, the necessary condition

$$\int_B \delta\phi \wedge *\nu_0 = 0, \qquad \nu_0 \in K,$$

is sufficient for the existence of a solution of the integral equations. The final result then follows from our remark that μ_0 is a harmonic form.

THEOREM II. *There exists a solution ϕ_p of $\Delta\phi = 0$, $t\phi = t\xi$, $t\delta\phi = t\delta\xi$, if and only if*

$$\int_B \delta\xi \wedge *\rho = 0 \tag{4.12}$$

for every harmonic field ρ for which $t\rho = 0$ on B.

In particular, if $R_p(M, B) = 0$, the condition (4.12) is satisfied.

5. Co-closed harmonic forms. The K problem is solvable if the given values for $t\delta\phi$ are all zero. We will show that in this case the solution ϕ is co-closed throughout M. Since $\delta\Delta\phi = \delta d\delta\phi = 0$, we have

$$N(d\delta\phi) = (d\delta\phi, d\delta\phi) = (\delta\phi, \delta d\delta\phi) + \int_B \delta\phi \wedge *d\delta\phi = 0,$$

by Green's formula, and since $t\delta\phi$ vanishes. Thus $d\delta\phi = 0$ identically, and so also $\delta d\phi = 0$. Now

$$N(\delta\phi) = (\delta\phi, \delta\phi) = (\phi, d\delta\phi) - \int_B \delta\phi \wedge *\phi = 0,$$

since $d\delta\phi = 0$ and $t\delta\phi = 0$. Thus $\delta\phi = 0$ as stated.

THEOREM III. *There exists a co-closed harmonic form having a given tangential boundary value.*

This result is similar to but not identical with Theorem 2 of **(2)** which states the existence of a solution to the problem $\delta d\phi = 0$, $t\phi = t\xi$ given on B, and that if ξ is of the form $\delta\chi$, then there exists a unique solution ϕ of the form $\delta\psi$. The theorem just established is stronger than the first of these two statements, and more general than the second since the values of $t\phi$ are restricted only by continuity.

If we further restrict $t\phi$ to be equal to the values of a derived form $d\alpha$ defined on B, we can show that $d\phi$ is zero in M. The restriction on $t\phi$ will be satisfied if $d_B t\phi = 0$, and if $t\phi$ has zero periods on all p-cycles of B. **(1.a)** We see, in fact, that

$$N(d\phi) = (d\phi, d\phi) = (\phi, \delta d\phi) + \int_B \phi \wedge *d\phi.$$

The volume term disappears since $\delta d\phi = 0$. Since $t\phi = td\alpha$, we have

$$\begin{aligned} d_B t(\alpha \wedge *d\phi) &= td(\alpha \wedge *d\phi) \\ &= t(d\alpha \wedge *d\phi) + (-1)^p t(\alpha \wedge d*d\phi) \\ &= t(\phi \wedge *d\phi), \end{aligned}$$

again since $d*d\phi = \pm*\delta\, d\phi = 0$. Thus, by Stokes' theorem,

$$\int_B \phi \wedge *d\phi = \int_{bB} \alpha \wedge *d\phi = 0$$

since the boundary bB of B is zero. Finally, $d\phi = 0$ in M. This result is a weaker form of the Dirichlet theorem for harmonic fields **(2**, Theorem **3)**.

Another type of condition which ensures that the orthogonality condition (4.12) be satisfied is that the assigned values of $t\delta\phi$ on B should be equal to a derived form $d_B\chi_{p-2}$ defined on B. If ρ is an eigenform, we have $d*\rho = 0$, and

$$\begin{aligned} d_B t(\chi \wedge *\rho) &= d_B(\chi \wedge t*\rho) \\ &= d_B\chi \wedge t*\rho \pm \chi \wedge d_B t*\rho \\ &= t\delta\phi \wedge t*\rho \pm \chi \wedge td*\rho \\ &= t(\delta\phi \wedge *\rho). \end{aligned}$$

Hence, by Stokes' theorem,

$$\int_B \delta\phi \wedge *\rho = \int_{bB} \chi \wedge *\rho = 0,$$

since B is closed. That is, (4.12) is satisfied as stated.

When the assigned boundary values of $t\delta\phi$ are of this type, and when also the values of $t\phi$ vanish, the corresponding solution of the K problem is closed. In fact, we have

THEOREM IV. *There exists a closed harmonic form $\phi = \phi_p$ with $t\phi = 0$, $t\delta\phi = d_B\chi_{p-2}$, where χ_{p-2} is a $p - 2$ form of class HC^1 defined on B.*

The orthogonality condition being satisfied, there exists a harmonic form ϕ with $t\phi = 0$, $t\delta\phi = d_B\chi$. To show that ϕ is closed, we first calculate

$$N(d\delta\phi) = (\delta\phi, \delta d\delta\phi) - \int_B \delta\phi \wedge *d\delta\phi.$$

The volume term vanishes since $\delta d\delta\phi = \delta\Delta\phi = 0$. For the surface integral, we have

$$\begin{aligned} d_B t(\chi \wedge *d\delta\phi) &= d_B(\chi \wedge t*d\delta\phi) \\ &= d_B\chi \wedge t*d\delta\phi \pm \chi \wedge td*d\delta\phi \\ &= d_B\chi \wedge t*d\delta\phi \pm \chi \wedge t*\delta d\delta\phi \\ &= t\delta\phi \wedge t*d\delta\phi + 0 \\ &= t(\delta\phi \wedge *d\delta\phi). \end{aligned}$$

The integrand being a derived form on B, the surface integral over the closed boundary manifold B vanishes, by Stokes' theorem. That is, $N(d\delta\phi) = 0$, so $d\delta\phi \equiv 0$. Again, since $\Delta\phi = 0$ we have also $\delta d\phi = 0$. Thus

$$N(d\phi) = (\phi, \delta d\phi) + \int_B \phi \wedge *d\phi = 0,$$

since $\delta d\phi = 0$ and $t\phi = 0$. Therefore, finally, $d\phi \equiv 0$ and Theorem IV is proved.

This proof demonstrates that a harmonic form with $t\phi = 0$, $t\delta\phi = d_B\chi$ is necessarily closed. However, the sufficient condition $t\phi = 0$ may be replaced by the condition $nd\phi = 0$ which is clearly necessary, and the remark $d\phi = 0$ still holds. To show this we refer to the Neumann boundary value theorem (**1.b**, Theorem II). It is a consequence of this result that there exists a harmonic form ϕ with $nd\phi = 0$ and $t\delta\phi = d_B\chi$ if and only if

$$\int_B d_B\chi \wedge *\tau = 0$$

for every harmonic field τ defined throughout M. But

$$\begin{aligned} d_B t(\chi \wedge *\tau) &= d_B\chi \wedge t*\tau \pm \chi \wedge td*\tau \\ &= t(d_B\chi \wedge *\tau), \end{aligned}$$

since $d*\tau$ is zero, and so this condition of orthogonality is satisfied. Therefore a harmonic form ϕ satisfying the indicated boundary conditions exists. Now we have $\Delta\phi = 0$, and therefore

$$t*d\delta\phi = -t*\delta d\phi = \pm td*d\phi = \pm d_B t*d\phi = 0,$$

since $t*d\phi = *nd\phi$ is zero. Thus the surface integral in (5.2) vanishes; and, since the volume integral is again zero, we conclude that $d\delta\phi$ vanishes. Then, since also $\delta d\phi$ must be zero, we find that

$$N(d\phi) = (\phi, \delta d\phi) + \int_B \phi \wedge *d\phi = 0,$$

since $t{*}d\phi = *nd\phi$ is zero. That is, $d\phi = 0$ holds in this case as well.

6. Green's operators for Laplace's equation. From (4.10) we observe that the equation $\Delta\phi = \beta$ is solvable in M with the homogeneous boundary conditions $t\phi = 0$, $t\delta\phi = 0$, if and only if $(\rho, \beta) = 0$ for all $\rho \in K$. Thus the modified equation $\Delta\phi = \beta - K\beta$ is solvable for arbitrary β. Since the solution is undetermined to the extent of an additive eigenform $\rho \in K$, uniqueness may be secured by the additional requirement $K\phi = 0$. Writing $\phi = G_K\beta$, we define the Green's operator G_K for the K problem on the given domain. As in **(1.b)**, **(5)**, we see that the kernel $g_K(x, y)$ of this operator is a double p-form with a singularity asymptotic to the local fundamental singularity of $\Delta\phi = 0$, as $x \sim y$. Also $g_K(x, y)$ satisfies the differential equation

$$\Delta_y\, g_K(x, y) = -k(x, y), \qquad x \neq y, \tag{6.1}$$

where

$$k(x, y) = \sum_{\nu=1}^{R_p(M,B)} \rho_\nu(x)\, \rho_\nu(y) \tag{6.2}$$

is the reproducing kernel of the eigenspace K.

Since $tG_K\phi = 0$ and $t\delta\, G_K\phi = 0$, we find that $t_y\, g_K(x, y) = 0$ and $t_y\, \delta_y g_K(x, y) = 0$. Also, to the orthogonality relation $KG = 0$ corresponds the formula $(\rho(x), g_K(x, y)) = 0$ for each $\rho \in K$. The symmetry property $g_K(x, y) = g_K(y, x)$ now follows in the usual way from Green's formula. Thus G_K is self-adjoint.

Let ϕ be any p-form; we calculate

$$\begin{aligned} G_K\Delta\phi &= (g_K, \Delta\phi) \\ &= \phi + (\Delta g_K, \phi) \\ &\quad + \int_B (\phi \wedge *dg_K - \delta g_K \wedge *\phi - g_K \wedge *d\phi + \delta\phi \wedge *g_K) \\ &= \phi - K\phi + \int_B (\phi \wedge *dg_K + \delta\phi \wedge *g_K). \end{aligned} \tag{6.3}$$

Since $tG_K = 0$, $t\delta G_K = 0$, and $tK = 0$, $t\delta K = 0$, we see that the surface integral

$$P_K\phi = -\int_B (\phi \wedge *dg_K + \delta\phi \wedge *g_K) \tag{6.4}$$

satisfies

$$tP_K\phi = t\phi, \quad t\delta\, P_K\, \phi = t\delta\phi. \tag{6.5}$$

Moreover,

$$\Delta P_K\phi = + \int_B \delta\phi \wedge *k,$$

and this surface integral is zero if and only if the orthogonality condition (5.1)

https://doi.org/10.4153/CJM-1954-043-x Published online by Cambridge University Press

holds. That is, if $t\phi$, $td\phi$ satisfy (5.1), then $P_K\phi$ is the harmonic form which solves the K problem with these boundary values. From (6.2), (6.3) and (6.4) we deduce the operator equation

$$\Delta G_K - G_K\Delta = P_K. \tag{6.6}$$

Similar formulae are valid for the dual M problem. There exists a unique self-adjoint operator G_M with kernel $g_M(x, y)$, satisfying relations corresponding and dual to (6.2)-(6.10). Moreover,

$$*_x*_y\, g_K^p\,(x, y) = g_M^{N-p}(x, y).$$

7. Examples. To find concrete examples of the existence theorems here proved, we turn to Euclidean space of two or of three dimensions. In the plane, a 1-form ϕ, in Cartesian coordinates, is a differential

$$\phi = P\,dx + Q\,dy.$$

The dual is

$$*\phi = Q\,dx - P\,dy,$$

while

$$d\phi = (Q_x - P_y)\,dx\,dy, \quad \delta\phi = P_x + Q_y.$$

Thus the main existence theorem may be stated in terms of the two coefficients P and Q which will be harmonic functions if ϕ is harmonic. Let R denote a simply-connected region, bounded by a smooth curve C of arc length parameter s. We see that there exist unique harmonic functions P and Q, such that the quantities

$$P\frac{dx}{ds} + Q\frac{dy}{ds}, \qquad P_x + Q_y$$

take assigned Hölder continuous values on C. For in this case the K problem has no eigenforms. Also, if $P_x + Q_y$ is given as zero on C, it is everywhere zero. The dual problem may be stated by replacing P, Q with Q, $-P$, respectively.

On the annulus $a \leqslant r \leqslant b$, $r^2 = x^2 + y^2$, the K problem has the eigenform $\rho = dr$. Since $t*\rho = t*dr = rd\theta = ds$, the condition of solvability is

$$\int_{r=a} (P_x + Q_y)\,ds = \int_{r=b} (P_x + Q_y)\,ds.$$

In Euclidean three-space, a 1-form ϕ and the related quantities may be written, in Cartesian coordinates,

$$\begin{aligned} \phi &= P\,dx + Q\,dy + R\,dz, \\ *\phi &= P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy, \\ d\phi &= (R_y - Q_z)\,dy\,dz + (P_z - R_x)\,dz\,dx + (Q_x - P_y)\,dx\,dy, \\ \delta\phi &= -(P_x + Q_y + R_z); \end{aligned}$$

the differential corresponding to the curl and the co-differential being the divergence, of the vector (P, Q, R). Two mixed boundary value problems may be stated for this vector, on a multiply-connected region R of space, bounded by a boundary surface B.

First, consider the K problem for the 1-form ϕ (or, equivalently, the M problem for its dual $*\phi$). Here we assign two tangential components T_1, T_2 of ϕ:

$$T_i = P\frac{dx}{ds^i} + Q\frac{dy}{ds^i} + R\frac{dz}{ds^i}, \qquad i = 1, 2,$$

where s^1 and s^2 are parameters of B; and the value of $\delta\phi$, or of

$$P_x + Q_y + R_z.$$

The number of independent eigenforms is equal to the number of independent relative 1-cycles R; and in particular is zero if the boundary surface is connected and if R is simply-connected. In this case we may state that there exist unique harmonic functions P, Q, R satisfying the above boundary conditions.

Furthermore, if $(P_x + Q_y + R_z)$ is given as zero on B, then it is zero throughout R; that is, the vector (P, Q, R) is solenoidal. If also the condition $t\phi = td\chi$ is satisfied as in §5, then $d\phi = 0$ which in this case implies that the vector (P, Q, R) is irrotational as well. The condition $t\phi = td\chi$ takes here the form

$$T_1\,ds^1 + T_2\,ds^2 = dF(s^1, s^2),$$

which will hold if $\partial T_1/\partial s^2 = \partial T_2/\partial s^1$ on B, and if also for each absolute 1-cycle A^1 of B we have

$$\int_{A^1} t\phi \equiv \int_{A^1}[T_1\,ds^1 + T_2\,ds^2] = 0.$$

Secondly, we see that the M problem for ϕ is equivalent to the K problem for $*\phi$, and that in this problem there are assigned values of the normal component

$$N = P\frac{dx}{dn} + Q\frac{dy}{dn} + R\frac{dz}{dn}$$

(n denoting normal distance from B, locally), and the two components

$$\frac{\partial T_i}{\partial n} - \frac{\partial N}{\partial s^i}, \qquad i = 1, 2,$$

of the curl or differential $d\phi$. The eigenvectors of this problem are harmonic vectors (irrotational and divergenceless) with vanishing normal components on B. That is, they are the secondary flows or circulations of Kelvin **(4)**. The number of independent flows of this kind is equal to the number of absolute 1-cycles (irreducible circuits), or, equivalently, to the minimum number of 2-dimensional diaphragms needed to make the region simply-connected. Thus, if R is simply-connected, we may assert the existence of unique harmonic functions P, Q, and R which satisfy the boundary conditions. In the more general case when R is not simply connected, the orthogonality condition is easily written down in terms of components. Again, we see from the dual of Theorem III that if the two assigned components of the curl are zero on B, then the curl vanishes inside.

We may here also apply the second result of §5 to the dual vector $*\phi$: thus if $t*\phi = td*\chi$ on B, we conclude that $d*\phi = 0$, i.e. that the divergence of (P, Q, R) vanishes. This necessary condition takes the form

$$N\, ds^1\, ds^2 = d(f_1\, ds^1 + f_2\, ds^2),$$

where f_1 and f_2 are single-valued on B. Since B is two-dimensional, the two-form $Nds^1\, ds^2$ is automatically closed; by de Rham's second theorem it is derived if its periods are all zero; that is, if

$$\int_{B_i} Nds^1 ds^2 = 0$$

for each component B_i of B.

The preceding remarks may be summed up as follows. Suppose that each Cartesian component of a vector field (P, Q, R) is harmonic in a multiply-connected region R of space. Then if the vorticity vector on the boundary is everywhere normal to the boundary, the vector field is irrotational, while if there is zero net inflow over each boundary component, the vector field is solenoidal in R.

References

1. G. F. D. Duff,
(*a*) *Differential forms in manifolds with boundary*, Ann. Math. *56* (1952), 115–127.
(*b*) *Boundary value problems associated with the tensor Laplace equation*, Can. J. Math., *5* (1953), 196–210.
(*c*) *A tensor equation of elliptic type*, Can. J. Math., *5* (1953), 524–535.

2. G. F. D. Duff and D. C. Spencer, *Harmonic tensors on Riemannian manifolds with boundary*, Ann. Math., *56* (1952), 128–156.

3. G. Giraud, *Equations et systèmes d'équations où figurent des valeurs principales des intégrales*, C. R. Acad. Sci. Paris, *204* (1937), 628–630.

4. Lord Kelvin, *Papers*, *4*, *13* (Cambridge, 1911).

5. G. de Rham and K. Kodaira, *Harmonic integrals*, Mimeographed notes, Institute for Advanced Study (Princeton, 1950).

University of Toronto