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ON TOPOLOGICAL CLASSIFICATION OF FINITE CYCLIC ACTIONS ON BORDERED SURFACES

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Abstract. In Hirose (Tohoku Math. J. **62** (2010), 45–53), Susumu Hirose showed that, except for a few cases, the order N of a cyclic group of self-homeomorphisms of a closed orientable topological surface S_g of genus $g \geqslant 2$ determines the group up to a topological conjugation, provided that $N \geqslant 3g$. Gromadzki *et al.* undertook in Bagiński *et al.* (Collect. Math. **67** (2016), 415–429) a more general problem of topological classification of such group actions for $N > 2(g-1)$. In Gromadzki and Szepietowski (Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **110** (2016), 303–320), we considered the analogous problem for closed nonorientable surfaces, and in Gromadzki *et al.* (Pure Appl. Algebra **220** (2016), 465–481) – the problem of classification of cyclic actions generated by an orientation-reversing self-homeomorphism. The present paper, in which we deal with topological classification of actions on bordered surfaces of finite cyclic groups of order $N > p-1$, where p is the algebraic genus of the surface, completes our project of topological classification of *"large"* cyclic actions on compact surfaces. We apply obtained results to solve the problem of uniqueness of the actions realizing the solutions of the so-called minimum genus and maximum order problems for bordered surfaces found in Bujalance *et al.* (*Automorphisms Groups of Compact Bordered Klein Surfaces: A Combinatorial Approach*, Lecture Notes in Mathematics **1439**, Springer, 1990).

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§1. Introduction

By an action of a group G on a surface S we understand an embedding of G into the group $\mathrm{Homeo}(S)$ of homeomorphisms of S. Two such actions are *topologically conjugate*, or of the same *topological type*, if the images of G are conjugate in $\mathrm{Homeo}(S)$.

In [10], it was shown that, except for a few cases, the order N of the finite cyclic group $\mathbb{Z}_N$ acting on a closed orientable topological surface S_g of genus $g \geqslant 2$ determines the topological type of the action, provided that $N \geqslant 3g$. In [2], Grzegorz Gromadzki *et al.* undertook a more general problem of topological classification of such actions for $N > 2(g-1)$. This is an essential extension, because between $3g$ and $4g+2$ only $3g+1$, $3g+2$, $3g+4$ and $4g$ can stand as the period of a single self-homeomorphism of S_g, whereas there are infinitely many rational values of a, b such that for $N = ag+b$ we have $N > 2g-2$ and N is the period of a self-homeomorphisms of S_g for infinitely many g. In [7], we considered analogous problem for cyclic actions generated by an orientation-reversing self-homeomorphism, while in [6] – a similar problem for closed nonorientable surfaces, obtaining a classification of topological types of action of $\mathbb{Z}_N$ on a surface S in function of a possible type of the quotient orbifold $S/\mathbb{Z}_N$, provided that N is sufficiently big.

The present paper, in which we deal with topological classification of actions of $\mathbb{Z}_N$ on a bordered surface of algebraic genus p, where $N > p-1$, completes our project of topological classification of big cyclic actions on compact surfaces. The lower bound $p-1$ for the order of an action is essential for two reasons. The first is that we again cover a quite large class of actions, since there are infinitely many rational values of a, b, for which there are infinitely many values of p, such that a bordered surface of algebraic genus p admits a cyclic action of order $N = ap+b$ and $N > p-1$. The second reason for the bound $N > p-1$ is that it is satisfied for all the actions realizing the solutions of the so-called minimum genus and maximum order problems for bordered surfaces found in [4], and the question about their topological rigidity partially motivated the present paper.

Our results can be seen as a topological classification of cyclic group actions of order N on bordered surfaces of algebraic genus $p \leqslant N$. Another problem, suggested by the referee of this paper, would be to obtain a similar classification for surfaces of large genera that is p bigger than N. We remark that there are many results in the literature about the spectrum of genera of surfaces admitting a given finite group as a group of self-homeomorphisms. As an example of such results in the case of closed orientable surfaces let us mention the important paper [14] by Kulkarni. While we believe that it should not be difficult to obtain similar results for bordered surfaces, it seems that it would be a rather difficult problem to classify topologically actions of order $N < p$. The main reason is that the orbit spaces which occur in the case $N < p$ may have much bigger and much more complicated mapping class groups (see Section 5 for a definition) than for $p \leqslant N$, and in such a case our method, based on a good understanding of these mapping class groups, is not effective.

There are two more interesting features of the actions considered in this paper. The first is that finite group actions on compact surfaces of negative Euler characteristic may be realized by analytic actions on Riemann surfaces, or dianalytic actions on Klein surfaces, due to the Hurwitz–Nielsen, Kerkjarto and Alling–Greenleaf geometrizations mentioned in Section 3.1. The loci in the moduli spaces of Klein surfaces composed of the points classifying the surfaces dianalytically realizing the actions considered here have dimensions 1, 2 or 3 (this follows from Lemma 4.3, formula (4) and classical formula of Fricke and Klein for dimension of Teichmüller spaces of Fuchsian groups cf. [4, Theorem 0.3.2]). This is similar as in the case of actions on unbordered nonorientable Klein surfaces [6], or orientation-reversing automorphisms of classical Riemann surfaces [7], but in contrast to the classical case of orientation-preserving cyclic actions of order $>2g-2$ described in [2], where the loci of such structures in the moduli space are 0-dimensional, which means, in particular, that the topological type of an action of an orientation-preserving self-homeomorphism of such order usually uniquely determines the conformal type of a Riemann surface on which it acts as an automorphism. Finally, observe that our results can be stated in terms of birational actions on real algebraic curves due to the functorial equivalence between bordered Klein surfaces and such curves described in [1].

This paper is organized as follows. In Section 2, we state our main results. Section 3 contains necessary preliminaries concerning finite topological actions on bordered surfaces from the combinatorial point of view. In particular, we review non-Euclidean crystallographic groups. In Section 4, we determine the possible topological types of the orbit space (orbifold) of a cyclic action of order N on a bordered surface of algebraic genus $p < N + 1$. We obtain ten different topological types here, all of which are either a disc or an annulus or a Möbius band, with some cone points in the interior and some corner points on the boundary. In Section 5, we review the relationship between the groups of automorphisms of non-Euclidean crystallographic groups and mapping class groups. We also compute the mapping class groups of three surfaces: once-punctured annulus, once-punctured Möbius band and twice-punctured disc, which are needed for Section 6, where we prove our main results. Finally, in Section 7, we apply our results to study topological rigidity of the solutions of the so-called minimum genus and maximum order problems for cyclic actions on bordered surfaces, solved over 30 years ago in [4].

§2. Statement of the main results

Suppose that a cyclic group of order N acts on a bordered surface S of algebraic genus p, where $N > p - 1$. We show in Section 4 that the orbit space $S/\mathbb{Z}_N$ is one of the following orbifolds:

(1) disc with 6 corner points;
(2) annulus with 2 corner points;
(3) Möbius band with 2 corner points;
(4) 1-punctured disc with 2 corner points;
(5) 1-punctured disc with 4 corner points;
(6) 1-punctured Möbius band;
(7) 2-punctured disc;
(8) 1-punctured annulus;
(9) 3-punctured disc;
(10) 2-punctured disc with 2 corner points.

Our classification of cyclic actions of big order is split into ten cases and the results are presented in ten consecutive subsections. Their proofs are given in Section 6 which is also divided in ten subsections with the same titles for the reader's convenience. Throughout the whole paper φ will denote the Euler totient function. We also need similar function ψ defined in [2] as

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$\psi(1) = 1$ and given a prime factorization $C = p_1^{\alpha_1} \cdots p_r^{\alpha_r} > 1$

$$\psi(C) = \prod_{i=1}^{r} (p_i - 2) p_i^{\alpha_i - 1}.$$

Observe the analogy with the Euler function φ which is defined for such C as

$$\varphi(C) = \prod_{i=1}^{r} (p_i - 1) p_i^{\alpha_i - 1}.$$

2.1 Actions with a disc with 6 corner points as the quotient orbifold

THEOREM 2.1. *There is an action of a cyclic group of order N on a bordered surface S with a disc having 6 corner points as the quotient orbifold if and only if $N = 2$ and S is a 3-holed sphere. Furthermore, such action is unique up to topological conjugation.*

2.2 Actions with annulus with 2 corner points as the quotient orbifold

THEOREM 2.2. *There is an action of a cyclic group of order N on a bordered surface S with an annulus having two corner points as the quotient orbifold if and only if N is even and S is one of the following surfaces:*

- *$N/2$-holed Klein bottle;*
- *$N/2$-holed torus, where $N/2$ is odd;*
- *$(N/2 + 1)$-holed projective plane;*
- *$(N/2 + 2)$-holed sphere, where $N/2$ is odd.*

Furthermore, such action is unique up to topological conjugation for each of these surfaces.

2.3 Actions with Möbius band with 2 corner points as the quotient orbifold

THEOREM 2.3. *There is an action of a cyclic group of order N on a bordered surface S having a Möbius band with 2 corner points as the quotient orbifold if and only if N is even and S is either $N/2$-holed Klein bottle or $N/2$-holed torus, the latter being possible only for odd $N/2$. Furthermore, in both cases the action is unique up to topological conjugation.*

2.4 Actions with a 1-punctured disc with 2 corner points as the quotient orbifold

THEOREM 2.4. *There is an action of a cyclic group of order N on a bordered surface S with a disc having one cone point of order m and 2 corner points as the quotient orbifold if and only if either*

- *m is even, $N = m$ and S is $N/2$-holed projective plane; or*
- *m is odd, $N = 2m$ and S is $N/2$-holed sphere.*

Furthermore in both cases the action is unique up to topological conjugation.

2.5 Actions with a 1-punctured disc with 4 corner points as the quotient orbifold

THEOREM 2.5. *There is an action of a cyclic group of order N on a bordered surface with a disc having one cone point of order m and 4 corner points as the quotient orbifold if and only if either*

- *m is even, $N = m$ and S is N-holed projective plane; or*
- *m is odd, $N = 2m$ and S is N-holed sphere.*

Furthermore, in both cases the action is unique up to topological conjugation.

2.6 Actions with 1-punctured Möbius band as the quotient orbifold

We consider the actions on orientable and nonorientable surfaces separately.

THEOREM 2.6. *There is an action of a cyclic group of order N on a bordered orientable surface with k boundary components, with a Möbius band having 1 cone point of order m as the quotient orbifold, if and only if k divides N, $N = 2\mathrm{lcm}(m, N/k)$, and either $t = (m, N/k)$ is odd, or $N/2t$ is even. Furthermore, in such case the algebraic genus of the surface is equal to $1 + (m-1)N/m$ and there are $\lceil \varphi(t)/2 \rceil$ conjugacy classes of such actions.*

THEOREM 2.7. *There is an action of a cyclic group of order N on a bordered nonorientable surface with k boundary components, with a Möbius band having 1 cone point of order m as the quotient orbifold, if and only if k divides N, $N = \mathrm{lcm}(m, N/k)$, and for $t = (m, N/k)$, N/t is odd. Furthermore, in such case the algebraic genus of the surface is equal to $1 + (m-1)N/m$ and the number of topological conjugacy classes of such actions is $\varphi(t)$ or $\lceil \varphi(t)/2 \rceil$ if N is even or odd, respectively.*

2.7 Actions with a 2-punctured disc as the quotient orbifold

THEOREM 2.8. *There exists an action of a cyclic group of order N on a bordered surface S with k boundary components, having a disc with two cone points of orders m and n as the quotient orbifold if and only if S is orientable and*

- $N = \operatorname{lcm}(m, n)$;
- k *divides* $t/(t, N/t)$, *where* $t = (m, n)$;
- *if* N *is even and* N/t *is odd, then* k *is even.*

In such case the algebraic genus of S is equal to $1 + N(1 - 1/m - 1/n)$ and if C denotes the biggest divisor of t/k coprime to Nk/t, then the number of equivalence classes of such actions is

- $\varphi(t/kC)\psi(C)$ *if* $n \neq m$;
- $\varphi(n/kC)\psi(C)/2 + 1$ *if* $n = m$ *and* $n/kC = 2^z$, *where* $z > 1$;
- $\lceil \varphi(n/kC)\psi(C)/2 \rceil$ *otherwise.*

2.8 Actions with a 1-punctured annulus as the quotient orbifold

First we deal with the actions on nonorientable surfaces.

THEOREM 2.9. *There exists an action of a cyclic group $\mathbb{Z}_N$ on a nonorientable surface S with k boundary components, with an annulus having one cone point of order m as the quotient orbifold, if and only if k divides N and $N = \operatorname{lcm}(m, N/k)$. Furthermore, in such case the algebraic genus of the surface is equal to $1 + N(m - 1)/m$ and there are $\varphi(t)$ different topological types of such action, where $t = (m, N/k)$.*

The case of orientable S considered in the next theorem is much more involved. It has two parts. The first describes the necessary and sufficient conditions for existence of the actions, whereas the second has quantitative character and provides the numbers of equivalence classes of such actions. These numbers are expressed in terms of BSK-maps and therefore a reader less familiar with the study of periodic actions on compact surfaces from a combinatorial point of view should postpone the reading of (ii)–(iv) until Section 3, where these maps are introduced.

THEOREM 2.10.

(i) *There exists an action of a cyclic group $\mathbb{Z}_N$ on an orientable surface S with k boundary components, with an annulus having one cone point of order m as the quotient orbifold if and only if either*

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(1) *k divides N, $N = 2\mathrm{lcm}(m, N/k)$ and $N/2$ is odd; or*
(2) *m divides N and there exits an integer n, $1 \leqslant n < k$, such that:*

(a) *n and $k - n$ divide m;*
(b) *N/m, n and $k - n$ are pairwise relatively prime;*
(c) *if N is even then one of N/m, n, $k - n$ is even.*

In such case the algebraic genus of the surface is equal to $1 + N(m - 1)/m$.

(ii) *Suppose that N, m, k satisfy* (1) *and $t = (m, N/k)$. Then there are $\varphi(t)$ equivalence classes of BSK-maps $\theta^1 : \Lambda \to \mathbb{Z}_N$ such that $\theta^1(c_1) \neq \theta^1(c_2)$.*
(iii) *Suppose that N, m, k satisfy* (2)*. Then the number of equivalence classes of BSK-maps $\theta^2 : \Lambda \to \mathbb{Z}_N$ such that $\theta^2(c_1) = \theta^2(c_2) = 0$, and $\theta^2(e_1)$ and $\theta^2(e_2)$ have orders N/n and $N/(k - n)$ is*

- *$\varphi(m/Cn(k - n))\psi(C)$ if $k \neq 2$;*
- *$\varphi(m/C)\psi(C)/2 + 1$ if $k = 2$ and $m/C = 2^z$, where $z > 1$;*
- *$\lceil\varphi(m/C)\psi(C)/2\rceil$ otherwise;*

where C is the biggest divisor of $m/n(k - n)$ coprime to $Nn(k - n)/m$.
(iv) *Every BSK-map corresponding to a $\mathbb{Z}_N$-action on S is equivalent either to some θ^1 from the assertion* (ii)*, or to some θ^2 from the assertion* (iii).

2.9 Actions with a 3-punctured disc as the quotient orbifold

The orders of the three cone points are either $2, 2, m$, $m \geqslant 2$, or $2, 3, m$, where $m \in \{3, 4, 5\}$. We consider these two cases separately.

THEOREM 2.11. *There is an action of a cyclic group of order N on a bordered surface S with a disc having 3 cone points of orders $2, 2, m$ as the quotient orbifold if and only if $N = \mathrm{lcm}(2, m)$. In such case S is orientable, it has N/m boundary components, its genus is equal to $g = 1 + (m - 2)N/2m$ and such an action is unique up to topological conjugation.*

THEOREM 2.12. *There is an action of a cyclic group of order N on a bordered surface S with a disc having 3 cone points of orders $2, 3, m$, where $m \in \{3, 4, 5\}$, as the quotient orbifold if and only if $N = \mathrm{lcm}(2, 3, m)$ and S is an orientable surface of topological genus g with k boundary components, where*

- *if $m = 3$ then $(g, k) = (3, 1)$ or $(2, 3)$;*
- *if $m = 4$ then $(g, k) = (6, 1)$;*

- *if* $m = 5$ *then* $(g, k) = (15, 1)$.

In each case the action is unique up to topological conjugation.

2.10 Actions with a 2-punctured disc with two corners as the quotient orbifold

The orders of the cone points are either $2, m$, $m \geqslant 2$, or $3, m$, where $m \in \{3, 4, 5\}$. We consider these two cases separately.

THEOREM 2.13. *There is an action of a cyclic group of order* N *on a bordered surface* S *with a disc having* 2 *cone points of orders* $2, m$ *and* 2 *corners as the quotient orbifold if and only if* $N = \mathrm{lcm}(2, m)$. *In such case* S *is nonorientable, it has* $N/2$ *boundary components, its genus is equal to* $g = 2 + (m-2)N/2m$, *and such an action is unique up to topological conjugation.*

THEOREM 2.14. *There is an action of a cyclic group of order* N *on a bordered surface* S *with a disc having* 2 *cone points of orders* $3, m$, *where* $m \in \{3, 4, 5\}$, *and* 2 *corners as the quotient orbifold if and only if* $N = \mathrm{lcm}(2, 3, m)$, S *has* $N/2$ *boundary components, and*

- *if* $m = 3$ *then* S *is orientable of genus* 2;
- *if* $m = 4$ *then* S *is nonorientable of genus* 7;
- *if* $m = 5$ *then* S *is orientable of genus* 8.

Furthermore, for $m = 3$ *there are two different topological types of such action, and for* $m = 4, 5$ *the action is unique up to topological conjugation.*

§3. Preliminaries

In principle, we use a combinatorial approach, based on Riemann unformization theorem for compact Riemann surfaces, its generalization for nonorientable or bordered surfaces with dianalytic structures of Klein surfaces, good knowledge of discrete group of isometries of the hyperbolic plane and some elementary covering theory. For the reader's convenience, we review the terminology of [4] used in this paper.

3.1 Hurwitz–Nielsen geometrization and its generalizations

Let G be a finite group of orientation-preserving self-homeomorphisms of a closed orientable surface S_g of genus g, $g \geqslant 2$.

By [11] and [15], there exists a structure of a Riemann surface on S_g, with respect to which the elements of G act as conformal automorphisms.

This result was generalized to the case of actions containing orientation-reversing self-homeomorphisms and to closed nonorientable surfaces by Kerejarto [12], and for bordered surfaces in a more recent monograph of Alling and Greenleaf [1], who introduced the concept of a Klein surface. Thus, although the paper concerns topological classification of topological actions, we assume, whenever necessary, that a surface has such a structure of a bordered Klein surface, and the elements of G act on it as dianalytic automorphisms. This assumption allows for effective conformally algebraic methods described in the following subsections.

3.2 Non-Euclidean crystallographic groups

By a non-Euclidean crystallographic group (NEC-group in short) we mean a discrete and cocompact subgroup of the group $\mathcal{G}$ of all isometries of the hyperbolic plane $\mathcal{H}$. The algebraic structure of such a group Λ is encoded in its signature:

$$s(\Lambda) = (g; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}), \tag{1}$$

where the brackets $(n_{i1}, \ldots, n_{is_i})$ are called *the period cycles*, the integers n_{ij} are the *link periods*, m_i *proper periods* and finally g the *orbit genus* of Λ. A group Λ with signature (1) has the presentation with the following generators

x_i	for $1 \leqslant i \leqslant r$,
c_{ij}, e_i	for $1 \leqslant i \leqslant k, 0 \leqslant j \leqslant s_i$,
a_i, b_i	for $1 \leqslant i \leqslant g$ if the sign is $+$,
d_i	for $1 \leqslant i \leqslant g$ if the sign is $-$,

subject to the relations

$x_i^{m_i} = 1$	for $1 \leqslant i \leqslant r$,
$c_{ij}^2 = (c_{ij-1}c_{ij})^{n_{ij}} = 1$	for $1 \leqslant i \leqslant k, 0 \leqslant j \leqslant s_i$,
$c_{is_i} = e_i c_{i0} e_i^{-1}$	for $1 \leqslant i \leqslant k$,
$x_1 \cdots x_r e_1 \cdots e_k [a_1, b_1] \cdots [a_g, b_g] = 1$	if the sign is $+$,
$x_1 \cdots x_r e_1 \cdots e_k d_1^2 \cdots d_g^2 = 1$	if the sign is $-$,

where $[x, y] = xyx^{-1}y^{-1}$. Elements of any system of generators satisfying the above relations will be called *canonical generators*. The elements x_i are elliptic transformations, a_i, b_i hyperbolic translations, d_i glide reflections and c_{ij} hyperbolic reflections. Reflections c_{ij-1} and c_{ij} are called consecutive. It is essential for applications that every element of finite order in Λ is conjugate either to a canonical reflection, or to a power of some canonical elliptic element x_i, or else to a power of the product of two consecutive canonical reflections.

The orbit space $\mathcal{H}/\Lambda$ is a hyperbolic orbifold, with underlying surface of topological genus g with k boundary components, and it is orientable if the sign is $+$ and nonorientable otherwise. The image in $\mathcal{H}/\Lambda$ of the fixed point of the canonical elliptic generator x_i is called *cone point* of order m_i, whereas the image of the fixed point of the product of two consecutive canonical reflections $c_{ij-1}c_{ij}$ is called *corner point* of order n_{ij}.

Now, an abstract group with such presentation can be realized as an NEC-group Λ if and only if the value

$$\varepsilon g + k - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{ij}}\right) \tag{2}$$

is positive, where $\varepsilon = 2$ if the sign is $+$, or $\varepsilon = 1$ otherwise. This value turns out to be the normalized hyperbolic area $\mu(\Lambda)$ of an arbitrary fundamental region for such a group, and we have the following Hurwitz–Riemann formula

$$[\Lambda : \Lambda'] = \frac{\mu(\Lambda')}{\mu(\Lambda)} \tag{3}$$

for a subgroup Λ' of finite index in an NEC-group Λ.

Finally, NEC-groups without orientation-reversing elements are Fuchsian groups. They have signatures $(g; +; [m_1, \ldots, m_r]; \{-\})$ usually abbreviated as $(g; m_1, \ldots, m_r)$. Given an NEC-group Λ containing orientation-reversing elements, its subgroup Λ^+ consisting of the orientation-preserving elements is called the *canonical Fuchsian subgroup of* Λ, and by [16], for Λ with signature (1), Λ^+ has signature

$$(\varepsilon g + k - 1; m_1, m_1, \ldots, m_r, m_r, n_{11}, \ldots, n_{ks_k}). \tag{4}$$

A torsion free Fuchsian group Γ is called a *surface group* and it has signature $(g; -)$.

We also use other results concerning relationship between the signatures of an NEC-group and its finite index subgroup proved in [4, Chapter 2].

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3.3 Bordered Riemann surfaces and their groups of automorphisms

By the Riemann uniformization theorem, every closed Riemann surface S of genus $g \geqslant 2$ can be identified with the orbit space $\mathcal{H}/\Gamma$ of the hyperbolic plane with respect to an action of a Fuchsian group Γ isomorphic to the fundamental group of S. A Klein surface is a compact bordered topological surface equipped with a dianalytic structure – historically it is also called bordered Riemann surface. For a given Klein surface S, Alling and Greenleaf [1] constructed certain canonical double cover S^+ being a Riemann surface, such that S is the quotient of S^+ by an action of an antiholomorphic involution with fixed points. The *algebraic genus* $p = p(S)$ of S is defined as the genus of S^+ and it follows from the construction that p coincides with the rank of the fundamental group of S, and so for a surface of topological genus g having k boundary components it is equal to $p = \varepsilon g + k - 1$, where $\varepsilon = 2$ if S is orientable and $\varepsilon = 1$ otherwise. It is well known (see [4] for example) that any compact Klein surface S of algebraic genus $p \geqslant 2$ can be represented as $\mathcal{H}/\Gamma$ for some NEC-group Γ. If S has topological genus g and k boundary components, then Γ can be chosen to be a *bordered surface group*, that is, an NEC-group with the signature

$$(g; \pm; [\,]; \{(\,), \overset{k}{\ldots}, (\,)\}), \tag{5}$$

whose only elements of finite order are reflections. It has the presentation

$$\begin{gathered}\langle a_1, b_1, \ldots, a_g, b_g, e_1, \ldots, e_k, c_1, \ldots, c_k \mid \\ c_i^2, [e_i, c_i], e_1 \ldots e_k [a_1, b_1] \ldots [a_g, b_g] \rangle\end{gathered}$$

if the sign is $+$, or

$$\langle d_1, \ldots, d_g, e_1, \ldots, e_k, c_1, \ldots, c_k \mid c_i^2, [e_i, c_i], e_1 \ldots e_k d_1^2 \ldots d_g^2 \rangle$$

otherwise. Finally, a finite group G is a group of automorphisms of $S = \mathcal{H}/\Gamma$ if and only if $G \cong \Lambda/\Gamma$ for some NEC-group Λ. A convenient way of defining an action of a group G on a bordered surface S is by means of an epimorphism $\theta : \Lambda \to G$ whose kernel is a bordered surface group. In such a case $S = \mathcal{H}/\Gamma$, where $\Gamma = \ker \theta$. We shall refer to such an epimorphism as to a bordered-surface-kernel epimorphism (BSK in short) or smooth epimorphism.

Two actions of G on S are topologically conjugate (by a homeomorphism of S) if and only if the associated smooth epimorphisms are equivalent

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in the sense of the next definition (see [3, Proposition 2.2]). We say that two smooth epimorphisms $\theta_i : \Lambda \to G$, $i = 1, 2$, are *equivalent* if and only if there exist automorphisms $\phi : \Lambda \to \Lambda$ and $\varphi : G \to G$ such that the following diagram is commutative.

$$\begin{tikzcd} \Lambda \arrow[r, "\phi"] \arrow[d, "\theta_1"'] & \Lambda \arrow[d, "\theta_2"] \\ G \arrow[r, "\varphi"] & G \end{tikzcd} \tag{6}$$

3.4 Some elementary algebra

For integers a, b we denote by (a, b) their greatest common divisor and we use additive notation for cyclic groups $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ throughout the whole paper. Furthermore, by abuse of language, we write $a \in \mathbb{Z}_N$ for a nonnegative integer $a < N$. To avoid unnecessary parentheses, we denote expressions of the form $a/(bc)$ simply as a/bc.

We need the following version of the classical Chinese remainder theorem.

LEMMA 3.1. *Given integers a, b, the system of congruences*

$$\begin{cases} x \equiv a\ (m), \\ x \equiv b\ (n), \end{cases}$$

has a solution if and only if $a \equiv b\ (t)$, where $t = (m, n)$ and this solution is unique up to $\mathrm{lcm}(m, n)$. □

The following useful result can be proved using Dirichlet's theorem on arithmetic progression (see [7] for a more elementary, direct, argument).

LEMMA 3.2. *Given an integer N and its divisor n, the reduction map $\mathbb{Z}_N^* \to \mathbb{Z}_n^*$ is a group epimorphism.*

Proof. Let $a \in \mathbb{Z}_n^*$, then $(a, n) = 1$ and so by Dirichlet theorem on arithmetic progression there exists infinitely many primes A of the form $a + bn$ and so $A \in \mathbb{Z}_N^*$ and its reduction modulo n is equal to a. □

We also need

LEMMA 3.3. (Harvey, [8]) *The group $\mathbb{Z}_N$ is generated by three elements a, b, c of orders m, n, l and such that $a + b + c = 0$ if and only if*

(i) $N = \mathrm{lcm}(m, n) = \mathrm{lcm}(m, l) = \mathrm{lcm}(n, l)$*; and*
(ii) *if N is even, then exactly one of the numbers N/m, N/n, N/l is even.*

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The condition (i) of Lemma 3.3 is equivalent to existence of pairwise relatively prime integers A, A_1, A_2, A_3 for which

$$m = AA_2A_3, \qquad n = AA_1A_3, \qquad l = AA_1A_2, \qquad N = AA_1A_2A_3. \tag{7}$$

The condition (ii) of Lemma 3.3 is that one of the numbers A_1, A_2, A_3 is even if N is even. The quadruple (A, A_1, A_2, A_3) is called *Maclachlan decomposition* of the triple (m, n, l) after Hidalgo [9].

§4. Periodic self-homeomorphisms of compact bordered surfaces of big periods

From (3.1.0.1) and (3.1.0.2) in [4, p. 61] we immediately obtain the following result.

LEMMA 4.1. *There exists a structure of a bordered Klein surface $S = S^k_{g,\pm}$ of topological genus h, with k boundary components and orientable or not according to the sign being plus or minus having a dianalytic automorphism φ of order N if and only if $\mathbb{Z}_N \cong \Lambda/\Gamma$, where Γ and Λ have signatures respectively*

(8)
$$(g; \pm; [\,]; \{(\,), \overset{k}{\ldots}, (\,)\}) \qquad \textit{and} \qquad (g'; \pm; [m_1, \ldots, m_r]; \{C_1, \ldots, C_{k'}\}),$$

where each cycle C_i is either empty or consists of an even number of periods equal to 2. Furthermore, nonempty cycles do not appear for odd N.

For the rest of this section we assume that Λ and Γ are as in Lemma 4.1 and we denote by $\theta : \Lambda \to \Lambda/\Gamma$ the canonical projection, where Λ/Γ will be identified with $\mathbb{Z}_N$. Recall that θ is called BSK-epimorphism. In order to state the next lemma, we need one definition and some notation. We define a *nonorientable word* to be a word w in the canonical generators of Λ and their inverses, such that w defines an orientation-reversing isometry of $\mathcal{H}$ and the reflections c_{ij} of Λ which belong to Γ do not appear in w. For $i = 1, \ldots, k'$ let s_i be the length of the cycle C_i in the signature of Λ. Let l_i denote the order of $\theta(e_i)$ in $\mathbb{Z}_N$ and

$$t_i = \begin{cases} 0 & \text{if } s_i = 0 \text{ and } c_{i0} \notin \Gamma, \\ N/l_i & \text{if } s_i = 0 \text{ and } c_{i0} \in \Gamma, \\ s_i N/4 & \text{if } s_i > 0. \end{cases}$$

The following lemma can be deduced from [4].

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LEMMA 4.2. *Let S be the surface $\mathcal{H}/\Gamma$. Then:*

(a) *S is nonorientable if and only if Γ contains a nonorientable word;*
(b) *the number of boundary components of S is $k = t_1 + \cdots + t_{k'}$.*

Proof. Indeed (a) follows from [4, Theorems 2.1.2 and 2.1.3]. Now, with the above notation, each empty period cycle of Λ whose corresponding canonical reflection belongs to Γ, produces N/l_i empty period cycles in Γ by [4, Theorem 2.3.3] (see also Theorems 2.4.2 and 2.4.4 therein). Next, let $c_0, c_1, \ldots, c_{2s}$ be a cycle of canonical reflections of Λ corresponding to a nonempty period cycle $(2, \overset{2s}{\ldots}, 2)$. Then $\theta(c_i) = 0$ or $N/2$. Observe however that two consecutive canonical reflections c_{i-1}, c_i have different images, since otherwise $c_{i-1}c_i$ would be an orientation-preserving torsion element of Γ. So this cycle of reflections is mapped either on $0, N/2, 0, N/2, \ldots, 0$ or on $N/2, 0, N/2, \ldots, 0, N/2$. In the former case each c_i for even i produces in Γ $N/2$ empty period cycles, while in the latter case the same is true for every odd i, and so each nonempty period cycle of length $2s$ produces in Γ $s(N/2)$ empty period cycles in virtue of [4, Theorem 2.3.2] (see also Theorem 2.4.4 therein). ∎

Observe that we can determine the topological type of the surface $\mathcal{H}/\Gamma$ by using Lemma 4.2 together with the Hurwitz–Riemann formula.

LEMMA 4.3. *If Γ is a bordered surface group of algebraic genus p and $N > p - 1$ then Λ has one of the following signatures:*

(1) $(0; +; [\,]; \{(2, 2, 2, 2, 2, 2)\})$,
(2) $(0; +; [\,]; \{(\,), (2, 2)\})$,
(3) $(1; -; [\,]; \{(2, 2)\})$,
(4) $(0; +; [m]; \{(2, 2)\})$,
(5) $(0; +; [m]; \{(2, 2, 2, 2)\})$,
(6) $(1; -; [m]; \{(\,)\})$,
(7) $(0; +; [m, n]; \{(\,)\})$,
(8) $(0; +; [m]; \{(\,), (\,)\})$,
(9*a*) $(0; +; [2, 3, m]; \{(\,)\}), m = 3, 4, 5$,
(9*b*) $(0; +; [2, 2, m]; \{(\,)\})$,
(10*a*) $(0; +; [3, m]; \{(2, 2)\}), m = 3, 4, 5$,
(10*b*) $(0; +; [2, m]; \{(2, 2)\})$.

Proof. By the Hurwitz–Riemann formula, $N > p - 1$ is equivalent to $\mu(\Lambda) < 1$. For Λ as in Lemma 4.1, we have

$$\mu(\Lambda) = \varepsilon g' + k' - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{s}{4},$$

where s is the sum of lengths of nonempty period cycles. Observe that s is even and $k' > 0$. We have $-1 \leqslant \varepsilon g' + k' - 2 \leqslant 0$.

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Suppose that $\varepsilon g' + k' - 2 = 0$. Then, since $(1 - 1/m_i) \geqslant 1/2$, $0 \leqslant r \leqslant 1$. If $r = 0$, then $s = 2$ and Λ has signature (2) or (3). If $r = 1$, then $s = 0$ and Λ has signature (6) or (8).

Suppose that $\varepsilon g' + k' - 2 = -1$, hence $(g', k') = (0, 1)$. Then $0 \leqslant r \leqslant 3$. If $r = 0$, then $s = 6$ and Λ has the signature (1). If $r = 1$, then $s = 2$ or 4 and Λ has signature (4) or (5). If $r = 2$, then $s = 0$ or 2. In the former case Λ has signature (7). In the latter case $1/m_1 + 1/m_2 > 1/2$ and Λ has signature ($10a$) or ($10b$). Finally, if $r = 3$ then $s = 0$, $1/m_1 + 1/m_2 + 1/m_3 > 1$ and Λ has signature ($9a$) or ($9b$). ∎

We close this section by a technical but simple lemma, which will be very useful in the next section.

LEMMA 4.4. *Suppose that C_i is a nonempty cycle in the signature of Λ. Then for some $\phi \in \mathrm{Aut}(\Lambda)$, $\theta \circ \phi$ maps the corresponding reflections $(c_{i0}, c_{i1}, \ldots, c_{is_i})$ on $(N/2, 0, \ldots, N/2, 0, N/2)$.*

Proof. By Lemma 4.1, all the periods in C_i are equal to 2 and by the proof of Lemma 4.2, consecutive canonical reflections $c_{i0}, c_{i1}, \ldots, c_{is_i}$ are mapped either on $N/2, 0, \ldots, N/2, 0, N/2$, or on $0, N/2, \ldots, 0, N/2, 0$. In the former case we take ϕ to be the identity, while in the latter case we define ϕ by $\phi(c_{ij}) = c_{ij-1}$ for $j = 1, \ldots, s_i$, $\phi(c_{i0}) = e_i^{-1} c_{is_i-1} e_i$, and the identity on the remaining generators of Λ. ∎

§5. Automorphisms of NEC-groups versus mapping class groups

From diagram (6) in Section 3 we see that for a topological classification of group actions via smooth epimorphisms we need to know how to calculate automorphisms groups of NEC-groups Λ. As we shall see, we need to know these automorphisms up to conjugation, which means that we actually need the groups $\mathrm{Out}(\Lambda)$ of outer automorphisms of Λ's. From the previous section we see that in this paper we need them only for three signatures $(1; -; [m]; \{(\,)\})$, $(0; +; [m]; \{(\,), (\,)\})$, $(0; +; [m, n]; \{(\,)\})$, and the outer automorphism groups for these NEC-groups were found in [3, Section 4] by using a connection between $\mathrm{Out}(\Lambda)$ and the mapping class group of the orbifold $\mathcal{H}/\Lambda$. For reader's convenience we review these results and their proofs (illustrated with figures and easier to follow then the proofs in [3]). For an NEC-group Λ, let $\mathrm{Mod}(\mathcal{H}/\Lambda)$ be the group of isotopy classes of homeomorphisms over $\mathcal{H}/\Lambda$ which map a cone point to a cone point of the same order, and analogously for the corner points, and $\mathrm{PMod}(\mathcal{H}/\Lambda)$ be the group of isotopy classes of homeomorphisms over $\mathcal{H}/\Lambda$ which fix the

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cone points and the corner points. For two elements ϕ_1, ϕ_2 of $\mathrm{Mod}(\mathcal{H}/\Lambda)$, $\phi_1\phi_2$ means applying ϕ_2 first and then applying ϕ_1. Let $\mathrm{Out}_0(\Lambda)$ be the subgroup of $\mathrm{Out}(\Lambda)$ which acts trivially on the set of conjugacy classes of the stabilizers of the fixed points of elliptic elements of Λ. Observe that these conjugacy classes are in one to one correspondence with the integers $m_1, \ldots, m_r, n_{11}, \ldots, n_{ks_k}$, and hence $\mathrm{Out}_0(\Lambda)$ is a subgroup of finite index of $\mathrm{Out}(\Lambda)$. The following lemma is proved in [3, Corollary 4.4].

LEMMA 5.1. *If* $\mathrm{PMod}(\mathcal{H}/\Lambda)$ *has finite order* n*, then* $\mathrm{Out}_0(\Lambda)$ *has order at most* n*.*

In order to obtain presentations of these groups, we review mapping class groups of two elementary surfaces. Let $S_{0,3}$ be the sphere with three marked points p_1, p_2 and p_3, and $\mathrm{PMod}(S_{0,3})$ be the group of isotopy classes of orientation-preserving diffeomorphisms over the sphere preserving each of these three points. It is well known that $\mathrm{PMod}(S_{0,3})$ is trivial (see, for example, [5, the proof of Proposition 2.3]). Let $N_{1,2}$ be the real projective plane with two marked points p_1 and p_2, and $\mathrm{PMod}(N_{1,2})$ be the group of isotopy classes of diffeomorphisms over the real projective plane preserving each of these two points. Let β_1 and β_2 be the oriented circles shown in Figure 1, and ν_i be the element of $\mathrm{PMod}(N_{1,2})$ obtained by sliding p_i once along β_i $(i = 1, 2)$. Korkmaz [13, Corollary 4.6] showed that $\mathrm{PMod}(N_{1,2})$ is generated by ν_1 and ν_2, and $\nu_1^2 = \nu_2^2 = (\nu_2\nu_1)^2 = 1$. Let ρ be the reflection indicated in Figure 1. By investigating the action on the fundamental group, we can see $\rho = \nu_1\nu_2$. Therefore, we see that $\mathrm{PMod}(N_{1,2})$ is generated by ν_1 and ρ, and $\nu_1^2 = \rho^2 = (\rho\nu_1)^2 = 1$.

LEMMA 5.2. [3, Proposition 4.12] *Let* Λ *be an NEC-group with signature* $(1; -; [m]; \{(\,)\})$ *and canonical generators* x*,* d*,* c*, satisfying the relations* $x^m = c^2 = 1, d^2xc = cd^2x$*. Then* $\mathrm{Out}(\Lambda)$ *is isomorphic to the Klein four-group and is generated by classes of automorphisms* γ*,* δ *defined by*

$$\gamma\colon \begin{cases} x \mapsto x^{-1}, \\ d \mapsto x^{-1}d^{-1}x, \\ c \mapsto c, \end{cases} \qquad \delta\colon \begin{cases} x \mapsto x, \\ d \mapsto (dx)^{-1}, \\ c \mapsto (dx)^{-1}c(dx). \end{cases}$$

Proof. Under the correspondence p_1 to the boundary, and p_2 to the cone point, $\mathrm{PMod}(N_{1,2})$ is isomorphic to $\mathrm{Mod}(\mathcal{H}/\Lambda) = \mathrm{PMod}(\mathcal{H}/\Lambda)$. The action of ρ on Λ is γ and that of ν_1 is δ and these actions are of order 2 and not inner automorphisms of Λ. By Lemma 5.1, the order of $\mathrm{Out}_0(\Lambda) = \mathrm{Out}(\Lambda)$ is

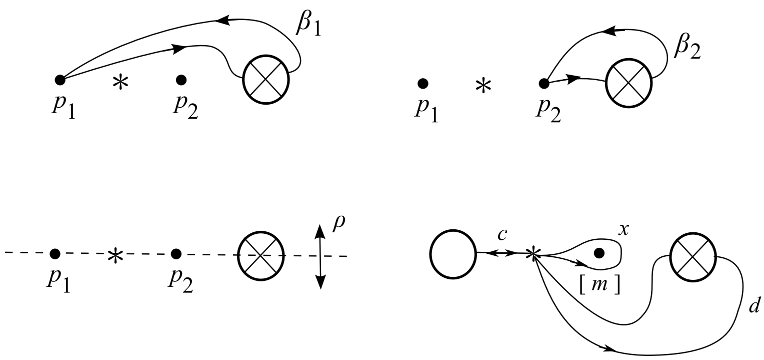

Figure 1.
$\otimes$ indicates the place to attach a Möbius band. The loops x, c and d represent the generators of Λ with signature $(1; -; [m]; \{(\,)\})$ or the orbifold fundamental group of $\mathcal{H}/\Lambda$ whose base point is $*$.

at most 4. Therefore, we see that $\mathrm{Out}(\Lambda)$ is the Klein four-group generated by γ and δ. ∎

LEMMA 5.3. *Let Λ be an NEC-group with signature $(0; +; [m]; \{(\,), (\,)\})$ with canonical generators x, e, c_1, c_2 satisfying the following defining relations: $x^m = c_1^2 = c_2^2 = 1$, $ec_1 = c_1e$, $xec_2 = c_2xe$. Then $\mathrm{Out}(\Lambda)$ is isomorphic to the Klein four-group and is generated by classes of automorphisms α, β defined by*

$$\alpha: \begin{cases} x \mapsto e^{-1}x^{-1}e, \\ e \mapsto e^{-1}, \\ c_1 \mapsto c_1, \\ c_2 \mapsto c_2, \end{cases} \qquad \beta: \begin{cases} x \mapsto e^{-1}xe, \\ e \mapsto (xe)^{-1}, \\ c_1 \mapsto c_2, \\ c_2 \mapsto c_1. \end{cases}$$

Proof. Let f be a homeomorphism over $\mathcal{H}/\Lambda$ fixing boundaries and a cone point, then we can regard f as an element of $\mathrm{PMod}(S_{0,3}) = 1$. Therefore, every element of $\mathrm{PMod}(\mathcal{H}/\Lambda) = \mathrm{Mod}(\mathcal{H}/\Lambda)$ is determined by its action on the boundary of $\mathcal{H}/\Lambda$. Let ρ be the reflection about the axis shown in Figure 2, and σ be the π-rotation about the cone point as shown in Figure 2. $\mathrm{PMod}(\mathcal{H}/\Lambda)$ is generated by ρ and σ, and its defining relations are $\rho^2 = \sigma^2 = (\rho\sigma)^2 = 1$. The action of ρ on Λ is α and that of σ is β and these actions are not inner automorphisms of Λ. By Lemma 5.1, the order of $\mathrm{Out}_0(\Lambda) = \mathrm{Out}(\Lambda)$ is at most 4. Therefore, we see that $\mathrm{Out}(\Lambda)$ is the Klein four-group generated by α and β. ∎

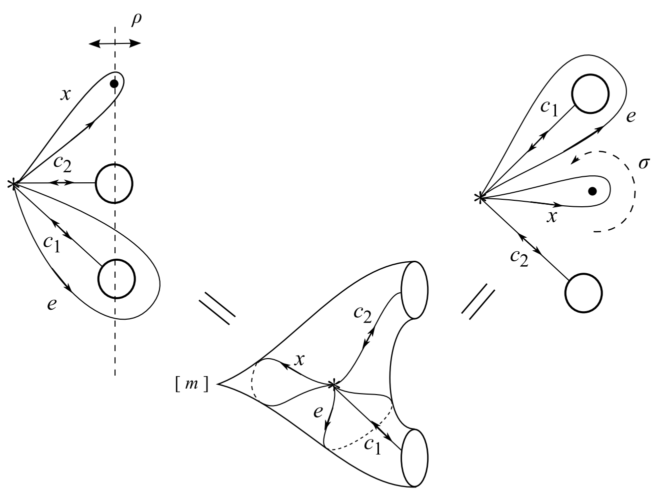

Figure 2.
The loops x, c_1, c_2 and e represent the generators of Λ with signature $(0;+;[m];\{(\,),(\,)\})$ or the orbifold fundamental group of $\mathcal{H}/\Lambda$ whose base point is $*$.

LEMMA 5.4. *Let Λ be an NEC-group with a signature $(0;+;[m,n];\{(\,)\})$ and generators x_1, x_2, c, satisfying the following defining relations: $x_1^m = x_2^n = c^2 = 1$, $x_1x_2c = cx_1x_2$. Then if $m \neq n$ then $\mathrm{Out}(\Lambda)$ has order 2 and is generated by the class of automorphism α while if $m = n$ and the Klein four-group generated by α, β in the other case, where*

$$\alpha\colon \begin{cases} x_1 \mapsto x_1^{-1}, \\ x_2 \mapsto x_1x_2^{-1}x_1^{-1}, \\ c \mapsto c, \end{cases} \qquad \beta\colon \begin{cases} x_1 \mapsto x_2, \\ x_2 \mapsto x_2^{-1}x_1x_2, \\ c \mapsto c. \end{cases}$$

Proof. Let f be a homeomorphism over $\mathcal{H}/\Lambda$ fixing boundaries and a corner, then we can regard f as an element of $\mathrm{PMod}(S_{0,3}) = 1$. Therefore, every element of $\mathrm{PMod}(\mathcal{H}/\Lambda)$ is determined by its action on the boundary of $\mathcal{H}/\Lambda$. Let ρ be the reflection about the axis shown in Figure 3. Then ρ reverses the orientation of the boundary of $\mathcal{H}/\Lambda$. $\mathrm{PMod}(\mathcal{H}/\Lambda)$ is generated by the involution ρ. The action of ρ on Λ is α and is not inner automorphism of Λ. By Lemma 5.1, the order of $\mathrm{Out}_0(\Lambda)$ is at most 2. Therefore, $\mathrm{Out}_0(\Lambda)$ is generated by α and $\mathrm{Out}_0(\Lambda)$ is isomorphic to $\mathbb{Z}_2$. If $n \neq m$, $\mathrm{Out}_0(\Lambda) = \mathrm{Out}(\Lambda)$. If $n = m$, there is a short exact sequence $1 \to \mathrm{Out}_0(\Lambda) \to \mathrm{Out}(\Lambda) \to \mathbb{Z}_2 \to 1$, where $\mathbb{Z}_2$ is the group of permutation of cones and is generated by τ

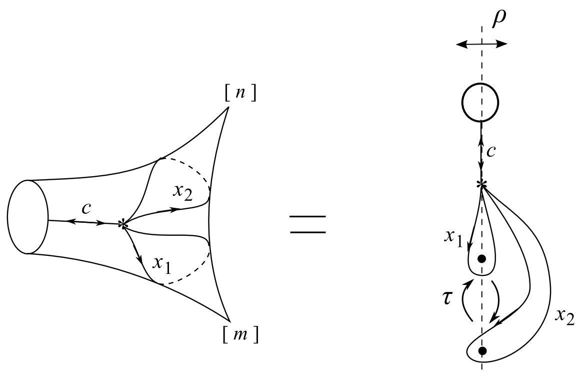

Figure 3.
The loops x_1, x_2, c are the generators of Λ with signature $(0; +; [m, n]; \{(\,)\})$ or the orbifold fundamental group of $\mathcal{H}/\Lambda$ whose base point is $*$.

in Figure 3. The action of τ on Λ is β. We conclude that $\mathrm{Out}(\Lambda)$ is generated by α and β, and its defining relations are $\alpha^2 = \beta^2 = (\alpha\beta)^2 = 1$. ∎

§6. Proofs of the main results

In this section we prove the results stated in Section 2, that is we classify, up to topological conjugation, cyclic actions corresponding to the signatures given in Lemma 4.3.

6.1 Actions with a disc with 6 corner points as the quotient orbifold

This is the easiest case concerning an NEC-group Λ with the signature $(0; +; [\,]; \{(2, 2, 2, 2, 2, 2)\})$ from Lemma 4.3. We denote the canonical reflections c_{1i} simply by c_i for $i = 0, 1, \ldots, 6$. We have $e_1 = 1$ and $c_0 = c_6$. It follows that Λ has the presentation

$$\langle c_0, \ldots, c_5 \mid c_0^2 = \cdots = c_5^2 = (c_0c_1)^2 = \cdots = (c_4c_5)^2 = 1\rangle.$$

Proof of Theorem 2.1. By Lemma 4.4 there is only one, up to equivalence, BSK-epimorphism $\theta : \Lambda \to \mathbb{Z}_N$, mapping $(c_0, c_1, c_2, c_3, c_4, c_5)$ on $(N/2, 0, N/2, 0, N/2, 0)$. In particular, we see that $N/2$ generates $\mathbb{Z}_N$, hence $N = 2$. By Lemma 4.2 and the Hurwitz–Riemann formula, S is a 3-holed sphere. ∎

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6.2 Actions with annulus with 2 corner points as the quotient orbifold

This case concerns an NEC-group Λ with the signature

$$(0; +; [\,]; \{(\,), (2,2)\})$$

from Lemma 4.3 which has the presentation

$$\langle e_1, e_2, c_{10}, c_{20}, c_{21}, c_{22} \mid \\ e_1e_2 = c_{ij}^2 = (c_{20}c_{21})^2 = (c_{21}c_{22})^2 = 1, e_1c_{10} = c_{10}e_1, e_2c_{20} = c_{22}e_2\rangle.$$

Proof of Theorem 2.2. Let $\theta : \Lambda \to \mathbb{Z}_N$ be a BSK-epimorphism and let $\theta(e_1) = a$. By Lemma 4.4 we may assume

$$\theta(e_1) = a, \qquad \theta(e_2) = -a, \qquad \theta(c_{20}) = \theta(c_{22}) = N/2,$$
$$\theta(c_{21}) = 0 \qquad \text{and} \qquad \theta(c_{10}) = 0 \text{ or } N/2.$$

Since a and $N/2$ generate $\mathbb{Z}_N$, we have $(a, N/2) = 1$ and it follows that the order of a is either N or $N/2$, the latter being possible only for odd $N/2$.

Suppose that the order of a is N. Then after composing θ with a suitable automorphism of $\mathbb{Z}_N$ we can assume that $\theta(e_1) = 1$. By Lemma 4.2 S is nonorientable, since $e_1^{N/2}c_{20}$ is a nonorientable word in $\Gamma = \ker\theta$, and its number of boundary components is either $N/2$ if $\theta(c_{10}) \neq 0$, or $N/2 + 1$ if $\theta(c_{10}) = 0$. From the Hurwitz–Riemann formula we easily compute that the genus of S is respectively 2 or 1. Now suppose that the order of a is $N/2$ which is odd. As above, after composing θ with a suitable automorphism of $\mathbb{Z}_N$ we can assume that $\theta(e_1) = 2$ and therefore again we obtain two nonequivalent BSK-maps, which give rise to two topologically nonconjugate actions. Observe however, that this time S is orientable by Lemma 4.2, and it has either $N/2$ or $N/2 + 2$ boundary components. By the Hurwitz–Riemann formula the genus of S is respectively 1 or 0. $\square$

6.3 Actions with Möbius band with 2 corner points as the quotient orbifold

This case concerns an NEC-group Λ with signature $(1; -; [\,]; \{(2,2)\})$ from Lemma 4.3. We denote the canonical reflections c_{1i} simply by c_i for $i = 0, 1, 2$. After ruling out the redundant generator e_1, we can write a presentation for Λ as

$$\langle d, c_0, c_1, c_2 \mid c_0^2 = c_1^2 = c_2^2 = (c_0c_1)^2 = (c_1c_2)^2 = 1, c_0d^2 = d^2c_2\rangle.$$

Proof of Theorem 2.3. The proof is very similar to that of Theorem 2.2 above. By Lemma 4.4 every BSK-map $\theta : \Lambda \to \mathbb{Z}_N$ is equivalent to one of the form

$$\theta(c_0) = \theta(c_2) = N/2, \qquad \theta(c_1) = 0, \qquad \theta(d) = a,$$

for some $a \in \mathbb{Z}_N$. By (b) of Lemma 4.2, S has $N/2$ boundary components, and from the Hurwitz–Riemann formula we compute that its genus is either 2 if it is nonorientable, or 1 otherwise. There are two cases, according to the order of a, which is either N or $N/2$. If a has order N, then by composing θ with a suitable automorphism of $\mathbb{Z}_N$ we can assume that $\theta(d) = 1$. Observe that here $N/2$ can be arbitrary, but S is nonorientable if and only if $N/2$ is even, since only then $d^{N/2}c_0$ is a nonorientable word in $\ker\theta$. If the order of a is $N/2$ then it must be odd, and by composing θ with a suitable automorphism of $\mathbb{Z}_N$ we can assume that $\theta(d) = 2$. In such case $d^{N/2}$ is a nonorientable word in $\Gamma = \ker\theta$ and hence S is nonorientable. □

6.4 Actions with a 1-punctured disc with 2 corner points as the quotient orbifold

This is the case concerning an NEC-group Λ with the signature

$$(0; +; [m]; \{(2, 2)\})$$

from Lemma 4.3. We denote the canonical reflections c_{1i} simply by c_i for $i = 0, 1, 2$. After ruling out the redundant generator e_1, we can rewrite the presentation for Λ as

$$\langle x, c_0, c_1, c_2 \mid x^m = c_0^2 = c_1^2 = c_2^2 = (c_0c_1)^2 = (c_1c_2)^2 = 1, c_0x = xc_2\rangle.$$

Proof of Theorem 2.4. Let $\theta : \Lambda \to \mathbb{Z}_N$ be a BSK-epimorphism. Then $\mathbb{Z}_N$ is generated by $\theta(x)$, which has order m, and $\theta(c_i)$ for some canonical reflection c_i which has order 2. Thus either $N = m$ if m is even, or $N = 2m$ if m is odd. By Lemma 4.4, we can assume that $\theta(c_0) = \theta(c_2) = N/2$, $\theta(c_1) = 0$. If $m = N$ then after composing θ with a suitable automorphism of $\mathbb{Z}_N$ we can assume that $\theta(x) = 1$, and hence the action is unique. By (a) of Lemma 4.2, S is nonorientable in this case, since $x^{N/2}c_0$ is a nonorientable word in $\ker\theta$. Now assume that $N = 2m$. Then after composing θ with a suitable automorphism of $\mathbb{Z}_N$ we can assume that $\theta(x) = 2$, and hence also in this case the action is unique up to topological conjugation. Observe that now S is orientable, by (a) of Lemma 4.2. Finally, by (b) of Lemma 4.2, S has $N/2$ boundary components in both cases, and its genus can be easily computed from the Hurwitz–Riemann formula. □

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6.5 Actions with a 1-punctured disc with 4 corner points as the quotient orbifold

This is the case concerning an NEC-group Λ with the signature

$$(0; +; [m]; \{(2, 2, 2, 2)\})$$

from Lemma 4.3. This case is very similar to that from the previous section. Now Λ has the presentation

$$\langle x, c_0, \ldots, c_4 \mid \\ x^m = c_0^2 = \cdots = c_4^2 = (c_0c_1)^2 = (c_1c_2)^2 = (c_2c_3)^2 = (c_3c_4)^2 = 1, c_0x = xc_4\rangle.$$

The proof of Theorem 2.5 is almost identical as that of Theorem 2.4. We leave details to the reader.

6.6 Actions with 1-punctured Möbius band as the quotient orbifold

These actions correspond to an NEC-group Λ with signature

$$(1; -; [m]; \{(\,)\})$$

from Lemma 4.3 which has the presentation

$$\langle x, d, c, e \mid xed^2 = x^m = c^2 = 1, ec = ce\rangle.$$

We have $\mu(\Lambda) = (m-1)/m$.

Proof of Theorem 2.6. Suppose that $\theta : \Lambda \to \mathbb{Z}_N$ is a BSK-epimorphism, such that $\Gamma = \ker\theta$ is an orientable bordered surface group with k empty period cycles. Since Γ contains a reflection, we have $\theta(c) = 0$, and by (b) of Lemma 4.2, $\theta(e)$ has order $n = N/k$. In particular, k divides N. Note that $\theta(x)$ and $\theta(e)$ generate a subgroup of index at most 2 of $\mathbb{Z}_N$. Since S is orientable, this has to be a proper subgroup of $\mathbb{Z}_N$, for otherwise Γ would contain a nonorientable word of the form wd, where w is a word in x and e such that $\theta(w) = -\theta(d)$. It follows that $N = 2\mathrm{lcm}(m, n)$.

By multiplying θ by an element of $\mathbb{Z}_N^*$, we can assume that

$$\theta(x) = N/m, \qquad \theta(e) = aN/n, \qquad \theta(d) = b, \tag{9}$$

for some $a \in \mathbb{Z}_n^*$ and some b for which

$$N/m + aN/n + 2b \equiv 0\ (N). \tag{10}$$

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We denote such BSK-map by θ_a, bearing in mind that the parameter a does not always determine it uniquely. Indeed, we have

$$b = -N/2m - aN/2n + \varepsilon N/2 = -n/t - am/t + \varepsilon N/2$$

for some $\varepsilon \in \mathbb{Z}_2$.

If $N/2$ is odd then for arbitrary $a \in \mathbb{Z}_n^*$ we have a unique odd b, namely $\varepsilon = 0$ if a is even, and $\varepsilon = 1$ if a is odd. If $N/2$ is even but t is odd, then b is odd for arbitrary $a \in \mathbb{Z}_n^*$ and $\varepsilon \in \mathbb{Z}_2$. Finally, suppose that t is even. Then a must be odd since $(a, n) = 1$, and b is odd if and only if n/t and m/t have opposite parity. But since n/t and m/t are relatively prime, b is odd if and only if $mn/t^2 = N/2t$ is even and ε arbitrary.

Summarizing the above paragraph, we conclude that if $N/2$ is odd, then $\theta_a(d)$ is uniquely determined by a, whereas if $N/2$ is even, then $\theta_a(d)$ is determined only modulo $N/2$. However, the two different possibilities for $\theta_a(d)$ define equivalent BSK-epimorphisms. Indeed, set $c = 1 + N/2$ and note that $c \equiv 1\,(m)$, $c \equiv 1\,(n)$ and c is odd, hence $c \in \mathbb{Z}_N^*$. Furthermore, we have $c\theta_a(x) = N/m$, $c\theta_a(e) = aN/n$ and $c\theta_a(d) = \theta_a(d) + N/2$, because $\theta_a(d)$ is odd.

Now, we determine the number of equivalence classes of BSK-epimorphisms. For ϕ representing an element of $\mathrm{Out}(\Lambda)$ and a BSK-map $\theta : \Lambda \to \mathbb{Z}_N$, let $\theta^\phi = \theta \circ \phi$. Since θ is into an abelian target, for the generators of $\mathrm{Out}(\Lambda)$ given in Lemma 5.2 we have

$$\theta^\gamma : \begin{cases} x \mapsto -\theta(x), \\ e \mapsto -\theta(e), \\ d \mapsto -\theta(d), \end{cases} \qquad \theta^\delta : \begin{cases} x \mapsto \theta(x), \\ e \mapsto -\theta(e), \\ d \mapsto -\theta(d) - \theta(x). \end{cases}$$

In particular, for every ϕ we have $\theta^\phi(x) = \pm\theta(x)$ and $\theta^\phi(e) = \pm\theta(e)$.

We claim that θ_a and $\theta_{a'}$ are equivalent if and only if $a \equiv \pm a'\,(t)$. For suppose that $\theta_{a'} = c\theta_a^\phi$ for some $c \in \mathbb{Z}_N^*$ and $\phi \in \mathrm{Aut}(\Lambda)$. Then, by replacing c by $-c$ if necessary, we may assume that $\theta_{a'}(x) = c\theta_a(x)$ and $\theta_{a'}(e) = \pm c\theta_a(e)$, which gives $c \equiv 1\,(m)$ and $a' \equiv \pm ca\,(n)$. It follows that $a \equiv \pm a'\,(t)$. Conversely, suppose that $a \equiv \pm a'\,(t)$. Then by Lemma 3.1 there exists c, such that $c \equiv 1\,(m)$ and $c \equiv \pm a' a^{-1}\,(n)$, where a^{-1} denotes the inverse of a modulo n. Note that such c is relatively prime to m and n, and hence to $N/2$. If $N/2$ is odd, then we can take c to be odd as well, so that $c \in \mathbb{Z}_N^*$. Now $\theta_{a'}(x) = c\theta_a^\phi(x)$ and $\theta_{a'}(e) = c\theta_a^\phi(e)$, where $\phi = \mathrm{id}$ or $\phi = \delta$. If $N/2$ is odd, then necessarily also $\theta_{a'}(d) = c\theta_a^\phi(d)$, whereas if $N/2$ is even, then possibly $c\theta_a^\phi(d) = \theta_{a'}(d) + N/2$, in which case it suffices to replace c by $c + N/2$.

Summarizing, on one hand each element of $\mathbb{Z}_t^*$ is the residue mod t of some $a \in \mathbb{Z}_n^*$ defining BSK-map θ_a by Lemma 3.2. On the other hand θ_a and $\theta_{a'}$ are equivalent if and only if $a \equiv \pm a'\ (t)$. So the elements of the quotient group $\mathbb{Z}_t^*/\{\pm 1\}$ parametrize the equivalence classes of BSK-maps (although it might happen that for a particular representative $x \in \mathbb{Z}_t^*$, θ_x is not a BSK-epimorphism, because $x \notin \mathbb{Z}_n^*$). Therefore, we have $\varphi(t)/2$ classes for $t > 1$ and 1 class for $t = 1$. $\square$

Proof of Theorem 2.7. Suppose that $\theta : \Lambda \to \mathbb{Z}_N$ is a BSK-epimorphism, such that $\Gamma = \ker\theta$ is a nonorientable bordered surface group with k empty period cycles. As in the proof of the previous theorem, we have $\theta(c) = 0$ and $\theta(e)$ has order $n = N/k$. Since Γ contains a nonorientable word, $\theta(d)$ is equal to $\theta(w)$ for some word w in x and e. It follows that $\mathbb{Z}_N$ is generated by $\theta(x)$ and $\theta(e)$, hence $N = \mathrm{lcm}(m, n)$. By multiplying θ by an element of $\mathbb{Z}_N^*$, we can assume that $\theta = \theta_a$ defined by (9) for some $a \in \mathbb{Z}_n^*$ and some b for which (10) is satisfied.

Now, if N is odd then for arbitrary $a \in \mathbb{Z}_n^*$ we have a unique b satisfying (10). By the same argument as in the previous proof, θ_a is equivalent to $\theta_{a'}$ if and only if $a \equiv \pm a'\ (t)$, and hence there are $\varphi(t)/2$ equivalence classes of BSK-epimorphisms if $t > 1$, and one such class if $t = 1$.

For the rest of the proof assume that N is even. By (10), $N/m + aN/n = n/t + am/t$ is even, which is possible if and only if n/t and m/t are both odd, hence $nm/t^2 = N/t$ is odd. Now $\theta_a(d)$ is determined by a only modulo $N/2$:

$$b = -\frac{1}{2}\left(\frac{N}{m} + a\frac{N}{n}\right) + \varepsilon\frac{N}{2}$$

for some $\varepsilon \in \mathbb{Z}_2$. We claim that given $a, a' \in \mathbb{Z}_n^*$, θ_a and $\theta_{a'}$ are equivalent if and only if either

(1) $a \equiv a'\ (t)$ and $\theta_{a'}(d) = c\theta_a(d)$, where c is the unique element of $\mathbb{Z}_N^*$ satisfying $c \equiv 1\ (m)$ and $ca \equiv a'\ (n)$; or
(2) $a \equiv -a'\ (t)$ and $\theta_{a'}(d) = c(\theta_a(d) + \theta_a(x))$, where c is the unique element of $\mathbb{Z}_N^*$ satisfying $c \equiv -1\ (m)$ and $ca \equiv a'\ (n)$.

To prove the claim suppose that $\theta_{a'} = c\theta_a^\phi$ for some $c \in \mathbb{Z}_N^*$ and $\phi \in \mathrm{Aut}(\Lambda)$. By Lemma 5.2, we may suppose that $\phi \in \{1, \gamma, \delta, \delta\gamma\}$. If $\phi = 1$ or $\phi = \gamma$, then after replacing c by $-c$ in the latter case, we have $\theta_{a'}(x) =$

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$c\theta_a(x)$, $\theta_{a'}(e) = c\theta_a(e)$ and $\theta_{a'}(d) = c\theta_a(d)$. Thus c satisfies $c \equiv 1\ (m)$ and $ca \equiv a'\ (n)$. By Lemma 3.1, such (unique) c exists if and only if $a \equiv a'\ (t)$. Similarly, if $\phi = \delta$ or $\phi = \delta\gamma$, then after replacing c by $-c$ in the former case, we have $\theta_{a'}(x) = -c\theta_a(x)$, $\theta_{a'}(e) = c\theta_a(e)$ and $\theta_{a'}(d) = c\big(\theta_a(d) + \theta_a(x)\big)$. Such (unique) c again exists if and only if $a \equiv -a'\ (t)$. This completes the proof of the claim.

Suppose $t > 2$. It follows from the previous paragraph that there is a surjection π from the set of equivalence classes of BSK-maps onto $\mathbb{Z}_t^*/\{\pm 1\}$, defined by $\pi([\theta_a]) = [[a]_t]$, where $a \in \mathbb{Z}_n^*$. We claim that π is a 2-over-one map. For let θ_a be a BSK-map defined by (9) and define θ'_a by

$$\theta'_a(x) = N/m, \qquad \theta'_a(e) = aN/n, \qquad \theta'_a(d) = b + N/2.$$

Evidently $\pi(\theta_a) = \pi(\theta'_a)$, but θ_a is not equivalent to θ'_a. For if they were equivalent, then (1) would be satisfied with $c = 1$, hence $b = b + N/2$. Now if $\pi(\theta_{a'}) = \pi(\theta_a)$ for some $a' \in \mathbb{Z}_n^*$, then $\theta_{a'}$ is equivalent either to θ_a or to θ'_a, by (1) if $a' \equiv a\ (t)$, or by (2) if $a' \equiv -a\ (t)$.

Finally, suppose $t = 2$. By (1) every BSK-map is equivalent to $\theta : \Lambda \to \mathbb{Z}_N$ such that $\theta(x) = N/m$ and $\theta(e) = N/n$. Fix such θ and define θ' by $\theta'(x) = \theta(x)$, $\theta'(e) = \theta(e)$, and $\theta'(d) = \theta(d) + N/2$. We have to show that θ and θ' are equivalent. Let c be the unique element of $\mathbb{Z}_N^*$ such that $c \equiv -1\ (m)$ and $c \equiv 1\ (n)$. By (2) it suffices to show that $\theta'(d) = c(\theta(d) + \theta(x))$. We have

$$2c\theta(d) = -c(\theta(x) + \theta(e)) = \theta(x) - \theta(e) = 2(\theta(x) + \theta(d)).$$

Either $c\theta(d) = \theta(d) + \theta(x)$ or $c\theta(d) = \theta(d) + \theta(x) + N/2$. The former equality is not possible, because $\theta(x) = N/m$ is odd and $\theta(d)(c-1)$ is even. Hence

$$c(\theta(d) + \theta(x)) = c\theta(d) - \theta(x) = \theta(d) + N/2 = \theta'(d).$$

It follows that all BSK-maps $\Lambda \to \mathbb{Z}_N$ are equivalent. ∎

6.7 Actions with a 2-punctured disc as the quotient orbifold

This case concerns an NEC-group Λ with signature $(0; +; [m, n]; \{(\,)\})$ from Lemma 4.3 which has the presentation

$$\langle x_1, x_2, c, e \mid x_1^m = x_2^n = c^2 = x_1 x_2 e = 1, ec = ce\rangle.$$

We have $\mu(\Lambda) = 1 - 1/m - 1/n$.

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Proof of Theorem 2.8. Suppose that $\theta : \Lambda \to \mathbb{Z}_N$ is a BSK-map. Since $\ker\theta$ contains a reflection, $\theta(c) = 0$ and it follows by (a) of Lemma 4.2 that S is orientable. By (b) of Lemma 4.2 we have $k = N/l$, where l is the order of $\theta(e)$. Since θ is a surjection, $\mathbb{Z}_N$ is generated by $\theta(x_1)$, $\theta(x_2)$ which have orders m and n respectively and $\theta(e) = -(\theta(x_1) + \theta(x_2))$ has order l. It follows that the conditions (i) and (ii) of Lemma 3.3 are satisfied, in particular $N = \mathrm{lcm}(m, n)$. Let (A, A_1, A_2, A_3) be the Maclachlan decomposition of (m, n, l), as above. We have $t = AA_3$, $A_1 = n/t$, $A_2 = m/t$ and $A_3 = k$. We see that k divides t, and because A_3 is relatively prime to A_1A_2, k is relatively prime to $nm/t^2 = N/t$. It follows that k divides $t/(t, N/t)$. Finally, if N is even, then by (ii) of Lemma 3.3, one of the numbers k, n/t, m/t must be even. It follows that k must be even if $N/t = nm/t^2$ is odd. Conversely, having k, n, m, N satisfying the conditions of the theorem, one can easily define, using Lemma 3.3, an appropriate BSK-map defining a surface and an action in question.

Every BSK-map is equivalent to $\theta_a : \Lambda \to \mathbb{Z}_N$ defined by

$$\theta_a(x_1) = A_1, \qquad \theta_a(x_2) = aA_2, \qquad \theta_a(e) = -(A_1 + aA_2), \qquad \theta_a(c) = 0$$

for some $a \in \mathcal{L}$, where

$$\mathcal{L} = \{a \in \mathbb{Z}_n^* \mid A_1 + aA_2 \text{ has order } l\}.$$

Suppose first that $m \neq n$. Let $S = \{c \in \mathbb{Z}_N^* \mid c \equiv 1\,(m)\}$. For $a, b \in \mathcal{L}$, we claim that θ_a and θ_b are equivalent if and only if $b \equiv ca\,(n)$ for some $c \in S$. Indeed, suppose that $\theta_b = c\theta_a^{\phi}$ for some $\phi \in \mathrm{Aut}(\Lambda)$ and $c \in \mathbb{Z}_N^*$. By Lemma 5.4, either $\theta_a(\phi(x_i)) = \theta_a(x_i)$ for $i = 1, 2$, or $\theta_a(\phi(x_i)) = -\theta_a(x_i)$ for $i = 1, 2$. By changing c to $-c$ in the latter case, we have $A_1 = cA_1$ and $bA_2 = caA_2$, and the claim follows. Thus, the equivalence classes of BSK-maps are parametrized by the orbits of the action of S on $\mathcal{L}$. Since this action is free, the number of orbits is $|\mathcal{L}|/|S|$. Let $B = A/C = t/kC$ and write $B = B_1B_2B_3$, where for $i = 1, 2, 3$ each prime dividing B_i divides A_i. By [2, Theorem 3.4] we have

$$|\mathcal{L}| = \varphi(A_1B)\psi(C) = \varphi(A_1B_1)\varphi(B_2)\varphi(B_3)\psi(C).$$

We also have

$$|S| = \frac{\varphi(N)}{\varphi(m)} = \frac{\varphi(A_1B_1)\varphi(A_2B_2)\varphi(A_3B_3)\varphi(C)}{\varphi(B_1)\varphi(A_2B_2)\varphi(A_3B_3)\varphi(C)} = \frac{\varphi(A_1B_1)}{\varphi(B_1)}$$

$$|\mathcal{L}|/|S| = \varphi(B_1)\varphi(B_2)\varphi(B_3)\psi(C) = \varphi(B)\psi(C).$$

This completes the proof in the case $m \neq n$.

Now suppose that $m = n$. This common value is equal to N and we have $A_1 = A_2 = 1$ and

$$\mathcal{L} = \{a \in \mathbb{Z}_N^* \mid 1 + a \text{ has order } l\}.$$

Now $\theta_a(\beta(x_1)) = \theta_a(x_2)$ and $\theta_a(\beta(x_2)) = \theta_a(x_1)$ for $\beta \in \mathrm{Aut}(\Lambda)$ from Lemma 5.4. Consequently, θ_a and θ_b are equivalent if and only if either $a = b$ or $ab = 1$. It follows that the number of equivalence classes of BSK-maps is $(|\mathcal{L}| + I)/2$, where I is the number of $a \in \mathcal{L}$ for which $a^2 = 1$. As in the case $m \neq n$, we have $|\mathcal{L}| = \varphi(B)\psi(C)$, where C is the biggest divisor of l coprime with k, and $B = l/C$.

In order to compute I, suppose that $a^2 = 1$ for some $a \in \mathcal{L}$. We have $N = kBC$, and since kB and C are coprime, $\mathbb{Z}_N \cong \mathbb{Z}_{kB} \oplus \mathbb{Z}_C$. Under this isomorphism, we write $a = (a_1, a_2)$, where $a_1 \in \mathbb{Z}_{kB}^*$ and $a_2 \in \mathbb{Z}_C^*$. We have $a_1^2 \equiv 1\ (kB)$ and $a_2^2 \equiv 1\ (C)$. Since $1 + a$ has order l, we have $1 + a_1 = ks$ for some $s \in \mathbb{Z}_B^*$ and $1 + a_2 \in \mathbb{Z}_C^*$.

In suitable rings we have

$$0 = 1 - a_i^2 = (1 + a_i)(1 - a_i).$$

Since $(1 + a_2)$ is invertible in $\mathbb{Z}_C$, $a_2 = 1$. In $\mathbb{Z}_{kB}$ we have

$$0 = (1 + a_1)(1 - a_1) = ks(2 - ks).$$

Since s is invertible, it follows that B divides $2 - ks$, hence $(B, k) \leqslant 2$. Observe that every prime divisor of B divides k, hence also (B, k). It follows that B is a power of 2. If $B \leqslant 2$ then $s = 1$ and $a = (a_1, a_2) = (k - 1, 1)$. If $B = 2^z$ for $z > 1$, then $k/2$ is coprime to $B/2$. Let $k' \in \mathbb{Z}_{B/2}^*$ denote the inverse of $k/2$. Then, since $B/2$ divides $1 - sk/2$, we have $s \equiv k'\ (B/2)$, and hence $s = k'$ or $s = k' + B/2$. Summarizing, we have

$$I = \begin{cases} 2 & \text{for } B = 2^z, z > 1, \\ 1 & \text{for } B \leqslant 2, \\ 0 & \text{otherwise.} \end{cases}$$

To finish the proof, observe that since $\psi(C)$ is odd, and $\varphi(B)$ is even if and only if $B > 2$, we have $I \equiv \varphi(B)\psi(C) \pmod 2$. It follows that $(|\mathcal{L}| + I)/2 = \lceil |\mathcal{L}|/2 \rceil$ if $I \neq 2$. ∎

6.8 Actions with a 1-punctured annulus as the quotient orbifold

This case concerns an NEC-group Λ with the signature

$$(0; +; [m]; \{(\,), (\,)\})$$

from Lemma 4.3 which has the presentation

$$\langle x, e_1, e_2, c_1, c_2 \mid xe_1e_2 = x^m = c_1^2 = c_2^2 = 1, e_1c_1 = c_1e_1, e_2c_2 = c_2e_2\rangle.$$

We have $\mu(\Lambda) = (m-1)/m$.

Proof of Theorem 2.9. Suppose that $\theta : \Lambda \to \mathbb{Z}_N$ is a BSK-map, such that $\Gamma = \ker\theta$ is a nonorientable bordered surface group. Then, since Γ is bordered, some of the canonical reflections, say c_1 belongs to Γ. Furthermore, by (b) of Lemma 4.2, the order of $\theta(e_1)$ is N/k. But since Γ is nonorientable, it contains a nonorientable word, by (a) of Lemma 4.2, which is possible if and only if $\theta(c_2) = N/2$ and $N/2$ is in the subgroup of $\mathbb{Z}_N$ generated by $\theta(x)$ and $\theta(e_1)$, and thus we obtain the condition $N = \mathrm{lcm}(m, N/k)$. Conversely, if the last condition is satisfied, then for $a \in \mathbb{Z}_{N/k}^*$ we can define a BSK-map

$$\theta_a(x) = N/m, \qquad \theta_a(e_1) = ak, \qquad \theta_a(e_2) = -(N/m + ak),$$
$$\theta_a(c_1) = 0, \qquad \theta_a(c_2) = N/2,$$

and every BSK-map is equivalent to some θ_a. Let $a, a' \in \mathbb{Z}_{N/k}^*$ and suppose that $\theta_{a'} = c\theta_a\phi$ for some $c \in \mathbb{Z}_N^*$ and some $\phi \in \mathrm{Aut}(\Lambda)$. By Lemma 5.3, $\theta_a\phi$ maps (x, e_1) on $\pm(N/m, ak)$ and so by replacing c by $-c$ if necessary in the latter case, we obtain that

$$\theta_{a'}(x) = cN/m, \qquad \theta_{a'}(e_1) = cak, \qquad \theta_a(e_2) = -c(N/m + ak),$$
$$\theta_a(c_1) = 0, \qquad \theta_a(c_2) = N/2,$$

which give $c \equiv 1\ (m)$ and $a' \equiv ca\ (N/k)$. As in the proof of Theorem 2.6, we conclude that θ_a and $\theta_{a'}$ are equivalent if and only if $a' \equiv a\ (t)$, and hence, the number of equivalence classes of such BSK-maps is $\varphi(t)$. □

As we already mentioned in Section 2, the case of orientable S is much more involved.

Proof of Theorem 2.10. Suppose that an action exists and let $\theta : \Lambda \to \mathbb{Z}_N$ be the corresponding BSK-map. Since S is bordered, $\ker\theta$ contains at least one canonical reflection, and we can assume $\theta(c_1) = 0$. We consider two cases: (1) $\theta(c_2) \neq 0$ and (2) $\theta(c_2) = 0$.

Case 1. $\theta(c_2) \neq 0$. By (b) of Lemma 4.2, $\theta(e_1)$ has order N/k, in particular $k|N$. Since S is orientable, $\theta(x)$ and $\theta(e_1)$ generate the subgroup of index 2 of $\mathbb{Z}_N$, by (a) of Lemma 4.2. Hence $N = 2\text{lcm}(m, N/k)$, and $\theta(c_2) = N/2$ is odd. Conversely, if (1) is satisfied, then for each $a \in \mathbb{Z}^*_{N/k}$ we can define a BSK-map $\theta^1_a : \Lambda \to \mathbb{Z}_N$ by

$$\theta^1_a(x) = N/m, \qquad \theta^1_a(e_1) = ak, \qquad \theta^1_a(e_2) = -(N/m + ak),$$
$$\theta^1_a(c_1) = 0, \qquad \theta^1_a(c_2) = N/2.$$

Case 2. $\theta(c_2) = 0$. Let l_i denote the order of $\theta(e_i)$ and set $n_i = N/l_i$ for $i = 1, 2$. Then, by (b) of Lemma 4.2 we have $n_1 + n_2 = k$. Now $\theta(x)$, $\theta(e_1)$ and $\theta(e_2)$ generate $\mathbb{Z}_N$ and hence the triple (l_1, l_2, m) satisfies the conditions of Lemma 3.3. Consider the Maclachlan decomposition of (l_1, l_2, m)

$$l_1 = AA_2A_3, \qquad l_2 = AA_1A_3, \qquad m = AA_1A_2.$$

We have $A_1 = n_1 = n$, $A_2 = n_2$, $A_3 = N/m$ and the conditions (a), (b), (c) follow from the properties of the Maclachlan decomposition. Conversely, if (2) is satisfied, then by Lemma 3.3, $\mathbb{Z}_N$ is generated by three elements a, b and c of orders m, l_1 and l_2, respectively such that $a + b + c = 0$. We define $\theta^2 : \Lambda \to \mathbb{Z}_N$ by

$$\theta^2(x) = a, \qquad \theta^2(e_1) = b, \qquad \theta^2(e_2) = c, \qquad \theta^2(c_1) = \theta^2(c_2) = 0.$$

This completes the proof of assertion (i).

Now we shall find the number of conjugacy classes of actions. The proof of assertion (ii) is analogous to that of Theorem 2.9 and we omit it.

To prove (iii), consider the Maclachlan decomposition (A, A_1, A_2, A_3) of the triple $(N/n, N/(k-n), m)$. We have $A_1 = n$, $A_2 = k - n$, $A_3 = N/m$ and $A = m/(n(k-n))$. Every BSK-map is equivalent to $\theta^2_a : \Lambda \to \mathbb{Z}_N$ defined by

$$\theta^2_a(e_1) = A_1, \qquad \theta^2_a(e_2) = aA_2, \qquad \theta^2_a(x) = -(A_1 + aA_2),$$
$$\theta^2_a(c_1) = \theta^2_a(c_2) = 0$$

for some $a \in \mathcal{L}$, where

$$\mathcal{L} = \{a \in \mathbb{Z}^*_{N/(k-n)} \mid A_1 + aA_2 \text{ has order } m\}.$$

It follows form Lemma 5.3, that for $a, a' \in \mathcal{L}$, θ^2_a is equivalent to $\theta^2_{a'}$ if and only if $a' = ca$ for some $c \equiv 1\ (N/n)$, or $aa' = 1$, the latter being possible only for $n = n - k = 1$. Now, the formulas for the number of equivalence classes of BSK-maps can be obtained by repeating the calculations from the proof of Theorem 2.8. The assertion (iv) is evident. ∎

Theorem 2.10 has some delicate subtlety which we illustrate with two remarks and two examples.

REMARK 6.1. For some triples (N, m, k) both conditions (1) and (2) are satisfied, as for instance in Example 6.3 below. In such a case, θ^1 and θ^2 are not equivalent. For suppose that $\theta^1 = c\theta^2\phi$ for some $c \in \mathbb{Z}_N^*$ and some $\phi \in \mathrm{Aut}(\Lambda)$. By Lemma 5.3, ϕ preserves $\{c_1, c_2\}$, hence $\theta^1(c_i) = c\theta^2\phi(c_i) = 0$ for $i = 1, 2$. This is a contradiction, because $\theta^1(c_1) \neq \theta^1(c_2)$.

REMARK 6.2. Suppose that N, m, k satisfy the condition (2), and let θ_1^2 and θ_2^2 be BSK-maps, where $\theta_1^2(e_1)$, $\theta_1^2(e_2)$ have orders N/n_1, N/n_2, where $n_1 + n_2 = k$ and $\theta_2^2(e_1)$, and $\theta_2^2(e_2)$ have orders N/n_1', N/n_2', where $n_1' + n_2' = k$. If $\{n_1, n_2\} \neq \{n_1', n_2'\}$ then θ_1^2 and θ_2^2 are not equivalent. This follows from Lemma 5.3, because for every $\phi \in \mathrm{Aut}(\Lambda)$, $\theta_1^2\phi$ maps (e_1, e_2) on $\pm(\theta_1^2(e_1), \theta_1^2(e_2))$.

EXAMPLE 6.3. Suppose $k = 2$, $2m \mid N$ and $N/2$ is odd. Then both conditions (1) and (2) are satisfied ($n = 1$ in (2)). The number of BSK-maps of type (1) is $\varphi(m)$ by the assertion (ii), and the number of BSK-maps of type (2) is $\lceil \varphi(m/C)\psi(C)/2 \rceil$, where C is the biggest divisor of m coprime to N/m, by the assertion (iii). By adding up these two numbers we obtain the total number of topological types of $\mathbb{Z}_N$-action on S, with the prescribed quotient orbifold. By the Hurwitz–Riemann formula, the genus of S is $N(m-1)/2m$.

EXAMPLE 6.4. Consider $m = N = 12$ and $k = 7$. Then (1) is not satisfied, but (2) is by two different pairs $\{n, k - n\}$, namely $\{1, 6\}$ and $\{3, 4\}$. By the assertion (iii) of Proposition 2.10, for each of these pairs, the corresponding BSK-map is unique up to equivalence. Thus we have two different topological types of $\mathbb{Z}_{12}$-action on S, with the prescribed quotient orbifold. By the Hurwitz–Riemann formula, the genus of S is 3.

6.9 Actions with a 3-punctured disc as the quotient orbifold

This subsection concerns NEC-groups Λ with the signatures

$$(0; +; [2, 3, m]; \{(\,)\}) \quad \text{for } m = 3, 4, 5 \qquad \text{and} \qquad (0; +; [2, 2, m]; \{(\,)\})$$

from Lemma 4.3 which have the presentation

$$\langle x_1, x_2, x_3, e, c \mid x_1^2 = x_2^n = x_3^m = x_1x_2x_3e = c^2 = 1, ce = ec\rangle,$$

where $n = 2$ or $n = 3$.

Proof of Theorem 2.11. Suppose that $\theta : \Lambda \to \mathbb{Z}_N$ is a BSK-map, such that $\ker \theta$ is a bordered surface group. Then $\theta(c) = 0$, $\theta(x_1) = \theta(x_2) = N/2$, and by multiplying θ by an element of $\mathbb{Z}_N^*$, we may assume $\theta(x_3) = N/m$, and hence $\theta(e) = -N/m$. Evidently, such BSK-map is unique up to equivalence. Since θ is an epimorphism, we have $N = \mathrm{lcm}(2, m)$. By Lemma 4.2, S is orientable and has N/m boundary components. The genus of S is uniquely determined by the Hurwitz–Riemann formula. □

Proof of Theorem 2.12. Here the cyclic group $\mathbb{Z}_N$ is generated by three elements of orders $2, 3$ and m and hence $N = \mathrm{lcm}(2, 3, m)$. For any BSK-map $\theta : \Lambda \to \mathbb{Z}_N$ we have $\theta(c) = 0$, and it follows from Lemma 4.2, that S is orientable and its number of boundary components is N/l, where l is the order of $\theta(e) = -(\theta(x_1) + \theta(x_2) + \theta(x_3))$. We have $\theta(x_1) = N/2$, and by multiplying θ by a suitable element of $\mathbb{Z}_N^*$ we may assume $\theta(x_2) = N/3$.

For Λ with the signature $(0; +; [2, 3, 3]; \{(\,)\})$, any BSK-epimorphism $\theta : \Lambda \to \mathbb{Z}_6$ is equivalent to one mapping (x_1, x_2, x_3) either on $(3, 2, 2)$ or $(3, 2, 4)$. In the former case we have $\theta(e) = 5$ and S has 1 boundary component and genus 3. In the later case we have $\theta(e) = 3$ and S has 3 boundary component and genus 2.

If Λ has signature $(0; +; [2, 3, 4]; \{(\,)\})$ or $(0; +; [2, 3, 5]; \{(\,)\})$, then by Chinese reminder theorem, there is $c \in \mathbb{Z}_N^*$ such that $c\theta$ maps (x_1, x_2, x_3) on $(N/2, N/3, N/m)$. In both cases we have $\theta(e) = -1$, hence S has one boundary component. The genus of S is easily computed from the Hurwitz–Riemann formula. □

6.10 Actions with a 2-punctured disc with two corners as the quotient orbifold

This case concerns NEC-groups Λ with signatures

$$(0; +; [3, m]; \{(2, 2)\}) \quad \text{for } m = 3, 4, 5 \qquad \text{and} \qquad (0; +; [2, m]; \{(2, 2)\})$$

from Lemma 4.3 which have the presentation

$$\langle x_1, x_2, e, c_0, c_1, c_2 \mid x_1^n = x_2^m = c_0^2 = c_1^2 = c_2^2 = (c_0c_1)^2 = (c_1c_2)^2 = x_1x_2e = 1, c_2e = ec_0\rangle,$$

where $n = 2$ or $n = 3$.

Proof of Theorem 2.13. Suppose that $\theta : \Lambda \to \mathbb{Z}_N$ is a BSK-map, such that $\ker\theta$ is a bordered surface group. By Lemma 4.4, we may assume that $\theta(c_0) = \theta(c_2) = N/2$ and $\theta(c_1) = 0$. We have $\theta(x_1) = N/2$, and by multiplying θ by an element of $\mathbb{Z}_N^*$, we may assume $\theta(x_2) = N/m$, and hence $\theta(e) = N/2 - N/m$. Evidently, such BSK-map is unique up to equivalence. Since θ is an epimorphism, we have $N = \mathrm{lcm}(2, m)$. By Lemma 4.2, S has $N/2$ boundary components and is nonorientable, as x_1c_0 is a nonorientable word in $\ker\theta$. The genus of S is uniquely determined by the Hurwitz–Riemann formula. ∎

Proof of Theorem 2.14. Suppose that $\theta : \Lambda \to \mathbb{Z}_N$ is a BSK-map, such that $\ker\theta$ is a bordered surface group. As in the previous proof, we can assume $\theta(c_0) = \theta(c_2) = N/2$, $\theta(c_1) = 0$ and $\theta(x_1) = N/3$. Since θ is an epimorphism, $\mathbb{Z}_N$ is generated by three elements of orders 2, 3 and m and hence $N = \mathrm{lcm}(2, 3, m)$. By (b) of Lemma 4.2, S has $N/2$ boundary components.

For Λ with the signature $(0; +; [3, 3]; \{(\,)\})$, any BSK-epimorphism $\theta : \Lambda \to \mathbb{Z}_6$ is equivalent to one mapping (x_1, x_2) either on $(2, 2)$ or $(2, 4)$. By [3, Lemma 4.6 and Proposition 4.14], for every $\phi \in \mathrm{Aut}(\Lambda)$, $\phi(e)$ is conjugate to e or e^{-1}, and hence the order of $\theta(e)$ is an equivalence invariant. It follows that the two maps described above are not equivalent. Indeed, for the first one $\theta(e) = 2$ has order 3, whereas for the second one $\theta(e) = 0$ has order 1. In both cases S is orientable, because $\theta(x_1)$, $\theta(x_2)$ are even, whereas $\theta(c_0) = 3$ is odd, and hence there is no nonorientable word in $\ker\theta$.

If Λ has signature $(0; +; [3, 4]; \{(\,)\})$ or $(0; +; [3, 5]; \{(\,)\})$, then by Chinese reminder theorem, there is $c \in \mathbb{Z}_N^*$ such that $c\theta$ maps (x_1, x_2) on $(N/3, N/m)$. It follows that θ is unique up to equivalence. For $m = 4$ we have a nonorientable word $x_2^2c_0$ in $\ker\theta$, hence S is nonorientable. For $m = 5$ there is no such word, hence S is orientable. The genus of S is easily computed from the Hurwitz–Riemann formula. ∎

§7. On uniqueness of actions realizing the solutions of the minimum genus and maximum order problems

Throughout the rest of the paper the letter p, used before to denote algebraic genus, will be used also to denote a prime integer, which will not lead to any ambiguity; for the genus we assume $p \geqslant 2$.

7.1 The minimum genus and maximum order problems for finite groups acting on bordered surfaces

We start with the following easy proposition which justifies later definitions.

PROPOSITION 7.1. *Let G be a finite group. Then there exists a bordered topological surface S, which can be assumed to be orientable or not, such that G acts on S by homeomorphisms. Furthermore, if S is assumed to be orientable, then the action of G can be chosen to contain orientation-reversing elements if and only if G has a subgroup G' of index 2.*

Proof. Let $g_1, \ldots, g_r$ be a set of generators of G and $g_{r+1} = (g_1 \ldots g_r)^{-1}$. Clearly we can assume that $r \geqslant 2$. Let Λ be an NEC-group with the signature $(0; +; [\,]; \{(\,), \overset{r+1}{\ldots}, (\,)\})$ and let us define an epimorphism $\theta : \Lambda \to G$ mapping e_i to g_i for $i \leqslant r+1$ and all c_i to 1, the identity element of G. Then, for $\Gamma = \ker\theta$, we have $G \cong \Lambda/\Gamma$ acting as a group of conformal automorphisms on $S = \mathcal{H}/\Gamma$, which has the conformal structure of orientable bordered Klein surface inherited from the hyperbolic plane $\mathcal{H}$. Now assume that G contains a subgroup G' of index 2. Assume that G' is generated as above, let $x \in G \setminus G'$, consider an NEC-group Λ with signature $(1; -; [\,]; \{(\,), \overset{r+2}{\ldots}, (\,)\})$ and define an epimorphism $\theta : \Lambda \to G$ mapping all reflections c_i to 1, e_i to g_i for $1 \leqslant i \leqslant r+1$, e_{r+2} to x^{-2}, and d_1 to x. Then for $\Gamma = \ker\theta$, we have $G \cong \Lambda/\Gamma$ acting as a group of conformal or anticonformal automorphisms on $S = \mathcal{H}/\Gamma$, where x reverses orientation. If we need the action on a nonorientable surface, then it is sufficient to take an NEC-group Λ with signature $(1; -; [\,]; \{(\,), \overset{r+1}{\ldots}, (\,)\})$ and define θ on e_i, c_i as above and $\theta(d_1) = 1$, in virtue of (a) of Lemma 4.2. $\square$

So, let $\mathcal{K}_+(N)$ (resp. $\mathcal{K}_-(N)$) be the family of orientable (resp. nonorientable) bordered topological surfaces, admitting a self-homeomorphism of order N. Denote by $p = p(S)$ the algebraic genus of a bordered surface S and recall that it is the rank of the fundamental group of S equal to $\varepsilon g + k - 1$, where g is the topological genus of S, k is the number of its boundary components and $\varepsilon = 2$ or 1 if S is orientable or not. By $S_p^{\pm}$ will be denoted a bordered surface of algebraic genus p, orientable if the sign is $+$ and nonorientable otherwise, and similar meaning will have $S_{g,k}^{\pm}$.

Denote by $\mathrm{H_p}(S)$ the set of all periodic self-homeomorphisms of S and consider two of its subsets $\mathrm{H_p^+}(S)$ and $\mathrm{H_p^-}(S)$, consisting of respectively orientation-preserving and orientation-reversing self-homeomorphisms when

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S is orientable. Finally let

$$\mathcal{K}_+^+(N) = \{S \in \mathcal{K}_+(N) \mid \exists\, \varphi \in \mathrm{H}_\mathrm{p}^+(S) \text{ such that } \#(\varphi) = N\},$$
$$\mathcal{K}_+^-(N) = \{S \in \mathcal{K}_+(N) \mid \exists\, \varphi \text{ of } \mathrm{H}_\mathrm{p}^-(S) \text{ such that } \#(\varphi) = N\},$$

where the operator $\#$ stands for the order. With these notations we define:

$$p_+(N) = \min\{p(S) \mid S \in \mathcal{K}_+(N)\}, \qquad p_-(N) = \min\{p(S) \mid S \in \mathcal{K}_-(N)\}$$
$$p_+^+(N) = \min\{p(S) \mid S \in \mathcal{K}_+^+(N)\}, \qquad p_+^-(N) = \min\{p(S) \mid S \in \mathcal{K}_+^-(N)\}$$

and

$$p(N) = \min\{p_+(N), p_-(N)\}.$$

The calculation of the above five values is known as the minimal genus problem. A bordered surface S is called N-*minimal* if $p(S)$ attains $p(N)$, $p_\pm(N)$ or $p_+^\pm(N)$.

Another problem of a similar type is the maximum order problem which consists in finding, for a given p, the maximal order of a finite action on a bordered topological surface of algebraic genus p. For $G = \mathbb{Z}_N$ we refine this problem by considering

$$N_+^+(p) = \max\{N \mid S_p^+ \in \mathcal{K}_+^+(N)\}, \qquad N_+^-(p) = \max\{N \mid S_p^- \in \mathcal{K}_+^-(N)\}$$
$$N_+(p) = \max\{N_+^+(p), N_+^-(p)\}, \qquad N_-(p) = \max\{N \mid S_p^- \in \mathcal{K}_-(N)\}$$

and

$$N(p) = \max\{N_+(p), N_-(p)\}.$$

These problems, of minimal genus and maximal order, were solved in [4], and here we consider the question of uniqueness of topological type of self-homeomorphisms of maximal order and self-homeomorphisms acting on surfaces of minimal genus.

7.2 On topological type of cyclic actions of a given nonprime order on bordered orientable surfaces of minimal genus

THEOREM 7.2. [4, Theorem 3.2.5] *Let N be a nonprime odd integer and let p be the smallest prime dividing N. Then*

$$p_+^+(N) = p_+(N) = \begin{cases} (p-1)\dfrac{N}{p} & \text{if } p^2 \mid N, \\[2mm] (p-1)\dfrac{N-p}{p} & \text{if } p^2 \nmid N, \end{cases}$$

and the corresponding N-minimal surface has 1 *boundary component.*

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COROLLARY 7.3. *The action realizing $p_+^+(N)$ and $p_+(N)$ given in Theorem 7.2 is unique up to topological conjugation if p^2 does not divide N and there are $p-1$ classes of such action in the other case.*

Proof. In the proof of Theorem 7.2 given in [4] it is shown that the minimum genus is realized just for an NEC-group Λ having signature

$$(0; +; [p, q]; \{(\,)\}),$$

where $q = N$ if $p^2 \mid N$, and otherwise $q = N/p$. We have $k = 1$, and $t = (p, q)$ is equal to p and 1 respectively. By Theorem 2.8, there are $\varphi(t)$ topological types of action corresponding to this signature. ∎

THEOREM 7.4. [4, Theorem 3.2.6] *Let $N \neq 2$ be an even integer not divisible by 4. Then $p_+^+(N) = p_+^-(N) = N/2 - 1$. Moreover any N-minimal surface from $\mathcal{K}_+^+(N)$ has one boundary component, whilst any such surface from $\mathcal{K}_+^-(N)$ has $N/2$ boundary components.*

COROLLARY 7.5. *The actions realizing $p_+^+(N)$ and $p_+^-(N)$ given in Theorem 7.4 are unique up to topological conjugation.*

Proof. In the proof of Theorem 7.4 given in [4] it is shown that Λ determining the minimal genera must have signature $(0; +; [2, N/2]; \{(\,)\})$ in the case of $p_+^+(N)$ and $(0; +; [N/2]; \{(2, 2)\})$ in the case of $p_+^-(N)$. In the first case there is a unique class of such action by Theorem 2.8. In the second case the action is unique by Theorem 2.4. ∎

THEOREM 7.6. [4, Theorem 3.2.7] *Let 4 divide N. Then $p_+^+(N) = N/2$, $p_+^-(N) = N/2 + 1$. Moreover any N-minimal surface from $\mathcal{K}_+^+(N)$ has one boundary component, whilst any such surface from $\mathcal{K}_+^-(N)$ has 2 boundary components if 8 divides N, and otherwise 4 boundary components.*

COROLLARY 7.7. *The actions realizing $p_+^+(N)$ and $p_+^-(N)$ given in Theorem 7.6 are unique up to topological conjugation.*

Proof. Also here it was shown in [4] that the action realizing $p_+^+(N)$ is given just by an NEC-group Λ with signature $(0; +; [2, N]; \{(\,)\})$ and so the action is unique by Theorem 2.8. In turn the signature $(1; -; [2]; \{(\,)\})$ is the unique one realizing $p_+^-(N)$ and so this action is unique by Theorem 2.6. ∎

7.3 On topological type of cyclic actions of a given nonprime order on bordered nonorientable surfaces of minimal genus

THEOREM 7.8. [4, Theorem 3.2.8] *Let N be a nonprime odd integer and let p be the smallest prime dividing N. Then $p_-(N) = (p-1)N/p + 1$ and*

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the corresponding N-minimal surface has 1 boundary component if p^2 divides N, and 1 or p boundary components if p^2 does not divide N and both of these cases can actually occur.

COROLLARY 7.9. *The actions realizing $p_-(N)$ given in Theorem 7.8 are unique up to topological conjugation if $k=p$ and there are $(p-1)/2$ types of action for $k=1$.*

Proof. In the proof of Theorem 7.10 given in [4] it is shown that Λ realizing the minimum genus must have signature $(1;-;[p];\{(\,)\})$. So the corollary follows from Theorem 2.7. □

THEOREM 7.10. [4, Theorem 3.2.9] *Let N be even and $N \neq 2$. Then $p_-(N)=N/2$ and any N-minimal surface from $\mathcal{K}_-(N)$ is a projective plane with $N/2$ boundary components.*

COROLLARY 7.11. *The actions realizing $p_-(N)$ given in Theorem 7.10 is unique up to topological conjugation.*

Proof. Also here it was shown in [4], that Λ must be an NEC-group with the signature $(0;+;[N];\{(2,2)\})$ and so our corollary follows from Theorem 2.4. □

7.4 On topological type of actions of a prime order N on surfaces of minimal genus

Observe that all results from the previous section concerning minimal genus were formulated and proved for N being nonprime. For prime N, more general results concerning the minimum genus $p_+^+(N,k)$, $p_+^-(N,k)$ and $p_-(N,k)$ of surfaces with specified number k of boundary components are given in [4]. These functions are periodic with respect to k, and so their knowledge obviously gives an effective way to solve the minimum genus problem by simply taking the minimum of $p_*^*(N,k)$ for varying k, and in this way the problem was solved in [4]. One can however calculate the minimum genus for a given prime N directly or using results of the previous; which is more relevant for our purpose which is also topological classification of actions realizing p_*^*.

PROPOSITION 7.12. *We have $p_+^+(2)=p_+^-(2)=p_-(2)=2$. The topological type of a $\mathbb{Z}_2$-action on a bordered surface of algebraic genus 2 is determined by the surface and the quotient orbifold. Up to topological conjugacy there are:*

- 2 *actions realizing* $p_+^+(2)$*:* 1 *on* 1*-holed torus and* 1 *on* 3*-holed sphere;*
- 4 *actions realizing* $p_+^-(2)$*:* 2 *on* 1*-holed torus and* 2 *on* 3*-holed sphere;*
- 8 *actions realizing* $p_-(2)$*:* 3 *on* 2*-holed projective plane and* 5 *on* 1*-holed Klein bottle.*

Proof. We shall see that $p_+^+(2) = p_+^-(2) = p_-(2) = 2$ – the smallest admissible genus. Since in such case $N = 2 > 1 = p - 1$ all the possible involved signatures appear in Lemma 4.3. We are interested with the ones with the normalized area 1/2 and we list all of them here for the reader's convenience:

(1) $(0; +; [\,]; \{(2, 2, 2, 2, 2, 2)\})$,	(2) $(0; +; [\,]; \{(\,), (2, 2)\})$,
(3) $(1; -; [\,]; \{(2, 2)\})$,	(5) $(0; +; [2]; \{(2, 2, 2, 2)\})$,
(6) $(1; -; [2]; \{(\,)\})$	(8) $(0; +; [2]; \{(\,), (\,)\})$,
(9b) $(0; +; [2, 2, 2]; \{(\,)\})$	(10b) $(0; +; [2, 2]; \{(2, 2)\})$.

Now the signature (1) give rise to a reflection of the 3-holed sphere with the disk with 6 corner points as the orbit space which is unique up to topological conjugacy by Theorem 2.1. By Theorem 2.2 the signature (2) provides four actions of $\mathbb{Z}_2$: on 1-holed Klein bottle, 2-holed projective plane and orientation-reversing reflections of 1-holed torus and 3-holed sphere. By Theorem 2.3 the signature (3) provides two actions of $\mathbb{Z}_2$: on 1-holed Klein bottle and an orientation-reversing action on 1-holed torus. By Theorem 2.5 the signature (5) gives rise to one action on 2-holed projective plane. By Theorem 2.6 the signature (6) does not provide any action on bordered orientable surface, whereas by Theorem 2.7 it gives rise to one action on 1-holed Klein bottle. By Theorem 2.9, the signature (8) gives rise to two actions on 1-holed Klein bottle and 2-holed projective plane, whereas by Theorem 2.10 it gives rise to an orientation-preserving action on 3-holed sphere. By Theorem 2.11, the signature (9b) gives rise to the orientation-preserving action on 1-holed torus. Finally the signature (10b) gives rise to the unique action on 1-holed Klein bottle by Theorem 2.13. Observe also that any two actions corresponding to different signatures are not topologically conjugate. ∎

Observe now that for odd N, there are no surfaces admitting orientation-reversing self-homeomorphisms of order N, and so we have to look only for $p_-(N)$ and $p_+^+(N)$, and classify topologically all actions realizing them.

PROPOSITION 7.13. *Let N be an odd prime. Then $p_+^+(N) = N - 1$, and $p_-(N) = N$. Furthermore, in both cases the corresponding surface has $k = N$ or $k = 1$ boundary components. Up to topological conjugation, there are:*

- *2 actions of order N on the N-holed nonorientable surface of algebraic genus $p_-(N)$. The orbit spaces of these actions are 1-punctured Möbius band and 1-punctured annulus;*
- *$3(N-1)/2$ actions of order N on the 1-holed nonorientable surface of algebraic genus $p_-(N)$; $N-1$ with a 1-punctured annulus and $(N-1)/2$ with a 1-punctured Möbius band as orbit spaces of the actions;*
- *unique action of order N on the N-holed orientable surface of algebraic genus $p_+^+(N)$;*
- *$(N-1)/2$ actions of order N on the 1-holed orientable surface of algebraic genus $p_+^+(N)$.*

Proof. Let $\theta : \Lambda \to \mathbb{Z}_N$ be a BSK-epimorphism defining an action of $\mathbb{Z}_N$ on a bordered surface. Then Λ has an empty period cycle in order to produce holes in the corresponding surface $X = \mathcal{H}/\Gamma$ for $\Gamma = \ker\theta$. Now all periods in Λ, if exists, are equal to N and so $(0; +; [N, N]; \{(\,)\})$ is the signature of Λ with the minimal possible area here.

On the other hand Theorem 2.8 asserts that such an epimorphism indeed exist, and the corresponding surface is orientable and has N or 1 boundary components. Furthermore, up to topological equivalence, there is unique such action or $\lceil\psi(N)/2\rceil = (N-1)/2$ actions respectively. This completes the part of the proof concerning $p_+^+(N)$ and also shows that for the study of $p_-(N)$ and its attainments, we need to consider NEC-groups with bigger area.

The second smallest area in this case have NEC-groups Λ with signatures

$$(1; -; [N]; \{(\,)\}) \qquad \text{and} \qquad (0; +; [N]; \{(\,), (\,)\})$$

which indeed, due to the Hurwitz–Riemann formula, concern actions of $\mathbb{Z}_N$ on surfaces of algebraic genus $p = N$. In the first case, such action indeed exists by Theorem 2.7. Furthermore, the corresponding surface has N or 1 boundary components and therefore, up to topological conjugacy, the corresponding action is respectively unique or there are $(N-1)/2$ topological classes of such actions. The second signature may realize $\mathbb{Z}_N$-actions both on orientable (Theorem 2.10) and nonorientable (Theorem 2.9) surfaces and in the latter case either $k = N$ and the action is unique, or

$k = 1$ and there are $N - 1$ actions up to topological conjugation mentioned in Theorem 2.9. ∎

7.5 On topological type of cyclic actions of maximal order on bordered surfaces of given algebraic genus

THEOREM 7.14. [4, Theorem 3.2.18] *Let $p \geqslant 2$ be an integer. Then*

- $N_-(p) = 2p$;
- $N_+^+(p) = \begin{cases} 2(p+1) & \textit{if } p \textit{ is even}, \\ 2p & \textit{if } p \textit{ is odd}; \end{cases}$
- $N_+^-(p) = \begin{cases} 2(p+1) & \textit{if } p \textit{ is even}, \\ 2(p-1) & \textit{if } p \textit{ is odd}. \end{cases}$

In particular

$$N_+(p) = \begin{cases} 2(p+1) & \textit{if } p \textit{ is even}, \\ 2p & \textit{if } p \textit{ is odd}, \end{cases} \qquad \textit{and}$$

$$N(p) = \begin{cases} 2(p+1) & \textit{if } p \textit{ is even}, \\ 2p & \textit{if } p \textit{ is odd}. \end{cases}$$

COROLLARY 7.15. *All actions realizing the solutions of the maximum order problem described in Theorem 7.14 are unique up to topological conjugacy.*

Proof. By the proof of [4, Theorem 3.2.18], $N_-(p)$ for arbitrary p and $N_+^-(p)$ for even p are realized by NEC-groups with signatures

$$(0; +; [2p]; \{(2,2)\}) \qquad \text{and} \qquad (0; +; [p+1]; \{(2,2)\})$$

respectively, and so these actions are unique by Theorem 2.4. Next, $N_+^+(p)$ is realized by signatures

$$(0; +; [2, p+1]; \{(\,)\}) \qquad \text{and} \qquad (0; +; [2, 2p]; \{(\,)\})$$

for p even and odd respectively, the corresponding surface has one boundary component, and so these actions are unique by Theorem 2.8. Finally, $N_+^-(p)$ for odd p is realized by signature $(1; -; [2]; \{(\,)\})$, and the corresponding surface has 2 boundary components if 4 divides $p - 1$ and 4 boundary components otherwise, so this action is unique again by Theorem 2.6. ∎

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