# 3

# First Contact: the Proper Category

## 3.1 Overview

Having given some idea of the kinds of manifolds to which the Borel conjecture applies directly in Chapter 2, we consider now the effect of modifying Borel's heuristic. Taking light of Prasad's (1973) extension of Mostow rigidity to the case of nonuniform lattices, we ask whether topological rigidity holds in this context?

It was already noticed in the early 1980s that this is not the case. Making use of Borel's calculations of the stable cohomology of $\mathrm{SL}_n(\mathbb{Z})$, Farrell and Hsiang observed that for $n$ >200 and $\Gamma$ a torsion-free subgroup of finite index in $\mathrm{SL}_n(\mathbb{Z})$, the quotient $\mathrm{SO}_n\backslash\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ is a not "properly rigid;" i.e. there are infinitely many manifolds $M$ not homeomorphic to $\mathrm{SO}_n\backslash\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$, but proper homotopy-equivalent to it.

Actually this happens iff $n \geq 4$ (and, moreover, the same is true for any number rings in place of $\mathbb{Z}$) as we will §3.7.[1]

The goal of this chapter is to explain this in its natural setting, using it as an excuse to explain some aspects[2] of the structure of $K\backslash G/\Gamma$,[3], Property (T),[4] $L^2$ cohomology[5], and some surgery theory that we will need in later chapters. Not as critical on utilitarian grounds, but nevertheless important, are discussions of

[1] Actually, we will only explain the failure of proper rigidity if $n > 3$; its affirmative solution depends on the "Borel conjecture with coefficients" and will have to wait till later.

[2] The next several footnotes are intended for the more expert reader.

[3] The discussion of which is also relevant to the proof of the Novikov conjecture for linear groups explained in Chapter 8.

[4] Which we will use, as is traditional, in the construction of expanders, which are relevant to the failure of forms of the Baum–Connes conjecture.

[5] Which is used in the proof of the flexibility theorem later that affirms a consequence of the Farrell–Jones conjecture and of the Baum–Connes conjecture unconditionally.

the cohomology of arithmetic groups (ultimately these discussions go to the very meaning of the conjecture),[6] and superrigidity.

The outline of the chapter is as follows: we will first explain the overall shape of $K\backslash G/\Gamma$ (which is a far-reaching generalization of the classical nineteenth-century reduction theory of binary quadratic forms) and give some information about the Borel–Serre compactification of this manifold (Borel and Serre, 1973). Then we will discuss some generalities about the cohomology of arithmetic groups and describe Borel's results on these groups.

Assembling all of this with some surgery theory, we will see a critical role played by the $\mathbb{Q}$-rank. The case of $\mathbb{Q}$-rank $= 0$ corresponds to the compact manifolds, i.e. the Borel conjecture in its usual sense, and if $\mathbb{Q}$-rank $< 3$, it turns out that these noncompact manifolds behave (for the purposes of topological rigidity) just like the compact case, and results explained later in the book will give their proper rigidity. Nonrigidity will immediately follow from the combination of surgery theory with Borel's calculations for very large $n$ (as mentioned above, $n > 176$).

Both for the purpose of lowering $n$ and for allowing a wider range of Lie groups (and for the purposes of later developments) we digress and explain several important properties of lattices in higher-rank groups, and of certain linear groups.

The first of these topics is strong approximation. This property of linear groups will give us control on certain finite quotients of linear groups. We will need this only in this chapter, so our discussion will be brief.

We then turn to Kazhdan's Property (T). Our focus will merely be on definitions, and we leave to other sources serious discussions of the scope of this property and its remarkable applications. These ingredients are then assembled and combined with superrigidity[7] to show that any lattice that has $\mathbb{Q}$-rank $\geq 3$ has a finite sheeted cover that is not properly rigid.

This proper rigidity we thus obtain is somewhat weaker than one would hope: it asserts the existence of a proper homotopy equivalence $f: M \to K\backslash G/\Gamma$ that is not properly homotopic to a homeomorphism. We will need to work harder to ensure that $M$ is not homeomorphic to $K\backslash G/\Gamma$ (by some other map), and that $M$ is smoothable, and to get the set of such $M$s to be infinite. For these we will use a mix of tools from comparison to the Lie algebra mod $p$, to the Baily–Borel compactification in the Hermitian case, to the use of "generalized modular symbols" of Ash and Borel (1990), in order to give a definitive solution for all $\mathrm{SL}_n(O)$ (with $O$ a number ring) and for all $\Gamma$ of $\mathbb{Q}$-rank $> 3$. (Alas, at

[6] As the cohomology of groups gives rise to geometric consequences via the Novikov conjecture.
[7] The extension of linear representations from lattices to the semisimple Lie groups that contain them.

the time of this writing, for example, the proper rigidity properties of certain lattices in $E_7$ are still not well understood.)

We close the chapter by considering the morals of this story, a reexamination of the forest having focused on particular trees. Despite the failure of proper rigidity, we consider noncompact variations of rigidity that actually are true for these locally symmetric spaces. We also discover a role for functoriality in this problem – an aspect which could seem surprising given that the initial problem is purely about certain very specific and beautiful objects.

## 3.2 $K\backslash G\Gamma$ and its Large-Scale Geometry

... in which we encounter the Tits building and the Borel–Serre compactification[8]

If $G$ is a connected Lie group, then it has a maximal compact subgroup $K$, which is unique up to conjugacy. Topologically, $K\backslash G$ is contractible. Give $G$ a right invariant and $K$ bi-invariant metric. If $G$ is semisimple (i.e. has no normal solvable subgroups), then $K\backslash G$ gets a complete metric of non-positive curvature.

As discussed in Chapter 2, $G$ often contains lattices. We shall assume (for simplicity) that $G$ is given the structure of linear algebraic group defined over $\mathbb{Q}$. The first lattices one thinks of are $G(\mathbb{Z})$ and its congruence subgroups, i.e. matrices lying in $G(\mathbb{Z})$ that are $\equiv \mathbf{I} \bmod n$. (We have to do this if we want to restrict attention to torsion-free lattices so that $K\backslash G/\Gamma$ is a manifold – the quotient space being a manifold means that the action of $\Gamma$ on $K\backslash G$ is free: the isotropy of the action of $\Gamma$ on the right has to be a compact subgroup of the discrete group $\Gamma$, and hence finite, and will be trivial when $\Gamma$ is torsion-free. Conversely, when $\Gamma$ has torsion, each element of finite order has a fixed point in $K\backslash G$, making the quotient an orbifold.)

The possibility of other algebraic number fields is not essentially eliminated by this condition, because of the method of restriction of scalars: the group $SL_n(\mathbb{Z}\sqrt{2})$ is a lattice in $SL_n(\mathbb{R}) \times SL_n(\mathbb{R})$. For uniform lattices, as we saw in §2.2, there are other arithmetic lattices that come from $G$ having compact forms that are Galois conjugate to the given form – because a lattice in $G \times G'$ gives us one in $G$ by projecting if $G'$ is compact (or alternatively, $G$ and $G \times G'$ are isomorphic after modding out by their maximal compact subgroups). For the noncompact case, these more subtle lattices don't play a role – since all the forms must be noncompact (because $\Gamma$ contains unipotents and compact groups do not), so the definition of arithmeticity is somewhat less subtle in this case.

[8] With apologies to A.A. Milne

https://doi.org/10.1017/9781316529645.004 Published online by Cambridge University Press

While our focus in Chapter 2 was on the compact case, here we are interested in what occurs in the noncompact case. An important theorem of Borel and Harish-Chandra[9] "blames" noncompactness on a "$\mathbb{Q}$-split torus" for $G$.

Let us follow this subgroup around in the simplest situation $\mathrm{SL}_n(\mathbb{Z})$. We will see an even more precise picture than mere noncompactness.

In $\mathrm{SO}(n)\backslash\mathrm{SL}_n(\mathbb{R})$ we can consider the torus of diagonal matrices (such that the product of their entries is 1). As a space of tori, these are the "rectangular" tori. Taking the logs of these eigenvalues, we get a map to $\mathbb{R}^{n-1}$ (the elements of $\mathbb{R}^n$ that have the sum of their components equal to 0). The symmetric group $\Sigma_n$ acts on this by permutation – without loss of generality, we can assume that the eigenvalues are listed in increasing order. This gives us a polyhedral cone in $\mathbb{R}^{n-1}$ and a subset of $\mathrm{SO}(n)\backslash\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$. This subset gives us a very good large-scale picture of this quotient manifold: for example, this embedding is essentially undistorted, and every point in the quotient space is of uniformly bounded distance to a point of this sector. Moreover, this statement is true if $\mathbb{Z}$ is replaced by integers in a totally real field. Although the real Lie group this embeds in a product of $\mathrm{SL}_n(\mathbb{R})$s, the effect of taking the quotient by the action of $\mathrm{SL}_n(O)$ is to cuts it down to the size of the polyhedral cone that is the quotient of the maximal flat.[10] The proofs of these kinds of statements are the subject of "reduction theory," developed by C.L. Siegel (1988), A. Borel, and their successors (see Borel and Ji (2005) for a modern account).

For other lattices we will have to glue together copies of this sector according to some combinatorial description governed by the theory of Tits buildings – which records the combinatorics of the parabolic subgroups. All of this is first most easily observed in yet another, even simpler, example, the product of hyperbolic manifolds $\prod M_i$. After discussing this toy example, we will return to $\mathrm{SO}(n)\backslash\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ and the general case.

Each noncompact hyperbolic manifold $M$ has a core, with cusps coming off. Pick a base point, and a sequence of points going towards infinity in each of the cusps. The geodesics connecting this base point to those points converge to a finite union of geodesic rays, each of which is isometrically embedded in the manifold (see Figure 3.1).

This union of geodesics looks like an asterisk with one "prong" for each cusp; we denote this by $A$. (This is the direct analogue of the polyhedral cone from the $\mathrm{SL}_n(\mathbb{Z})$ case.)

One can imagine a map from $M$ to $A$, roughly mapping each point to the

[9] See Borel and Harish-Chandra (1961).

[10] This is very much like the phenomenon that occurs in the Dirichlet unit theorem, where all of the directions in logarithm space for the various embeddings of the units just curl it up into a torus.

https://doi.org/10.1017/9781316529645.004 Published online by Cambridge University Press

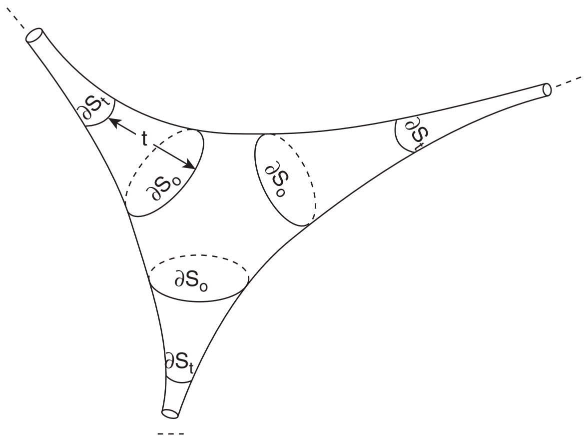


Figure 3.1 Adapted from Thurston's notes

point on the asterisk closest to it, (and then modifying it slightly on a compact set, arrange the map so that the inverse image of the base point is the core of $M$, and the inverse image of any point in one of the rays is a "flat manifold horospherical section" of the cusp.

Let me elaborate on the terminology.

The isometry group of hyperbolic space $\mathbb{H}^n$ is $O(n,1)$ – which we will imagine via the ball model. The isometries form three classes: elliptic, hyperbolic, and parabolic. Each elliptic element has fixed points in the interior, and lies in a maximal compact. (The action of the isometry group is transitive, so what fixes one point is conjugate to what fixes any other point: hence, the maximal compact subgroup is unique up to conjugacy.)

Hyperbolic elements act via translation along a geodesic (with some rotation in the normal direction.[11]) A parabolic element has a unique fixed point on the boundary sphere at $\infty$.

Given such a fixed point, the *horosphere* going through that point can be defined as follows. Choose a unit speed geodesic $\gamma$ going from $p$ to a specific point at $\infty$. Now consider the sphere of radius $R$ centered at $\gamma(R)$. The limit set of these spheres is an orbit $O(n,1)_p/O(n,1)_\infty$. The isotropy group is a *parabolic subgroup*, which is isomorphic to the semidirect product $O(n-1) \ltimes \mathbb{R}^{n-1}$ which is the isometry group of $\mathbb{R}^{n-1}$.

(In general, parabolic subgroups are those subgroups that contain a *Borel*

[11] Following Thurston (2002), we do not distinguish between hyperbolic isometries and "loxodromic" ones.

https://doi.org/10.1017/9781316529645.004 Published online by Cambridge University Press

*subgroup*, i.e. a maximal connected solvable group. They are the isotropy groups of points on the boundary of $K\backslash G$.)

Now let us return to our hyperbolic manifold with a number of cusps. Lifting the geodesics associated to the cusps gives a finite set of points on the boundary, which are fixed points of nontrivial parabolics. The subgroup of $\Gamma$ fixing a (lifted) cusp acts as a lattice on the horosphere. The quotient is a flat manifold (which is a cross section of the cusp – choosing another point $p$ on $\gamma$ would give a parallel cross section).

The product of a number of hyperbolic manifolds both contains and maps to the corresponding product of asterisks, which is a polyhedral cone whose dimension in the $\mathbb{Q}$-rank of this product lattice.[12]

Note that the inverse image of a point in this cone depends strongly on which face that point lies on. It will be a product of some number of cores and some number of flat manifolds. (Note that by taking finite covers of this product, we can mangle the product structure, but will still get a similar union of flat pieces approximating the manifold.)

For $\mathrm{SL}_n$ the picture is similar. We've seen the cone, and the inverse of a point in the interior of the top simplex is a nilmanifold: isomorphic to $\mathrm{UT}(n, \mathbb{R})/\mathrm{UT}(n, \mathbb{Z})$, where $\mathrm{UT}(n, ?)$ denotes the group of upper triangular matrices with (1s on the diagonal and) entries in '?'.

Recall that a point in the top simplex corresponds to a diagonal matrix, whose eigenvalues are distinct. This unitary group is the unipotent subgroup of the matrices that preserve the flag given by these subspaces. A point in a different simplex corresponds to some coincidences among eigenvalues. At these points, one has an incomplete flag and normal to it one has a "genuine" lattice part (corresponding to a product of SLs associated to the various combined eigenspaces) with a nilpotent bundle over that associated to the unipotents that are the identity module the flag.

As one moves towards infinity, the unipotent pieces have volume that decays rapidly to 0,[13], and that is what accounts for the finiteness of the volume of these nonuniform lattices. The lattice part stays bounded in size (but does not shrink[14]).

[12] Here by $\mathbb{Q}$-rank we merely mean the number of noncompact hyperbolic factors, whether or not they are arithmetic. As a consequence of Margulis's arithmeticity theorem, all, even non-arithmetic lattices, can be approximated by finite polyhedral cones, defining for us $\mathbb{Q}$-rank even when there is no $\mathbb{Q}$-structure! The reason is that there is such a structure for negatively curved manifolds, and everything is virtually a product of negatively curved homogeneous spaces and arithmetic ones.

[13] A nilmanifold is essentially "an iterated fiber bundle of torus on top of torus and so on". The layers shrink at different rates. Gromov (1978) has shown that manifolds with metrics of bounded curvature but diameter going to 0 are finitely covered by nilmanifolds.

[14] This is also similar to what occurs in the case of a product of hyperbolic manifolds – the

Another concrete case for which the calculations are not difficult is the case of Hilbert modular groups,[15] $\Gamma = \mathrm{SL}_2(O_F)$ where $F$ is a totally real field of degree $d$. In that case, there are finitely many cusps (equal to $h(O_F)$, the class number of the ring[16]). This group acts on a product of $d$ hyperbolic planes (where $d = [F; Q]$). The cusps are actually solvable manifolds.[17] The bounded part is a torus corresponding to $O_F^*$. The fiber is the torus $\mathbb{R}^d/O_F$ and the monodromy of this bundle is the action of $O_F^*$ on $O_F$. The base torus stays of bounded size as one goes down the cusp (it takes some distance to work up the twist corresponding to a nontrivial unit), while the fiber torus decays exponentially by homothety as one goes down the cusp.

Now let us work in general, guided by these special case. If $G$ is a linear algebraic group defined over $\mathbb{Q}$, we shall define a simplicial complex, the Tits building of $G$ using the parabolic subgroups of $G$. The minimal parabolic is $\mathbf{B}$, by definition, the Borel subgroup, and $G$ itself is the maximal parabolic.

To a parabolic $P$ we associate a simplex $\sigma_P$ so that $\sigma_P \subset \sigma_Q$ iff $Q \subset P$. The group $G$ corresponds to the empty simplex. The maximal simplices correspond to (conjugates of a) Borel[18] subgroup.

It is a very nice theorem of Solomon and Tits (proved rather geometrically: see, e.g., Abramenko and Brown, 2008) that this complex has the homotopy type of a wedge of spheres of dimension $q - 1$ (where $q = \mathbb{Q}$-rank).

The Borel–Serre compactification (Borel and Serre, 1973) of $K\backslash G/\Gamma$ is a compact manifold[19] with boundary so that $K\backslash G/\Gamma$ is its interior. Actually, it has a more refined structure: it has the structure of a manifold with corners – and this structure carries a great deal of geometry in it, but we will not need this.

The compactification takes place on $K\backslash G$, and is $G(\mathbb{Q})$- (but not $G(\mathbb{R})$-) equivariant. Associated to $P$ we have a Euclidean space $e_P$ so that $\dim e_P + \dim \sigma_P = q - 1$. These open cells are disjoint, but $e_P \subset \mathrm{cl}(e_Q)$ iff $P \subset Q$.

The corner structure comes like this. The unipotent subgroup of $P$ acts on $K\backslash G$ as a free $(\mathbb{R}_+{}^*)^{\dim(\sigma_P)+1}$-proper action. Include each orbit into the

inverse images of points that are not in a top simplex have bounded diameter, which does *not* go to 0 as the point moves to infinity. Of course, the volumes of these point inverses go to 0 very rapidly, or the locally symmetric manifold could not be finite volume.

15 See Freitag (1990) for a crystal clear explanation.

16 For congruence subgroups, the number of cusps is the order of a *ray class group*.

17 That non-nilmanifolds arise is because here $G$ has rank greater than 1, and we are dealing with nonpositive curvature rather than strict negative curvature.

18 It is not instantly obvious that this is a simplicial complex. A hint is that for simple algebraic groups, the conjugacy classes of parabolic subgroups are in a 1–1 correspondence with subsets of the nodes of the Dynkin diagram.

19 Actually, when $\Gamma$ has torsion, it is an orbifold.

$(\mathbb{R}_+{}^*)^{\dim(\sigma_P)+1}$-space $([0,\infty))^{\dim(\sigma_P)+1}$. One can thus compactify each orbit.[20] The relations among the parabolic subgroups enable one to glue these together to include $K\backslash G$ as the interior in a manifold with corners on which the $G(\mathbb{Q})$-action extends. Borel and Serre topologize this union as a manifold so that the action of $\Gamma$ on it is continuous and proper discontinuous. In particular, they see that down in the quotient, they obtain a compactification.

They also observe that the boundary of $K\backslash G$ so obtained has the Tits complex as its nerve and therefore the $\Gamma$ cover of the $\partial$ has the homotopy type of a wedge of spheres $\bigvee \mathcal{S}^{q-1}$.

In the case of a lattice of $\mathbb{Q}$-rank 1, the picture is the one of isolated cusps, and the compactification glues onto the end a copy of the slice of the horosphere. For a product of these manifolds, one obtains the product of these compactifications (and, of course, the corner structure is evident in this case).

Moreover, using the fact that the universal cover of these closures are contractible, it is quite easy to see that the boundaries look like joins of the boundaries of the universal covers of the original compactified factors – and hence an infinite wedge of spheres, $\bigvee S^{q-1}$ (where $q$ is the $\mathbb{Q}$-rank).

Note then the underlying homotopy type:

- If $\mathbb{Q}$-rank $= 0$, then we must be compact (and the homotopy type is that of $\varnothing$).
- If $\mathbb{Q}$-rank $= 1$, then the cover of the boundary is a union of copies of the universal cover of the boundary. Thus the Borel–Serre boundary is a (union of) aspherical manifold(s) whose fundamental group is a subgroup of $\Gamma$ (of course, it's a lattice in the parabolic associated to that cusp).
- If $\mathbb{Q}$-rank $= 2$, then we get a pleasant surprise, the boundary is connected – which means that every compact subset of $K\backslash G/\Gamma$ has a unique component with compact closure (i.e., it has one end).

Moreover, the boundary is a closed aspherical manifold, since it has an aspherical cover, namely the regular cover associated to $\Gamma$, which is homotopy equivalent to a wedge of circles.[21]

This is actually a very interesting aspherical manifold that is not a lattice in any Lie group! However it is not really a surprise to us – the Tits building in this situation is a graph, and we have lattices associate to the nodes, glued together according to "boundaries" along the edges[22]. Like 3-manifolds, these

[20] Formally, one should take an associated bundle to viewing $K\backslash G$ as a $(\mathbb{R}_+{}^*)^{\dim(\sigma_P)+1}$-principal bundle using this action on the octant $([0,\infty))^{\dim(\sigma_P)+1}$.

[21] Note that aspherical is equivalent to all higher homotopy groups vanishing, but higher homotopy groups are unchanged in covering spaces.

[22] For example for $\mathrm{SL}_3(\mathbb{Z})$ one gets two copies of $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ thought of as block $3 \times 3$ matrices

https://doi.org/10.1017/9781316529645.004 Published online by Cambridge University Press

boundaries have decompositions into geometric pieces, and it is not hard to generalize this construction to more complicated kinds of “graph manifolds”.

The connectedness of this cover means that the map from fundamental group at $\infty$ to $\Gamma$ is surjective. In other words, any loop in $K\backslash G/\Gamma$ can be pulled to $\infty$ (i.e. outside of any compact). However, to do this, one typically must increase the diameter of loops.[23]

If $\mathbb{Q}$-rank $> 2$, then we discover that the boundary is not aspherical ($\pi_{r-1}$ is nonzero) – our first hint that all is not well with a proper Borel conjecture.[24] As we will see in the coming sections, because of this, when $\mathbb{Q}$-rank $> 2$, proper rigidity typically fails. At the end of the chapter we will try to learn some lessons from this failure.

## 3.3 Surgery

Surgery theory is a framework for studying the classification of high-dimensional manifolds. In this section we will describe some of the features of surgery theory, and in particular, a situation where there are “no obstructions”. In particular, we will explain the observation of Farrell and Hsiang (1982) that for very large lattices the proper analogue of the Borel conjecture fails. Later sections will show that failure is actually ubiquitous and more dramatic than these examples show.[25]

Our presentation in this section is quick and dirty. Later on we will need and give more precise, and more conceptual, discussions: the need for better calculations requires alternative descriptions, from whose vantage point the very nature of our central problem changes.

Atiyah (1961) observed that:

**Theorem 3.1** *If one has a homotopy equivalence between closed manifolds $h\colon M' \to M$, then there is a kind of equivalence between their stabilized tangent bundles, namely* stable isomorphism of spherical fibrations.

Let me explain. Assume first that $M \ni m$ and $M' \ni m'$ are smooth so that they have tangent bundles, $\mathrm{T}M$ and $\mathrm{T}M'$ respectively, in the usual sense.

(with a $2 \times 2$ block either on the top left or bottom right). These intersect along the Heisenberg group $\mathrm{U}(3, \mathbb{Z})$ in $\mathrm{SL}_3(\mathbb{Z})$. The fundamental group of the boundary is this amalgamated free product. The kernel of the map of this group to $\mathrm{SL}_3(\mathbb{Z})$ is an infinite-rank free group.

[23] This will be (part of) the reason why we will ultimately succeed in proving a “bounded” topological rigidity for higher-rank locally symmetric manifolds – see the discussion in the morals, §3.8.

[24] Of course, the resolution *could have been* that there are some special non-aspherical manifolds that are rigid. There are some, but Borel–Serre boundaries turn out not to be among these.

[25] But as we said, there are also versions of rigidity that do apply to nonuniform lattices.

https://doi.org/10.1017/9781316529645.004 Published online by Cambridge University Press

An equivalence between tangent vector bundles in the usual sense would be a continuous family of linear isomorphisms (not necessarily the differential, $Dh$, of the map) $\mathrm{T}M'_{m'} \to \mathrm{T}M_{h(m)}$. A stable isomorphism of such vector bundles would be such a family $\mathrm{T}M'_{m'} \times \mathbb{R}^d \to \mathrm{T}M_{h(m)} \times \mathbb{R}^d$ for some $d$. A *stable isomorphism of spherical fibrations* is such a family of maps, not necessarily linear, but which is a degree-1 proper homotopy equivalence on each fiber. (This means that the map induces a homotopy equivalence between the fiberwise one-point compactifications, i.e. the stable spherical fibrations. Note that the one point compactification can be thought of as being the unit sphere of one stabilization further.)

This implies that some invariants of the tangent bundle are homotopy invariant, such as Stiefel–Whitney classes.[26] However, this equivalence relation on bundles is very weak: over a space $X$ of finite type,[27] there are only finitely many such equivalence classes.[28] However, characteristic classes, such as the Pontrjagin classes, allow for an infinite number of conceivable tangent bundles for manifolds within that homotopy type.

Just as (oriented) bundles can be thought of as maps into Grassmanians,[29] BSO, there is a classifying space for (oriented) spherical fibrations BSF, i.e. maps $E \to X$ whose homotopy fiber is a sphere are classified by maps $X \to$ BSF, so that we can interpret Atiyah's theorem as saying that the composite map

$$M \to \mathrm{BSO} \to \mathrm{BSF}$$

is a homotopy invariant of compact manifolds $M$. The proper analogue of Atiyah's theorem holds as well.

So, given $h\colon M' \to M$, taking into account the automatic equivalence of their stable tangent bundles in BSF, gives us a refined tangential data for a homotopy equivalence:

$$\nu(h)\colon M \to F/\mathrm{O},$$

where $F/\mathrm{O}$ is the fiber of the map BSF $\to$ BSO. This invariant of $h$ is called the *normal invariant of h* (since it is a stable invariant, and the stable normal

26 This fact also follows from the Wu formula that gives a homotopy-theoretic description of the Stiefel–Whitney classes in terms of the action of the Steenrod operations on the cohomology of a manifold.

27 That is, with the homotopy type of a finite CW-complex.

28 This follows immediately from an obstruction theory – induction over the skeleta of a triangulation – making use of Serre's result that the stable homotopy groups of spheres are finite.

29 That is, there is a universal bundle, and every bundle is the pullback of this bundle under a map that is well-defined up to homotopy.

bundle is adequate for its definition, rather than the more subtle, unstable tangent bundle).

Another way to say this is that the two tangent bundles combine to give a map from $M$ to the homotopy pullback of

$$\begin{array}{ccc} & & \mathrm{BSO} \\ & & \downarrow \\ \mathrm{BSO} & \to & \mathrm{BSF}, \end{array}$$

which, of course, is homotopy equivalent to $\mathrm{BSO} \times F/\mathrm{O}$, as we leave to the reader.

Now, I should say that there is a similar discussion possible in the category of nonsmooth, triangulable, or even topological, manifolds, which gives rise to classifying spaces – so in the topological case, we have $\nu(h)\colon M \to F/\mathrm{Top}$. *A first view of surgery theory* is that it is about the difficulty in realizing maps into $F/\mathrm{O}$ or $F/\mathrm{Top}$ from homotopy equivalences.

However, there is one situation where there is no obstruction at all:

**Theorem 3.2** ($\pi$–$\pi$ theorem) *Suppose that $M$ is a connected manifold with nonempty connected boundary,* $\dim M \geq 6$*, and $\pi_1(\partial M) \to \pi_1(M)$ is an isomorphism. Then every homotopy class of maps $M \to F/\mathrm{Cat}$ (for* $\mathrm{Cat} =$ Diff, PL, Top*) is realized by a homotopy equivalence of pairs* $(M', \partial M') \to (M, \partial M)$.

A relative version of this theorem actually implies a uniqueness result for the pair $(M', \partial M')$.[30] This theorem is immediately relevant to our situation, since the Borel–Serre compactification, when $\mathbb{Q}$-rank$(\Gamma) > 2$, satisfies the hypothesis of this theorem.

We shall now review some results about the nature of these classifying spaces.

First of all, the homotopy groups of BSF are finite, so the map $G/O \to \mathrm{BSO}$ is a rational homotopy equivalence.

The reason for this is not difficult: the homotopy groups of BSF corresponed to spherical fibrations over the sphere. A spherical fibration over $\mathcal{S}^n$ can be thought of (just like a bundle) as the result of gluing together two trivial bundles over the two hemispheres $\mathcal{D}^n_\pm$. The gluing is a map $\mathcal{S}^{n-1} \to$ self-homotopy equivalences of the fiber sphere $\mathcal{S}^i$, which is the iterated loop-space $\Omega^i \mathcal{S}^i$ of a sphere. A little thought then shows that the homotopy groups of BSF are therefore the same as the stable homotopy groups of spheres, and these are finite thanks to a theorem of Serre (see Serre, 1951).

30 It will be unique up to $h$-cobordism, or, if we work with simple homotopy equivalences, then it will be unique up to Cat-isomorphism.

https://doi.org/10.1017/9781316529645.004 Published online by Cambridge University Press

Characteristic class theory also tells us that Pontrjagin classes give us a rational homotopy equivalence $\mathrm{BSO} \to \prod K(\mathbb{Z}, 4i)$.

The theorem of Kervaire and Milnor (1963) on the finiteness of the number of smooth structures on a sphere can be translated into the statement that the homotopy of $\mathrm{Top}/\mathrm{O}$ is finite, or that $F/\mathrm{O} \to F/\mathrm{Top}$ is a rational equivalence.[31] Thus:

**Theorem 3.3** *There is a rational homotopy equivalence*

$$F/\mathrm{Cat} \to \prod K(\mathbb{Q}, 4i).$$

Remarkably, Sullivan gave a complete and precise analysis of $F/\mathrm{Top}$,[32] which we will explain in Chapter 4. See, for example, Rourke and Sullivan (1971) – in itself a historically interesting paper – for part of the proof of the following, and Madsen and Milgram (1979)) for a complete explanation.

**Theorem 3.4** *At the prime* 2*, there is an equivalence:*

$$F/\mathrm{Top}_{(2)} \to \prod K(\mathbb{Z}_{(2)}, 4i) \times K(\mathbb{Z}/2, 4i-2).$$

*Away from* 2*, there is an equivalence:*

$$F/\mathrm{Top}[1/2] \to \mathrm{BSO}[1/2].$$

**Remark 3.5** In writing things this way, we are using localization theory for simply connected spaces (or of $H$-spaces) which enables one to assign to such a space $X$, the localization of $X$ as a set $P$ of primes. This space $X_{(P)}$ is functorially associated to $X$, and its homotopy (and homology) groups are those of $X$, but tensored with $\mathbb{Z}[1/q]$, where $q$ runs over the primes *not* in $P$. So $X_{(2)}$ has as homotopy groups those of $X$, tensored with the group of rational numbers with odd denominators.

Localizing at a set of primes has the effect of ignoring contributions of the other primes. Part of the theory explains how to combine the information at the various primes with rational information to give information about ordinary homotopy classes of maps $[\ ; X]$. We refer the reader to Hilton *et al.* (1975) for

31 This is an outright lie of the worst kind: it is a misleading truth. To set up such an equivalence, one needs to be able to do enough topological topology (i.e. topology in the topological category) to be able to mimic many smooth constructions. In particular one requires topological transversality – which is indeed a theorem from Kirby and Siebenmann (1977). With transversality however, it is a simple matter to prove that rational Pontrjagin classes are topological invariants (a transparent consequence of the statements thrown about in the main text) – as we explain in §4.5. That was a major result of Novikov, for which he earned a Fields medal. In the next section we will return to this train of thought. In any case, for now, please bear with the inaccuracies above.

32 Actually, Sullivan did the PL case, but once the work of Kirby and Siebenmann mentioned in the previous footnote became available, the result for Top immediately follows.

an exposition of this theory and Bousfield and Kan (1972) for a more modern approach.

**Warning** Sullivan's map to BSO[1/2] is not transparently related to the tangent bundle of the underlying smooth manifolds (when one has a homotopy equivalence between closed manifolds) – and then forgetting their smooth structure – however, rationally it contains the same information as should be reasonable given our discussion above.[33]

Let us now combine our discussion into a proposition:

**Proposition 3.6** *If $M = K\backslash G/\Gamma$ is a locally symmetric manifold of dimension greater than* 5 *and* $\mathbb{Q}$-rank$(\Gamma) \geq 3$*, then there are infinitely many smooth manifolds proper homotopy-equivalent to $M$ that are not homeomorphic to $M$ (detected by their rational Pontrjagin classes) if, for some $i$, $H^{4i}(M;\mathbb{Q}) \neq 0$.*

(The reader who is familiar with Siebenmann's thesis can also reverse the argument we have given to prove the converse to this proposition.)

We can assume $M$ is replaced by the Borel–Serre compactified version. If the $\mathbb{Q}$-rank$(\Gamma) \geq 3$, this is a $\pi$–$\pi$ manifold, so Wall's theorem reduces it to a classifying space question – and the cohomological condition is exactly equivalent to the set of homotopy classes of maps $M \to F/\mathrm{Top}$ to be infinite (and infinitely many of these classes will automatically be smoothable).

Following Farrell and Hsiang, we presently observe that for $n \geq 176$, Borel's work gives on cohomology of arithmetic groups gives us this conclusion for $\mathrm{SO}(n)\backslash\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ (or more precisely a lattice in $\mathrm{SL}_n(\mathbb{Z})$ that is of finite index and torsion free). (We remark that for $\mathbb{Z}[i]$, Borel's results would have allowed the choice of $n > 32$.)

The proper setting for this work is the relation between cohomology of arithmetic groups and representation theory, but we will avoid a general discussion focusing on just the contribution of the trivial representation – which Borel (1974) showed was the whole story in a "stable range".

The result is that:

**Theorem 3.7** *For $K < \mathbb{Q}$-rank$(\Gamma)/4$, $H^k(K\backslash G/\Gamma;\mathbb{R})$ is represented by differential forms on $K\backslash G$ that are right $G$-invariant.*

In particular, the lattice itself is irrelevant! (We will see that however, above

[33] It turns out that BO $\to$ BTop is an isomorphism on homotopy groups rationally (the injectivity of this map being Novikov's theorem on topological invariance of rational Pontrjagin classes, and the rational surjectivity following from the finiteness of the number of differential structures on the sphere).

this value of $k$, the cohomology group can indeed change with the choice of lattice $\Gamma$.)

Here's a way to think about this. Suppose $L$ is a compact Lie group containing $K$; then, by the Hodge theorem, we can compute $H^*(K\backslash L)$ by means of harmonic forms, but by integrating with respect to $L$, and using the uniqueness of harmonic representatives, we can essentially identify the cohomology with the forms on $K\backslash L$ that are invariant under the action of $L$.

Now if $G$ is a real semisimple group, with $K$ its maximal compact, we denote by $G_{\mathbb{C}}$ its complexification, and by $G'$ the maximal compact of $G_{\mathbb{C}}$. The Cartan decomposition for $G'$ and $G_{\mathbb{C}}$ only differ by a multiplication by $i$. This implies that the $G$-invariant forms on $K\backslash G$ are essentially the same as the $G'$-invariant forms on $K\backslash G'$. We call $K\backslash G'$ the *compact dual* of $K\backslash G$.

For a uniform lattice, this copy of the cohomology of $K\backslash G'$ actually *embeds* in $H^k(K\backslash G/\Gamma;\mathbb{R})$.

For nonuniform lattices, this is not the case, and it is not easy to tell which of these cohomology are actually present in $H^*(K\backslash G/\Gamma)$ (e.g. the top class never survives). However, here Borel's theorem tells us that in the range mentioned above this *is* actually a complete description of the cohomology.

For $\mathrm{SL}_n(\mathbb{R})$, the complexification is $\mathrm{SL}_n(\mathbb{C})$, whose maximal compact is $\mathrm{SU}(n)$. Thus the compact dual is $\mathrm{SO}(n)\backslash\mathrm{SU}(n)$. Thus the cohomology is that of a product of spheres of dimensions $5, 9, 13, 17, \ldots$ The smallest dimension that is a sum of these and a multiple of 4 is 44, giving the result for $n > 176$.

For $\mathrm{SL}_n(\mathbb{C})$, thought of as a real Lie group, the complexification is $\mathrm{SL}_n(\mathbb{C}) \times \mathrm{SL}_n(\mathbb{C})$. Thus, the compact dual of $\mathrm{SU}(n)\backslash\mathrm{SL}_n(\mathbb{C})$ is $\mathrm{SU}(n)$ and therefore a product of spheres of dimension $3, 5, 7, 9, \ldots$ The first relevant cohomology is in dimension 8, so for $n > 32$ these produce examples.

This method shows failure of proper rigidity for $\mathrm{SL}_n(O_F)$ for $n > 32$ if $F$ has a complex embedding, and $n > 176$ when $F$ is totally real. These counterexamples are "stable" in at least two senses: (1) they do not go away if we stabilize the manifold by taking products with Euclidean space, $\mathbb{R}^k$; and (2) they survive on passing to any further finite cover.

However, this method is insensitive to the lattice in $\mathrm{SL}_n$, and for example, this cannot lead to the idea that as the volume of the symmetric space goes up, so does the size of this set of manifolds, which actually seems to be the typical behavior.

More precisely, we will soon see that there is a finitely generated abelian group structure on this set of topological manifolds, and that (via a nonlinear map related to the Pontrjagin classes but distinct from it) it is $\cong \oplus H^{4i}(\Gamma;\mathbb{Q})$

https://doi.org/10.1017/9781316529645.004 Published online by Cambridge University Press

after $\otimes\mathbb{Q}$.[34] We shall see that frequently the rank of this abelian group (even rationalized) grows with $\Gamma$.

However, the impatient reader who wants to move on to matters more directly concerned with the *validity* of rigidity can now skip to the end of this chapter or to the next (with occasional references to the skipped sections, especially about Property (T)).

## 3.4 Strong Approximation

Our first order of business is to give a fairly straightforward argument that, in the case of $\mathrm{SL}_n(O_F)$, $n > 4$, there is always a finite sheeted cover with a substantial amount of cohomology. In §3.7, we will use this to give an essentially elementary replacement for the work of Borel used in the previous section to disprove the proper Borel conjecture for $n > 4$. (The argument for $n = 4$ will not be quite as elementary and will require material from §3.6.) We will write down the argument in the case of $\mathbb{Z}$, but the arguments are completely general. Following this we will discuss strong approximation, which gives a good understanding of the quotients of quite general linear groups. Ultimately, this will imply that all $\mathbb{Q}$-rank $> 2$ lattices have finite covers that are not properly rigid.[35]

We begin by noting that $\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}_p)$ is a surjection. The kernel $\mathrm{SL}_n(Z;p)$ consists of matrices of the form $(\mathrm{I} + pA)$, where $A \in M_n(\mathbb{Z})$ is such that $(\mathrm{I} + pA)$ is invertible. The key thing as noted by Lee and Szczarba (1976) is that this congruence kernel has a homomorphism $\to M_n(\mathbb{Z}_p)$, assigning $A$ to $\mathrm{I} + pA$. Note that $\det(\mathrm{I} + pA) = \pm p^n p_A(-1/p)$ and hence we need that $A$ have trace $0 \bmod p$. (Of course, this is the Lie algebra of $G$ in general.)

Now we can write down explicitly a 3-cycle in the congruence subgroup that is $p$-torsion and detected by projection to this abelian $p$-group. It is a $\mathbb{Z}^3$ in $\mathrm{SL}_5(\mathbb{Z})$. There is a $\mathbb{Z}^2$ which consists of matrices that are 1s on the diagonal and the top row is $(1, 0, 0, pa, pb)$. This commutes with the Heisenberg group (Heis) of upper diagonal matrices in $\mathrm{SL}_3(\mathbb{Z}) \subset \mathrm{SL}_3(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{SL}_5(\mathbb{Z})$. We obtain a $\mathbb{Z}^3$ by taking the product of the $\mathbb{Z}^2$ with the central $p\mathbb{Z}$ in the level-$p$ congruence subgroup of the Heisenberg group.

This $\mathbb{Z}^3$ gives us a cycle in $H^3(\mathrm{SL}_n(\mathbb{Z};p);\mathbb{Z})$ which is nontrivial, because it

[34] The smooth version maps to the topological one so that the map is finite-to-one, and the image need not be a subgroup, but it contains a lattice in this cohomology group by an argument we will give in §3.7.

[35] But it will not imply stability in the second sense of the previous section. Indeed we will see a rank-3 reducible lattice where every proper homotopy equivalence to any finite sheeted cover becomes properly homotopic to a homeomorphism in a further cover.

https://doi.org/10.1017/9781316529645.004 Published online by Cambridge University Press

is detected by mapping to $M_n(\mathbb{Z}_p)$ (by the Künneth formula), but is $p$-torsion, because the central $\mathbb{Z}$ is of order $p$ in $H^1(\mathrm{Heis}_3(\mathbb{Z};p);\mathbb{Z})$ – i.e. the homology of the level-$p$ congruence subgroup of the Heisenberg group – since the $3 \times 3$ matrix

$$\begin{pmatrix} 1 & 0 & p^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a commutator in this group. Consequently we have found *an element of order $p$ in $H^4(\mathrm{SL}_n(\mathbb{Z};p);\mathbb{Z})$* by the universal coefficient theorem.

We will see in §3.7 below that for $p$ sufficiently large this element is the first Pontrjagin class of some manifold proper-homotopy equivalent to $\mathrm{SO}(n)\backslash\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z};p)$. Actually, *these elementary calculations* with Lie algebras and playing with congruence subgroups suffice to show that for $\mathbb{Q}$-rank $> 6$ one can always find a congruence cover where there are arbitrarily large finite number of manifolds that can be distinguished by $p_1$ – the first Pontrjagin class.[36]

Reduction modulo primes for linear groups over fields of characteristic 0 is a very powerful method and produces many useful homomorphisms. This is, for instance, used to prove (see e.g. Wehrfritz, 1973) that such groups are residually finite (Malcev) and also virtually torsion-free (Selberg).

Let us describe some easy homomorphisms if $\Gamma \subset \mathrm{GL}_n(F)$ is a finitely generated group over a field $F$ of characteristic 0. Consider the generators of $\Gamma$ as lying in a finitely generated ring over $\mathbb{Z}$. Its field of fractions is a finite (algebraic) extension of a field of finite transcendence degree. We can then "specialize" values for the transcendentals so that these matrices all lie in an algebraic extension (as the determinant will be a rational function that is not identically 0). Then the matrix entries really are algebraic numbers with finitely many primes in their denominators, and we can therefore reduce modulo large primes. However, for simplicity of exposition, we will imagine that our groups lie just over the integers, perhaps with finitely many denominators.

These congruence subgroups provide a natural sequence[37] of subgroups that converge to the trivial group. Amazingly, *the image of a linear group under such reductions is*, with finitely many exceptions, *governed by the Zariski closure of*

[36] As explained in §3.7, Novikov's theorem that rational Pontrjagin classes are topological invariants can be refined for $p_1$ to the statement that in $H^4(\mathrm{BSTop};\mathbb{Z}[1/2])$ it is definable for oriented topological bundles.

[37] Which corresponds to a tower of covering spaces if one chooses a sequence of moduli that divide one another. A different choice, which does not form a directed system but rather is just a sequence of covers, is the congruence kernels as one varies over different primes. Those still converge to the universal cover, for example, in the pointed Gromov–Hausdorff sense.

*the group*. (This is the content of the *strong approximation theorem*.) Thus, any Zariski-dense finitely generated subgroup of $\mathrm{SL}_n(\mathbb{Q})$ surjects onto $\mathrm{PSL}_n(\mathbb{Z}_p)$ for all but finitely many primes. Indeed, like in the Chinese remainder theorem, one can map onto almost any finite product $\times \mathrm{PSL}_n(\mathbb{Z}_{p_i})$.

Slightly more precisely, let $S$ be a finite set of primes. We consider $\mathbb{Z}[1/S]$ the ring of rational numbers whose denominators have all prime factors in $S$. Suppose that $\Gamma \subset \mathrm{GL}_n(\mathbb{Z}[1/S])$ with Zariski closure $G$. Strong approximation asserts that the closure of $\Gamma$ in $\prod G(\mathbb{Z}_p)$ is of finite index. Informally, strong approximation says that the closure of a linear group in the congruence topology is essentially determined by its closure in the Zariski topology.

A nice application of this is due to Lubotzky (1996). Recall that the start of the Gromov–Piatetski-Shapiro examples was the construction of a separating hypersurface in a hyperbolic manifold. Millson (1976) had noticed that on taking a finite cover, this hypersurface lifts to several components.

Actually this virtual disconnectedness is true in general, as the fundamental group of the hypersurface is not Zariski-dense in $\mathrm{O}(n, 1)$ – it lies in a smaller $\mathrm{O}(n-1, 1)$ – and therefore not congruence dense. A suitable deep finite congruence cover will therefore have the hypersurface disconnected.

As each of the sides is Zariski-dense in the group, these both have full image, which means that the complement of the union of the lifts of the hypersurface have two components.

A corollary of Van Kampen's theorem and these observations directly gives:

**Theorem 3.8** *Every hyperbolic manifold with a separating hyperbolic hypersurface has a finite index subgroup whose fundamental group surjects to a free group.*[38]

This then implies that such a lattice has *many* subgroups of finite index – indeed super-exponentially in the index (since nonabelian free groups do).

Another nice application of strong approximation, also due to Lubotzky (1987), is the following.

**Theorem 3.9** *Any finitely generated group linear group in a field of characteristic* 0 *always has subgroups of index divisible by d (for any given d).*

We refer to Lubotzky and Segal (2003) for a more thorough discussion of strong approximation, its literature and applications.

[38] Explicitly, let $M$ be a manifold containing two hypersurfaces $A$ and $B$ whose union does not separate $M$ and $*$ be a base point of $A \cup B$. Then, making a curve transverse to $A \cup B$, one can write a product $aabba^{-1} \cdots \in F_2$ recording the order and directions of the intersections. This gives a (surjective) homomorphism $\pi_1(M) \to F_2$.

## 3.5 Property (T)

In this brief section we will discuss the notion of Property (T), discovered by Kazhdan during the 1966 Moscow ICM (during a game of ping-pong with Atiyah). While it seems at first like a technical property about unitary representations, it has had applications – surely not all foreseen at that point – to many areas of mathematics, and (via the notion of expander graph) theoretical computer science.

We shall also discuss the opposite notion, amenability, originally introduced by von Neumann in his analysis of the Banach–Tarski paradox. These are both fascinating subjects deserving (and having received) book-length treatments; here they are merely introduced in recognition of the role they will play several times in what follows.

We will begin on the amenable side of the universe, since it is more familiar. For finite groups $G$, averaging the values of a real-valued function on $G$ is a general and straightforward algebraic procedure that involves no limiting procedures. If $G$ is compact then, at least for continuous functions, this can be done by integration with respect to Haar measure.

Remarkably, using weak-$*$ limits it is possible to define averaging processes on some infinite groups. Even for $\mathbb{Z}$ this is a remarkable statement: we are asserting that there is a functional

$$A\colon L^\infty(\mathbb{Z}) \to \mathbb{R}$$

that assigns a number to any bounded sequence of real numbers, agrees with ordinary limit when it exists, and is positive, linear, and translation invariant. Positivity means that $A(f) \geq 0$ if $f \geq 0$. Linear is obvious and translation invariant means that $A$ is invariant under the action of $\mathbb{Z}$ on $L^\infty(\mathbb{Z})$ by translation. Positivity and linearity can be achieved by extending any $f$ (since $\mathbb{Z}$ is discrete, any function is continuous) to $\beta\mathbb{Z}$, the Stone–Čech compactification and evaluating this extension on any point in $\beta\mathbb{Z} - \mathbb{Z}$.

The invariance requires using a bit of the geometry of $\mathbb{Z}$, but this is the key! Replace the sequence by its averages (i.e. like Cesàro means). Let $g(n) = 1/(2|n+1)| \sum f(m)$ (where the sum is over the interval $I_n = [-|n|, |n|]$.

**Observation** $A$, defined as the limit of the sequence $g(n)$, is translation invariant because the number of elements in the symmetric difference $I_n \triangle T I_n$ is $o(\# I_n)$.

**Remark** We made the construction using the Stone–Čech compactification. Sometimes (as hinted above) people construct $A$ as a weak-$*$ limit of the averaging functionals that define the values of $g$; sometimes non-principal ultrafilters

https://doi.org/10.1017/9781316529645.004 Published online by Cambridge University Press

are used in making this construction. These are just cosmetic differences – although they have somewhat different feels (point-set topology versus functional analysis versus logic).

Note the averaging procedure (and the limiting procedure) is well defined when the sequence has a limit. However, in general, it is very dependent on our choices. For example, suppose we had replaced the intervals $I_n = [-|n|, |n|]$ by intervals $J_n = [n - |n|, n + |n|]$; we still would obtain an averaging function that satisfies all the above properties, yet would have a much less democratic[39] feel than the $I_n$ seem to have – the values of $f$ at most integers (e.g. those outside of union of the $J_n$) will then be completely irrelevant.

Democracy put aside, the above consideration suggests defining a *Folner sequence*[40] to be a sequence of subsets $A_n$ of $\Gamma$, so that for any $\gamma$, $\#(\gamma A_n \triangle A_n)/\#A_n \to 0$. (This need only be checked for generators.) Under those conditions we can define a left-invariant positive linear functional by the procedure above. Folner (1956) proved the converse, that a group has a mean iff there is a sequence of such sets. Groups that have such a mean, or equivalently, an exhaustion[41] by subsets whose "boundaries" are asymptotically negligible, are called *amenable*.

(The boundary of a set in $\Gamma$ is precisely the the union symmetric difference of the set with its translates under a generating set of $\Gamma$. If we consider the volume of a set the number of elements it contains, then the last sentence is just a restatement in words of the formula of the previous one.)

There is a close connection between amenability and unitary representation theory. Consider the unitary action of $\Gamma$ on $L^2\Gamma$. It has a nontrivial fixed vector iff $\Gamma$ is finite.

However, $v_n = (1/\sqrt{\#A_n}) \sum \gamma$ where the sum is taken over $A_n$ is a sequence of *almost-invariant vectors*. That is, $||v_n|| = 1$ but for every $\gamma$, $||\gamma v_n - v_n|| \to 0$. One can describe this as saying that the trivial representation is weakly contained in the regular representation – another equivalent of amenability.

Yet another interpretation of amenability can be given in terms of the Laplacian on functions $\nabla \colon L^2\Gamma \to L^2\Gamma$ defined as follows. We shall consider $\Gamma$ as a graph, as usual, choosing a finite symmetric generating set $S$, and connecting two elements $g$ and $g'$ if there is an $s \in S$ such that $g = sg'$ (so that $\Gamma$ acts on the right by isometries). Define the Laplacian by $\nabla f(x) = f(x) - (1/\#S \sum f(sx))$.

39 And more fickle, in that $J_n$ is disjoint from the later sets averaged over.

40 These considerations do not explain why we would give *this* name to this class of subsets, only that we call attention to them. The last sentence in the paragraph is necessary for that point.

41 It is a very elementary fact that if a discrete metric space $X$ has a Folner sequence of subsets, then it has an exhaustion by Folner sets $B_i$; i.e., $B_i \subset B_{i+1}$ and $X = \bigcup B_i$.

It compares $f$ to its average. Note that $\nabla$ is a (bounded) self-adjoint and positive operator (by direct calculation of $\langle \nabla f, f \rangle$).

**Theorem 3.10** (Kesten, 1959) *$0 \in \mathrm{Spec}(\nabla)$ iff $\Gamma$ is amenable. This is equivalent to each of the following two statements:*

(1) *The symmetric random walk on $\Gamma$ does not have exponentially decaying return probabilities, i.e. $p_{2n}(e,e) \neq O(c^n)$ for any $c < 1$, where $e$ is the identity element of the group.*
(2) *The number of words (in the symmetric set of generators $S$) of length at most $2n$ representing the trivial element $W(n)$ satisfies $W(n)^{1/2n} \to \#S$.*

Note that the statement $0 \in \mathrm{Spec}(\nabla)$ does not mean that there are any eigenvectors with eigenvalue 0 (although that would be the simplest explanation), i.e. $\ker \nabla$ need not be nontrivial, because of the possibility of a nondiscrete spectrum. Indeed, 0 is an eigenvalue[42] iff $\Gamma$ is finite.

However, the almost-invariant vectors are test functions of norm 1 with $|\nabla f_n| \leq \sum \#(\gamma A_n \nabla A_n)/\#A_n$ (summed over the elements of $S$) showing that it is not true that $\langle \nabla f, f \rangle > c||f||^2$ for any $c > 0$.

The connection between random walk, heat flow, and the Laplacian is important. Note that $\nabla = I - \mathrm{M}$, where M is the Markov operator, defined by

$$\mathrm{M} f(x) = \mathrm{E}(f(\gamma x)),$$

where E means, as always, the expectation value of a random variable, and here it is $f$ of a random neighbor of $x$ (i.e. the translate by a random generator of $\Gamma$). Note $||\mathrm{M}|| \leq 1$, and equality holds iff $\Gamma$ is amenable. The probability of return is given by

$$p_n(e,e) = \langle \delta_e, \mathrm{M}^n \delta_e \rangle.$$

So if $0 \notin \mathrm{Spec}(\nabla)$, we get exponential decay of the return probabilities. (The converse is tricky.) The expression $W(n)/\#S^{2n}$ is simply another calculation of $p_{2n}(e,e)$ and hence statement (2) is equivalent to (1).

Property (T) is *opposite* to amenability (not its negation!) and it is quite nontrivial that there are any infinite groups at all that have this property.

**Definition 3.11** A group $\Gamma$ has Property (T) if *every* unitary representation that has almost-invariant vectors has a fixed vector. (In other words, given a generating set $S$, there is a Kazhdan constant $\varepsilon$ – that typically depends on $S$ – such that, for any nontrivial irreducible representation $\rho$ (or, equivalently, any

[42] There is a natural generalization of $\nabla$ to differential forms, and then as we will discuss in §3.6, $\nabla$ frequently has nontrivial kernel acting on $L^2$-forms.

https://doi.org/10.1017/9781316529645.004 Published online by Cambridge University Press

representation with no nontrivial fixed vectors ρ), the only $v$ with $||\rho(s)v - v|| \leq \varepsilon||v||$ is $v = 0$. [43]

An amenable discrete group has Property (T) iff it is finite – one can construct almost-invariant vectors by averaging over a sequence of Folner sets.

Margulis showed that higher-rank lattices have only finite or finite index normal subgroups by the crazy strategy of showing that all quotients are amenable and have Property (T). Obviously, arbitrary quotients of Property (T) groups have Property (T).

Kazhdan observed, in his original 1967 paper, via consideration of induced representations, the following.

**Proposition 3.12** *A locally compact group $G$ has Property (T) iff any (and hence every[44]) lattice $\Gamma \subset G$ does.*

He also showed

**Proposition 3.13** *A discrete group with Property (T) must be finitely generated.*

For suppose that $\Gamma = \bigcup \Gamma_n$ is an ascending union of proper subgroups. Then $\bigoplus L^2(\Gamma/\Gamma_n)$ is a unitary representation which has almost-invariant vectors (each $\gamma$ ultimately acts trivially, so a sequence of vectors that are nontrivial only in the components indexed by a large $n$ form an almost-invariant sequence of vectors), but it will have an invariant vector only if some $\Gamma_j = \Gamma$.

**Theorem 3.14** (Kazhdan) *Products of real simple Lie groups of rank greater than* 1 *have Property (T).*

He deduced that lattices in these groups were finitely generated.

We already know enough to see that $O(n, 1)$ does not have Property (T), because we know lattices that have nontrivial $\mathbb{Z}$ quotients, and note that Property (T) is (obviously!) inherited by quotients. Less simple is that $U(n, 1)$ also does not have Property (T). This is shown in Kostant (1975), as is the following positive result.

**Theorem 3.15** (Kostant) $Sp(n, 1)$ *has* Property (T)*, as does the real rank* 1*-form* $F_{4(-20)}$ *of the exceptional complex Lie group of type* $F_4$.

This gives us now negatively curved examples of Property (T) groups. We

43 The notation is supposed to indicate that the trivial representation $T$ is separated from all the other irreducible representations (by the parentheses).

44 Assuming there is at least one!

https://doi.org/10.1017/9781316529645.004 Published online by Cambridge University Press

can add large powers of all the elements one at a time,[45] and maintain negative curvature, giving (uncountably many![46]) Property (T) groups that are torsion.

The early history of Property (T) only had examples that came out of representation theory. Now there are completely different mechanisms for this of both algebraic and analytic geometric origin – so now there are many other Property (T) groups known. Before saying a little more about this, we digress to give another characterization of Property (T) (see Shalom, 2000; Bekka *et al.*, 2008).

**Theorem 3.16** (Delorme–Guichardet, Shalom) *A group has* Property (T) *iff every action of* $\Gamma$ *on a Hilbert space by affine isometries has a fixed point. If the group does not have* (T) *then there is an action where not only is there no fixed point, but the displacement* $\sum ||v - \gamma(v)||^2$ *has a realized minimum on the unit sphere (where* $\sum$ *is over the generating set).*

All amenable groups have affine isometric actions that are metrically proper, i.e. actions for which the orbits of vectors $\to \infty$ in norm (as $\gamma \to \infty$) as was shown by Bekka, Cherix, and Valette – yet another way in which Property (T) and amenable are at opposite poles.

A consequence of this theorem is that:

**Corollary 3.17** *If a group* $\Gamma$ *acts simplicially on a tree (without inversions) without fixing any vertex, then* $\Gamma$ *cannot have* Property (T).

This excludes nontrivial amalgamated free products and HNN extensions, as well as giving another argument for the finite generation of Property (T) groups (see Serre, 2003). We shall prove the corollary by noting that, if $\Gamma$ acts on a tree $T$, then it acts on $L^2(T)$.

**Proposition 3.18** (Cartan) *If* $\Gamma$ *acts on a tree* $T$ *and it has a bounded orbit, then it has a fixed point.*

Cartan was actually working on other spaces of nonpositive curvature.[47] The proof goes like this. Given a bounded set in a tree, it lies in a *unique* ball of smallest radius. As this the bounded set is $\Gamma$-invariant, so is that ball, and therefore its center is fixed.

If the action of $\Gamma$ on $T$ has no bounded orbit, then $L^2(T)$ has no fixed vectors, which is incompatible with Property (T).

[45] This is an application of the “Dehn filling” idea as in the previous chapter.

[46] And hence the fact that Property (T) does not force finite presentability.

[47] I believe that Cartan’s application was the uniqueness of the maximal compact in a semisimple group by considering the action of such a group on $G/K$, a complete manifold of nonpositive curvature. Incidentally, the analogous fact in the case of Lie groups over local fields makes use of the curvature properties of Tits buildings.

## Appendix: Property (T) and Expanders

Expander graphs are graphs that are hard to disconnect, i.e. require the removal of many edges to separate a large number of vertices from the rest. It (now) seems obvious that such graphs should be valuable for the construction of things like communication networks. But, in fact, they have legion applications in theoretical computer science (Hoory *et al.*, 2006) and pure mathematics (Lubotzky, 1984, 2012).

We consider finite $d$-regular graphs $\Gamma_i$ (for simplicity – a bound on valence is really all that's necessary). We consider the Cheeger constant of these graphs

$$h(\Gamma) = \inf(\#\partial A/\#A),$$

where $A$ is a subset of $\Gamma$ with fewer than half of the vertices, and $\partial A$ is the set of vertices of $A$ that share an edge with $\Gamma - A$. If we allowed $A$ to be big then setting $A = \Gamma$ we'd always get 0 as our infimum.

This notion makes sense for infinite graphs, as well as finite ones, if we *impose* the condition that $A$ is finite in the infinite case. Note that $\Gamma$ is amenable as a group iff $h(\Gamma) = 0$ viewing $\Gamma$ as a (Cayley) graph – and that this condition is equivalent to $0 \in \mathrm{Spec}(\nabla)$.

However, for expansion, we are interested in *finite* graphs, and we want the reverse, i.e. that $h(\Gamma_i) > \varepsilon > 0$.

To summarize:

**Definition 3.19** An expander sequence of $d$-regular graphs is a sequence $\Gamma_i$ (of $d$-regular graphs) such that $h(\Gamma_i) > \varepsilon > 0$.

These were first introduced and studied *explicitly* by M. Pinsker (in Bassalygo and Pinsker, 1973) – and in a paper presented at the 7th International Teletraffic Conference. He showed that they exist, by arguing that random graphs are expanders. They have been an important tool in theoretical computer science ever since, and you can find much interesting material and history in Hoory *et al.* (2006).

More recently, it was pointed out in Gromov and Guth (2012) that Pinsker was preceded by a paper of Kolmogorov and Barzdin that studied expanders as models for the brain (nodes on the surface and axons going through the bulk, without disjoint axons getting too close to one another), but then, alas, having an upper bound on size[48] to fit into our heads. Expanders were their examples of graphs that would be hard to fit in our heads.

Why this genericity of expansion should be true is clear if one considers a toy variant. Consider the graph $\Gamma$ with $n$ vertices determined by two permutations, using each permutation to connect $[i]$ to $[\pi i]$ (note that $[i]$ is also connected to $[\pi^{-1} i]$). Given a subset $A$, then the expected number of edges leaving $A$ is $\#A(1 - \#A/\#\Gamma)^4$ suggesting a bound of at most 1/16 independently of $\#\Gamma$. Of course, there are many choices of $A$, and we have to compute the expected extremal. This means one should look at subsets $A$ of size $n/2$ that contain significantly fewer edges leaving them, say $n/20$, and then estimating tail probabilities in a binomial distribution. The details are left to the reader.[49]

If one is interested in using this for building a network (or an error-correcting code or . . . ), then random methods are not so useful – buildings surely must be built from blueprints.[50] The applications in mathematics often require knowing that certain graphs form expander sequences.[51]

Now, for finite graphs, 0 is always in the spectrum of $\nabla$. Constant functions have $\nabla f = 0$. And 0 has multiplicity greater than 1 iff $\Gamma$ is disconnected (different constant functions on the different components). Graphs that are connected but easily disconnected should therefore be characterized as having an eigenvalue near 0. This is the content of the following basic theorem.

**Theorem 3.20** *A sequence of d-regular graphs is an expander sequence iff there is an* $\varepsilon > 0$ *so that the spectrum of* $\nabla$ *restricted to functions with* $\int f = 0$ *(the orthogonal complement of the constants) is bounded* $> \varepsilon > 0$.

We will denote by $L^2(\Gamma)^\circ$ this subspace of $L^2(\Gamma)$.

This theorem is inspired by Cheeger's theorem in Riemannian geometry (see Cheeger, 1970) that bounds the isoperimetric constant of a Riemannian manifold in terms of the spectrum of the Laplacian. Note that for a subset $A$, the modified characteristic function, $f_A = 1_A - \#A/\#\Gamma$ has $\int = 0$, and $\nabla$ related to $\#\partial A/\#A$. The isoperimetric constant is approximately realized by a level set of an eigenfunction for a small eigenvalue.

48 There is a bound to how much of an expander can be fit without distortion, even in Hilbert space. This will be of critical importance later for purposes of the Novikov conjecture. For science fiction purposes, the cognitive capacities of aliens elsewhere in the multiverse can be expected to be greater than ours, in the Kolmogorov–Barzdin model, only if the number of spatial dimensions increases (or they have better programming of their neural nets).

49 Actually, to the active reader. An inactive reader can find them written down in many places.

50 I expect this to be my *bon mot* quoted years after I have otherwise been forgotten, showing how shortsighted people were back at the beginning of the third millennium. Indeed, I almost deleted this comment during revision.

51 Many of these are closer to the Selberg example explained below than the Property (T) examples we begin with now. This is a good moment to mention that there are now many constructive methods of getting expanders that do *not* come out of Property (T).

A consequence of this theorem is that a random walk on an expander sequence is rapidly mixing.[52]

The following important result of Margulis (1988) is now perhaps anticlimactic, given our discussion.

**Theorem 3.21** *Suppose that* $\Gamma$ *is a group with* Property (T). *Then the Cayley graphs of* $\Gamma/\Gamma_i$, *for a sequence of normal subgroups of finite index in* $\Gamma$ *(using a common generating set S coming from* $\Gamma$*), gives a sequence of expander graphs.*

To see why the isoperimetric inequality is true, consider $\oplus L^2(\Gamma/\Gamma_i)^\circ$ (where the superscript $^\circ$ means that we are considering the orthogonal complement of the constant vectors) and, since there are no fixed vectors, there can be no almost invariant vectors, which means that $\nabla f_{A_i}$ is large, which means that $\partial A_i$ is also large.

Concretely we can set $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ for any $n > 2$ (and use the elementary matrices as a generating set) and obtain the expander sequence $\mathrm{SL}_n(\mathbb{Z}/m)$ – where $m$ is varying.

Note, by the way, that the representations arising in this proof are all (sums of) finite-dimensional representations of the group $\Gamma$, so we are nowhere near the full power of Property (T). Lubotzkyc and Zimmer have suggested the notion of Property $\tau$, which is Property (T) for finite-dimensional representations, or even restricting further to a class of finite quotients (say ones factoring through some finite quotient or some congruence quotient).

A good example of this is $\mathrm{SL}_n(\mathbb{Z})$ for $n = 2$. We shall work with a congruence subgroup of this group, which is a free group. Obviously, it does not have Property (T), as it has a $\mathbb{Z}$ quotient, and just as obviously covers corresponding to the subgroups $k\mathbb{Z}$, for a surjection of this group to $\mathbb{Z}$, have isoperimetric constant $\to 0$ (consider the inverse image of the interval $[0, k/2]$ in the cycle) and, again just as obviously, the bottom of the spectrum of these quotients of $\mathrm{SO}(2)\backslash\mathrm{SL}_2(\mathbb{R}) \to 0$ (by considering functions that are 1 on $[0, k/2]$ and $-1$ on $[k/2, k-1]$).

However, when we restrict our attention to the family of *congruence quotients*, then a theorem of Selberg asserts that, for all of these manifolds, $\mathrm{SO}(2)\backslash\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z}; k)$ has $\lambda_1 > 3/16$. One can translate between graphs and manifolds, and actually this is a family of expander graphs whose girth[53] grows[54] (logarithmically) with $k$.

52 Which perhaps suggests its application to de-randomization.

53 The girth of a graph is the length of the shortest cycle in the graph; it is an analog of the length of the shortest geodesic (= twice the injectivity radius) of a compact manifold.

54 Note that if we use the Property (T) expanders, relations in the fundamental group give bounded cycles everywhere in the graph. Random graphs will frequently have some short cycles, but relatively few of them.

https://doi.org/10.1017/9781316529645.004 Published online by Cambridge University Press

Finally, we close our discussion by mentioning one of the more recent methods for proving Property (T), because it turns our discussion on its head and uses expander properties as a way of obtaining Property (T).

**Theorem 3.22** *Let $\Gamma$ be a group generated by a finite symmetric set $S$, with $e \notin S$. Let $L(S)$ be the graph with vertex set $S$ and in which $\{s, s'\}$ is an edge if $s^{-1}s' \in S$. Suppose that $L(S)$ is connected and has spectral gap greater than $1/2$. Then $\Gamma$ has* Property (T).

As a nontrivial consequence of this, in some models of random groups, having Property (T) is generically the case – a far cry from the essentially Lie-theoretic origin of the first examples. Moreover, this method produces groups with very strong fixed-point properties, often stronger than those true for lattices in high-rank groups. See the notes in §3.9 for some more discussion of this important direction.

## 3.6 Cohomology of Lattices

The cohomology of lattices is a topic of endless fascination that can be studied from many viewpoints, from the geometric[55] (construction of explicit cycles) to the analytic (e.g. Hodge theory and $L^2$-cohomology) to the number-theoretic (such as Langands functoriality). In this section we will touch briefly on a few methods for producing cohomology classes motivated by purely utilitarian needs. For simplicity, we will divide our discussion into four parts:

(1) Property (T) and $H^1$;
(2) Matsushima formula and connection to representation theory;
(3) generalized modular symbols and geometric cycles;
(4) $L^2$-cohomology.

### 3.6.1 $H^1$ and Property (T)

We have already tacitly discussed $H^1(\Gamma; \mathbb{R})$ when discussing Property (T). Its vanishing is necessary if $\Gamma$ satisfies (T), because otherwise $\mathbb{Z}$ is a quotient of $\Gamma$, and (T) is inherited by quotients.

Actually we had, less obviously, given a cohomological interpretation of

[55] Not to mention the heroic geometric group-theoretic work of Agol (2013), Haglund and Wise (2007, 2012), and Kahn and Markovic (2012) that gives positive first Betti number (and even more, homomorphisms to $\mathbb{Z}$ with finitely generated kernels) for finite covers of lattices in O(3, 1). See the wonderful exposition by Bestvina (2014).

https://doi.org/10.1017/9781316529645.004 Published online by Cambridge University Press

Property (T) in characterizing those groups by the fixed-point property: any action of $\Gamma$ on a Hilbert space $H$ by affine isometries has a fixed point.

This statement can be expressed cohomologically. Any affine action has a unitary part $\rho\colon \Gamma \to \mathrm{U}(H)$. (It can be obtained by letting $\rho(\gamma)(v) = \lim t\gamma(t^{-1}v)$ as $t \to 0$.) Affine actions are associated to cocycles, and cohomologically trivial ones are the ones with fixed points (i.e., are actually unitary after conjugating by a suitable translation).

Thus, the Delorme–Guichardet fixed-point theorem can be viewed as the cohomological statement that:

**Theorem 3.23** $\Gamma$ *is a group with* Property (T) *iff, for any unitary representation $\rho$ of* $\Gamma$*, $H^1(\Gamma; \rho) = 0$.*[56]

The reason is this. The 1-cochains with values in the representation, $\mathrm{C}^1(\Gamma; \rho)$ is made of $H$-valued functions on $\Gamma$, and an element $\alpha \in \mathrm{C}^1(\Gamma; \rho)$ lying in $\ker \mathrm{d}\colon \mathrm{C}^1(\Gamma; \rho) \to \mathrm{C}^2(\Gamma; \rho)$ means that $\alpha(\gamma\gamma') = \rho(\gamma)\alpha(\gamma') + \alpha(\gamma)$. Associated to a cocycle is the affine isometric action on $H$ where $\gamma$ acts by $\gamma v = \rho(\Gamma)v + \alpha(\gamma)$. This cocycle is a coboundary of a vector $v \in H = \mathrm{C}^0(\gamma; \rho)$ if $\alpha(\gamma) = v - \rho(\gamma)v$. Then $\gamma v = v$ for every $\gamma$ and the action has a fixed point (and one can conjugate the action by a translation to a unitary action).

Part of the interest in such statements is because of their connection to deformations. The infinitesimal version of rigidity asks about deformations of the defining representation $\rho\colon \Gamma \to G$. Reasoning about deforming the defining representations and working modulo the deformations given by inner automorphisms leads one to want to prove vanishing of such cohomology groups.

Kazhdan's approach to Property (T) gives a representation-theoretic method, but other cohomological vanishing theorems have been proved by Hodge-theoretic methods or Bochner arguments. These methods were employed by Calabi and Vesentini (1960), Calabi (1961), Weil (1960, 1962, 1964), and Selberg (1960, 1965) to prove early local rigidity theorems. They still are useful – as rigidity moves into new settings (such as for non-lattices, and fixed-point properties for actions on spaces other than Hilbert spaces).

Another consequence of rigidity of representations is that the defining representations of such a group cannot have "essential" matrix coefficients that are transcendentals, because transcendentals can always be deformed (or specialized). (By "essential" I am ignoring the possibility of conjugacy of an algebraic

[56] The Shalom improvement we had mentioned above replaces this cohomology by its reduced version, where one mods $\ker \partial$ by the *closure* of $\mathrm{Im}\,\partial$. Often reduced and unreduced groups are different, and the reduced ones are easier to study, but it sometimes happens that they vanish simultaneously (at least in low dimensions) – see also Block and Weinberger (1992), where a similar phenomenon occurs in a characterization of amenability.

https://doi.org/10.1017/9781316529645.004 Published online by Cambridge University Press

matrix by a transcendental one.) To grossly simplify, this is why superrigidity (a vast generalization of Kazhdan's Property (T) for Γ) leads to arithmeticity theorems.[57]

It is worth noting that an immediate consequence of the theorem as stated is that all finite-dimensional irreducible representations are separated from unitary representations that don't contain them. In addition, although this is obvious in any case, as cohomology with coefficients in representations includes cohomology of covers, philosophically this study naturally leads us to consider the behavior in towers simultaneously with the cohomology of a given space, a theme we will return to in §3.6.4.

### 3.6.2 Matsushima Formula

The yoke binding representation-theoretic theory and cohomology is tightened by the Matsushima formula that extends the earlier observations connecting the cohomology of the compact dual to that of all locally symmetric manifolds with a given universal cover.

Unlike those previous observations, it has the virtue of being sensitive to the lattice. We will not directly make use of this material, but an awareness of it will make some discussions make more sense (or seem better motivated[58]).

The discussion is much simpler in the case of cocompact lattices, so we start by making this assumption.

The complex of differential forms on $K\backslash G/\Gamma$ can be identified with the cochain complex $C^*(\mathfrak{G}, \mathfrak{K}; C^\infty(G/\Gamma))$, where $\mathfrak{G}, \mathfrak{K}$ are the Lie algebras of $G$ and $K$, respectively, and we use the Chevalley–Eilenberg complex for relative Lie algebra cohomology. Thus $H^*(K\backslash G/\Gamma) \cong H^*(\mathfrak{G}, \mathfrak{K}; C^\infty(G/\Gamma))$.

It turns out (this is a kind of elliptic regularity result) that we can break $C^\infty(G/\Gamma)$ into pieces according to the decomposition of $L^2(G/\Gamma)$. This is a sum of pieces that are $G$-invariant irreducible representations, with finite multiplicity. Ultimately one gets a formula of the form

$$H^*(K\backslash G/\Gamma) \cong \bigoplus m(\Pi, \Gamma) H^*_C(\mathfrak{G}, C^\infty \Pi),$$

where the right-hand side is a sum over irreducible representations of $G$, with multiplicities according to the number of times that they appear in $L^2(G/\Gamma)$ and the cohomological term (which only involves $G$ and its representations, and not the lattice) being continuous cohomology with coefficients in the smooth vectors in the given representation. When $G$ is simple, the cohomological term vanishes whenever $* < \mathrm{rank}_{\mathbb{R}}(G)$ and there are no terms other than the compact

[57] And, for example, Property (T) itself implies that all finite-dimensional unitary representations are equivalent to ones defined over an algebraic number fields.

[58] Or less unmotivated.

https://doi.org/10.1017/9781316529645.004 Published online by Cambridge University Press

dual (which is the contribution of the trivial representation). (In the semisimple case, this vanishing holds below the lowest $\mathbb{R}$-rank of any of the factors.) These facts are responsible for the independence of rational cohomology in the stable range of the lattice – at least in the uniform case.

The place where the lattice enters is in the nontrivial representations because of the multiplicities $m(\Pi, \Gamma)$. These will frequently grow as $\Gamma$ shrinks (note that if $\Gamma'$ is a normal subgroup of $\Gamma$, the finite group $\Gamma/\Gamma'$ acts on any of these $\Pi$s, and since these representations don't have a trivial part, the multiplicities must be nontrivial). A geometric approach to this is the following. If $\Gamma$ is arithmetic, then it has non-normal subgroups that have a large number of symmetries (i.e., that do not cover the original manifold).[59] When one pulls a harmonic form up to such a cover, it can well be non-invariant under this action – causing the amount of cohomology to grow. If this would never happen, it would mean that the pullback to the universal cover would be invariant under $G(\mathbb{Q})$, which is exactly equivalent to it coming from the compact dual.

When $\Gamma$ is nonuniform, then the above analysis of cohomology does not work directly, but Borel (1974) showed that nevertheless there is a range depending on the $\mathbb{Q}$-rank where it does hold. This is enough for the applications to $\mathrm{SL}_n$ when we let $n \to \infty$, (which is important for $K$-theory), but this is not enough for our immediate needs. Some highly unstable classes in the nonuniform case that are always beyond the range of this isomorphism are the topic of the next subsection.

### 3.6.3 Generalized Modular symbols

A different and transparent example of how cohomology grows in covers that is visible in hyperbolic geometry occurs for nonuniform lattices (in all dimensions).

If $M$ is a noncompact finite volume hyperbolic $n$-manifold, then $\mathrm{cd}(\pi_1 M) = n - 1$ (because $M$ has the homotopy type of an $(n-1)$-dimensional complex, and it contains a $\mathbb{Z}^{n-1}$ in the fundamental group of the cusp).[60] It can certainly happen though that $H_{n-1}(M) = 0$ (e.g. this is true for all the hyperbolic knot

[59] This is related to the large *commensurator* of an arithmetic group. $G(\mathbb{Q})$ acts on the disjoint union of the $K\backslash G/\Gamma'$ where $\Gamma'$ is commensurable with $\Gamma$, but each of these individual manifolds is only acted on by the normalizer of their own *fundamental group in* $G(\mathbb{Q})$. If $\Gamma$ is arithmetic and $\Gamma'$ is a $G(\mathbb{Q})$-conjugate of $\Gamma$ (but not necessarily in the normalizer of $\Gamma$), we can take a subgroup $\Gamma''$ of finite index in $\Gamma \cap \Gamma'$ that is normal in $\Gamma'$ but not in $\Gamma$. The group $\Gamma'/\Gamma''$ acts on the $\Gamma''$ cover, which is a cover of $K\backslash G/\Gamma$, but the action does not cover the projection to $K\backslash G/\Gamma$. These *hidden symmetries* are responsible for the algebra of Hecke operators that acts on cohomology groups of arithmetic manifolds.

[60] Of course, this is, in the arithmetic case, a special case of the result of Borel and Serre that the cohomological dimension differs from $\dim(G/K)$ by the $\mathbb{Q}$-rank.

https://doi.org/10.1017/9781316529645.004 Published online by Cambridge University Press

complements in the 3-sphere). However, the fundamental group of the cusp is a proper "small" subgroup of the fundamental group, i.e. it is not Zariski-dense – it obviously lies in a proper parabolic, so by strong approximation we can find finite congruence quotients of $\pi_1 M$ onto which the cusp maps to a proper subgroup.

This means that these covers have multiple cusps (by covering space theory). Once you have more than one cusp, then $H_{n-1}(M) \neq 0$, because each cusp gives a cycle,[61] and the one relation among these is that the sum of all of these cycles vanishs. Associated to a pair of cusps there is a (number of) proper geodesic(s) lines going from one cusp to the other. These will have intersection number 1 and $-1$ on these two cusps (depending on ordering, and using a standard, say inward normal, convention for orientation of boundaries) and 0 with the other cusps. Each such proper geodesic gives a functional on homology which proves the nonvanishing of the individual cusps. (In fact, picking one cusp as a base, the lines connecting that cusp to all the others give #(cusps – 1) independent cycles.) As we go deeper in the group (or up a tower), the number of cusps increases and hence the size of the homology.

Of course, when the $\mathbb{Q}$-rank $> 1$, then this doesn't make sense as stated: the Borel–Serre boundary is connected in all covers, and $\pi_1^\infty \to \pi_1$ surjects. However, when we pay closer attention to the corners within the Borel–Serre boundary, which correspond to proper parabolic subgroups, none of these surjects, and the covers do indeed cause *these corners* to become multiple components, and then give rise to cycles.

**Theorem 3.24** (See Ash and Borel, 1990; Schwermer, 2010) *Let $G$ be an algebraic group defined over $\mathbb{Q}$, and let $P$ be a $\mathbb{Q}$-parabolic subgroup of $G$. If $P(\mathbb{R}) = M(\mathbb{R})A(\mathbb{R})N(\mathbb{R})$ is the Langlands decomposition of this parabolic,*[62] *then there are nontrivial cycles in $K\backslash G/\Gamma$ of the form $N(\mathbb{R})/N(\mathbb{R}) \cap \Gamma$ in dimension* $\dim(N(\mathbb{R}))$ *if $\Gamma$ is sufficiently deep. Passing to a congruence subgroup $\Gamma'$ then there are at least* $\#(\Gamma'\backslash\Gamma/(\Gamma' \cap P))$ *(double cosets) linearly independent cycles obtained this way.*

Using congruence subgroups we then get a large (i.e. growing like a positive power of the volume, but definitely sublinear in it) rank of Betti number.

**Remark 3.25** (In place of proof) Generalized modular symbols are examples of *geometric cycles*. Geometric cycles are associated to Lie subgroups of $G$, and give rise to some explicit cycles, when the lattice intersects them in a lattice.

[61] To get a well-defined cycle, one should adopt an orientation convention, i.e. making use of the normal direction pointing towards $\infty$.

[62] So that $M$ is reductive, $A$ is abelian and $N$ is nilpotent.

https://doi.org/10.1017/9781316529645.004 Published online by Cambridge University Press

To get an embedding, one often has to pass to a finite cover, and then when one passes to deep enough covers, they will (by strong approximation) typically produce a number of disjoint cycles.

The standard way to check that these cycles are nontrivial is to find another geometric cycle of the dual dimension that intersects it with nonzero intersection number. In the above theorem, Levi subgroups are the source of duals.

As in the case of modular symbols, pulling these up covers can give growth to the Betti numbers.

We did this with $H^1$ and the Millson example that uses codimension-1 geometric cycles in arithmetic hyperbolic manifolds associated to quadratic forms, and, following Lubotzky,localization observed that this even gave maps onto free groups. In this case this implies that there is then a further tower of covers (*not* converging to the trivial group) for which $b_1$ grows linearly with the index.[63]

### 3.6.4 $L^2$-cohomology

None of the methods discussed till this point has the potential of giving Betti numbers that grow linearly with volume (or, equivalently, with the index of the cover). However, the Euler characteristic tells us that this must happen *sometimes*. If $\chi(K\backslash G/\Gamma) \neq 0$, then, by multiplicativity of $\chi$ in finite covers, as one goes up any family of covers, some Betti number must increase linearly.[64]

In this section we will review the relevant facts about $L^2$-cohomology and especially a remarkable theorem of Lück that explains exactly when this rare situation occurs in towers of regular covers.[65]

This story begins without any particular interest in finite sheeted covers, but rather with the consideration of arbitrary regular covers.[66] For infinite complexes, there are alternatives to the usual simplicial chain complex: one can consider, for example, locally finite chains, which gives rise to Borel–Moore homology. This gives a non-homotopy-invariant homology theory: it is invariant under proper homotopy equivalences.

A more subtle choice is to consider the complex of $L^2$ simplicial chains[67] (or

63 This family of covers shows very different geometry than that associated with congruence covers.

64 Clearly, no Betti number for covers of a finite complex can grow faster than linearly, since these are bounded by the number of cells, which grows exactly linearly in the number of sheets of the cover.

65 It actually also applies to sequences of regular covers that Gromov–Hausdorff converge to the universal cover. (See the notes in §4.11 for a recollection of Gromov–Hausdorff space.)

66 Indeed, it can be developed in terms of arbitrary group actions.

67 We are tacitly weighting all simplices equally in our discussion.

cochains). If the complex is locally finite (as it will be in all of our applications), then the $\partial$ map is a bounded map. Its homology is an invariant of $X$. It is functorial with respect to maps that are Lipschitz and "uniformly proper," i.e. if one has a bound on the size of the inverse image of simplices (or else the pushforward of an $L^2$ chain need not be $L^2$).

It is perhaps worthwhile to consider the case of $\mathbb{R}$. The chain complex is then identified with $0 \to L^2(\mathbb{Z}) \to L^2(\mathbb{Z}) \to 0$, where the boundary map sends $f \to (t-1)f$, where $t$ is a generator of $\mathbb{Z}$. Obviously $H_1 = 0$, but $H_0$ is a large infinite-dimensional space (for example $\delta_0$ is not in the image) but it doesn't seem to have much structure to say anything about.

There are two parts to the solution of this problem. The first is basic. We considered $L^2$ to enable the use of Hilbert-space methods, in which case we should insist that the constructed homology groups be Hilbert spaces. The way to achieve this is to insist that we never quotient out by non-closed subspaces, i.e. to take the closure of the image of $\partial$ when forming the homology groups. We will denote this version, i.e. where we take the quotient by closures, by $\mathcal{H}$.

(An equivalent alternative to using closures is to form a Laplacian from the chain complex in the usual formal way following Hodge, and define homology to be the kernel of the Laplacian. The "torsion" (closure $\operatorname{Im}\partial)/\operatorname{Im}\partial$ thrown away by this method corresponds to spectrum of $\nabla$ near 0 that does not consist of harmonic forms.

The second part is to note that, following Atiyah (1974), when we are dealing with a universal cover,[68] the action of $\Gamma$ on these Hilbert spaces is appropriate for defining a normalized dimension (using the theory of von Neumann algebras) that can be (in principle) an arbitrary nonnegative real number.[69] This will then define $b_i^{(2)}(X)$ (we suppress the $\Gamma$ from our notation, unless needed) – the $L^2$-Betti numbers of $X$:

$$b_i^{(2)}(X) = \dim_\Gamma \underline{\mathcal{H}_i}^{(2)}(X).$$

We proceed informally. The idea shall be that we want to see what fraction of the regular representation some other unitary $\Gamma$ representation is. We restrict attention to unitary representations that are closed subrepresentations of some multiple of the regular representation, as ours naturally are (viewing the quotient as the orthogonal complement to the image of the boundary).

We want $\dim_\Gamma L^2(\Gamma) = 1$. If $P$ is a $\Gamma$-equivariant projection of $\bigoplus L^2(\Gamma) \to$

[68] or even a regular cover.

[69] In general the indices of $L^2(\mathbb{Z})$-modules can be any real number. However, not all of these arise as dimensions of kernels and cokernels of elliptic operators. In the special case of the de Rham operator on general finite complexes (or compact manifolds), this question is the very fruitful area of the Atiyah conjecture, which has deep positive and negative results. For other operators, such as the signature operator on manifolds with boundary, it is very easy to obtain transcendental numbers as such dimensions, even if the fundamental group in $\mathbb{Z}$.

https://doi.org/10.1017/9781316529645.004 Published online by Cambridge University Press

$V$, then the dimension is a trace of $P$. To figure out what the trace should be, consider first the case when $\Gamma$ is finite. In that case, $V$ is finite-dimensional in the ordinary sense, and

$$\dim_\Gamma V = \dim(V)/\#\Gamma.$$

We can consider the matrix of the projection to have coefficients in $\mathbb{C}[\Gamma]$. This dimension is then the sum of the coefficients of the identity (element of $\Gamma$) along the diagonal, i.e. the coefficient of the identity in the trace. Note that when $\Gamma' \subset \Gamma$ is a finite index subgroup, we have:

$$\dim_{\Gamma'} V = [\Gamma : \Gamma'] \dim_\Gamma V;$$

here $L^2(\Gamma)$ is a sum of $[\Gamma : \Gamma']$ copies of $L^2(\Gamma')$ when thought of as a $\Gamma'$ representation. It turns out that the dimension of any nontrivial representation is positive in this sense.

This has the property that $\dim_\Gamma V \oplus W = \dim_\Gamma V + \dim_\Gamma W$. Very useful is the property (almost obvious from the above heuristic)

$$\dim_{\Gamma'} V = \dim_\Gamma \operatorname{ind}_{\Gamma'}^{\Gamma} V.$$

The usual homological algebra shows that $K = X/\Gamma$ a finite complex, then one has

$$\chi_\Gamma(X) = \chi(K).$$

Atiyah went further and showed that if one takes any elliptic operator on a compact manifold, then the $\Gamma$-dimension of the kernel and cokernel on the universal cover make sense, and one has an equality of indices upstairs and down, but this is rather more delicate – it requires more geometry and analysis than the result on Euler characteristics, which is a result of pure algebra.

It is easy to see that for any infinite complex (and hence for any infinite group acting freely) $\mathcal{H}^0_{(2)}(X) = 0$; a constant map is $L^2$ iff it is 0. Applying the Euler characteristic relation, we see from setting $K$ to be a finite graph that #generators of the free group $F$ acting freely and cocompactly on it equals $1 - \dim_F \mathcal{H}^1_{(2)}$ (regular tree).

On the other hand, if $\Gamma$ is amenable, then Cheeger and Gromov (1986) showed that for $X = \mathrm{E}\Gamma$, the universal cover of $\mathrm{B}\Gamma$, we have that $\mathcal{H}^i_{(2)}(X) = 0$. They deduced from that that the same is true for any $\Gamma$ with an infinite amenable normal subgroup. And therefore $\chi(K) = 0$ if $K$ is an aspherical complex whose fundamental group has an infinite normal amenable subgroup.

All of this connects to finite covers for residually finite groups by a beautiful theorem of Lück.

**Theorem 3.26** (Lück, 1994) *If $K$ is a finite complex with residually finite fundamental group $\Gamma$ and universal cover $X$, and letting $\Gamma_i$ be a descending chain of normal subgroups (with $K_i$ the associated covers) then*

$$\lim H_k(K_i)/[\Gamma\colon \Gamma_i] = b_k^{(2)}(X).$$

Thus for finite complexes one can ascertain linearity of the growth of Betti numbers in terms of $b_k^{(2)}(X)$ in terms of the universal cover, i.e. are there *any* $L^2$-harmonic $k$-forms. This is, interestingly enough, a statement that does not depend on the uniform lattice that is acting, or the sequence of normal finite index subgroups used in defining, the normalized Betti numbers.

It turns out that one can use harmonic analysis[70] on Lie groups to obtain that the only cohomology $\mathcal{H}_k^2(K\backslash G)$ that can be nonzero is when $k = \frac{1}{2}\dim(G/K)$: see Olbrich (2002). In this dimension it will be nonzero iff the Euler characteristic $\chi \neq 0$ (which can also be determined from the $\chi$(compact dual) and which is iff $\mathrm{rank}_{\mathbb{C}} G = \mathrm{rank}_{\mathbb{C}} K$). So, for $\mathrm{SL}_n(\mathbb{R})$ this only happens for $n = 2$, but for $\mathrm{U}(m,n)$ it's always true (and for $\mathrm{O}(m,n)$ it depends on parity considerations of $m$ and $n$).

This theorem is adequate for the purposes of understanding uniform lattices; however for nonuniform lattices, while there is a finite complex for $K(\Gamma,1)$, – thanks to the Borel–Serre theory, it is *not* $K\backslash G/\Gamma$, which even has the wrong dimension. Thus the universal cover is not $K\backslash G$ and we cannot directly use the above calculation to learn about the growth of Betti numbers in towers. It is nevertheless true that the $L^2$-Betti numbers for nonuniform lattices are proportional (with the ratio of volumes being the proportionality constant) to those of the uniform lattices!

The most conceptual proof I know is due to Gaboriau (2002), who introduced notions of $L^2$-invariants for equivalence relations. Using this he showed that both $\Gamma$ and $\nabla$ act *to preserve measure* and that they also *commute* with each other on the same space $X$[71] with finite co-volume, then for every $k$,

$$b_k^{(2)}(\Gamma)/\mathrm{vol}\,(X/\Gamma) = b_k^{(2)}(\nabla)/\mathrm{vol}\,(X/\nabla).$$

We note that as a consequence of the theorems in this section, if $M = K\backslash G/\Gamma$, then $(-1)^{\frac{1}{2}\dim(G/K)}\chi(M) \geq 0$.

The Hopf conjecture asserts that this is true for all closed aspherical manifolds. It is not even known for (variable) negatively curved manifolds, although

70 OK – one can if one is Borel.

71 In this situation $X = G$, the ambient Lie group; the lattices can be viewed as acting in a commuting fashion by having one act on the left and the other on the right. The invariant measure exists, because the Lie group $G$ is unimodular (whenever it has a lattice). This idea of Gromov is called *measure equivalence*.

https://doi.org/10.1017/9781316529645.004 Published online by Cambridge University Press

Gromov (1991) did use $L^2$ ideas combined with Hodge theory to prove a Kähler version of this conjecture.[72]

In the next chapter we will discuss some other uses of $L^2$ to probe the Borel philosophy.

## 3.7 Mixing the Ingredients

We now wrap up our discussion and show the ubiquity of the failure of the naive proper analogue of the Borel conjecture. (Before jumping to conclusions, however, please go to §3.8 on morals!) All of the results and arguments in this section are joint work with Stanley Chang, and more details can be found in Chang and Weinberger (2003, 2007, 2015).

Our first result argument shows that we can use the completely elementary results about $H^4$ of congruence subgroups of $\mathrm{SL}_n(O)$ to show proper-nonrigidity for all $n > 4$.

**Theorem 3.27** *For every $n > 4$, for every lattice in $\mathrm{SL}_n(O)$, the associated locally-homogeneous manifold has a finite sheeted cover that is not properly rigid. Moreover, we can arrange for this cover, there is a proper homotopy-equivalent manifold that is smooth, and is distinguished (topologically) from the locally symmetric manifold by having a different $p_1$.*

*Proof* We shall just use the groups $\mathrm{SL}_n(O; p)$ studied above (every lattice contains these for large $p$, by the congruence subgroup theorem: Bass *et al.*, 1967). We turn to our classifying spaces armed with our knowledge about $p$-torsion in $H^4(\mathrm{SL}_n(O; p); \mathbb{Z})$:

$$\begin{array}{ccccccc} F/\mathrm{O} & \to & \mathrm{BSO} & \to & \prod K(\mathbb{Z}, 4i) & \to & K(\mathbb{Z}, 4) \\ \downarrow & & \downarrow & & & & \downarrow \\ F/\mathrm{Top} & \to & \mathrm{BSTop} & & \to & & K(\mathbb{Z}[1/2], 4). \end{array}$$

The leftmost square consists entirely of rational homotopy equivalences because BF has finite homotopy groups according to Serre's theorem on the finiteness of stable homotopy groups of spheres. The map $\mathrm{BSO} \to \prod K(\mathbb{Z}, 4i)$ is the total Pontrjagin class (interpreting cohomology classes as maps to Eilenberg–Mac Lane spaces).

The homotopy of BSO is known, thanks to Bott periodicity, and we have a $\mathbb{Z}$ in every fourth dimension. We shall ignore the prime 2. Bott periodicity,

[72] On the other hand, recent work of Avramidi (2018) calls this conjecture into question in general.

https://doi.org/10.1017/9781316529645.004 Published online by Cambridge University Press

via its connection to the Chern character (see e.g. Hatcher, 2017), implies that $p_k : \pi_{4k}\mathrm{BSO} \cong \mathbb{Z} \to \mathbb{Z}$ is multiplication by $(2k-1)!$.

Note that a Pontrjagin class $p_k$ can be defined in a topologically invariant fashion in $\mathbb{Z}[1/N]$ if we invert all primes that arise in $\pi_i(\mathrm{Top}/\mathrm{O})$ for $i \le 4k+1$. So $\pi_3(\mathrm{Top}/\mathrm{O}) \cong \mathbb{Z}/2$ and then the groups vanish till $\pi_7(\mathrm{Top}/\mathrm{O}) \cong \mathbb{Z}/28$ and forever after, they are isomorphic to the group of differentiable structures on spheres studied by Kervaire and Milnor. Thus, the question of which primes need to be inverted becomes related to Bernoulli numbers. However, we will just use $p_1$ and be happy to invert the prime 2 to obtain topological invariance.

Now, to lift a map $K\backslash G/\Gamma \to K(\mathbb{Z},4)$ to $F/\mathrm{O}$, note that in every dimension $d$ there is an $N(d)$ so that there is a map from the $d$-skeleton $K(\mathbb{Z},4)^{[d]} \to F/\mathrm{O}$ (making use of the rational homotopy equivalence $\mathrm{BSO} \to \prod K(\mathbb{Z},4i)$) so that the composition $K(\mathbb{Z},4)^{[d]} \to F/\mathrm{O} \to K(\mathbb{Z},4)$ is multiplication by $N(d)$. Letting $d \ge \dim(G/K)$, and multiplying by $N(d)$, e.g. choosing $p > N(d)$, we obtain a normal invariant that we can do smooth surgery to and obtain a smooth proper homotopy equivalence $f\colon M \to K\backslash G/\Gamma$ distinguished by the fact that $p_1(M) - f^*p_1(K\backslash G/\Gamma)$ is of order $p$.

Notice that $f^*p_1(K\backslash G/\Gamma)$ depends only on the map that $f$ induces on $\pi_1$, i.e. only on the homotopy class of the map, not the proper homotopy class. By Mostow rigidity, all automorphisms of $\Gamma$ come from isometries of $K\backslash G/\Gamma$ to itself, and hence $p_1(K\backslash G/\Gamma) \in H^4(K\backslash G/\Gamma)^{\mathrm{Out}(\Gamma)}$. Consequently, this manifold $M$ cannot be homeomorphic to $K\backslash G/\Gamma$ – it is not merely a proper homotopy equivalence that is not properly homotopic to a homeomorphism. □

Note that the above proof used the idea that smooth invariants are topological invariants if we ignore a few primes (whose number depends on dimension). It is an important fact that $F/\mathrm{Top}$ has an H-space structure, and $S^{\mathrm{Top}}(M)$ has an abelian group structure (for all manifolds) so that the map $S^{\mathrm{Top}}(M) \to [M\colon F/\mathrm{Top}]$ is a group homomorphism.[73] For $\pi$–$\pi$ manifolds (of dimension greater than 5) this map is an isomorphism. The group structure on $F/\mathrm{Top}$ is exactly the one that makes the maps arising in Sullivan's description of $F/\mathrm{Top}$ into $H$-maps.

**Proposition 3.28** *For all $d > 4$ there is an $M(d)$ such that if $M$ is a smooth manifold, the image of the map $S^{\mathrm{Diff}}(M) \to S^{\mathrm{Top}}(M)$ contains a subgroup of index bounded by $M(d)^{\mathrm{rank} H^*(M;\mathbb{Z})}$.*

*Proof* This is a formal consequence of the statement that there is an $M(d)$ so

[73] Siebenmann proved this in the last essay of Kirby and Siebenmann (1977). It is a consequence of a periodicity theorem that is a cousin of Bott periodicity for BO. For a geometric explanation, see Cappell and Weinberger (1987) and Weinberger (1994).

https://doi.org/10.1017/9781316529645.004 Published online by Cambridge University Press

that the composition $M(d)\colon F/\mathrm{Top} \to F/\mathrm{Top} \to \mathrm{Top}/\mathrm{O}$ is null-homotopic on the $d$-skeleton. The $d$-skeleton of $F/\mathrm{Top}$ is a finite complex, and $\mathrm{Top}/\mathrm{O}$ is a cohomology theory with finite homotopy groups. Therefore $[F/\mathrm{Top}_{(0)}\colon \mathrm{Top}/\mathrm{O}] = 0$ and hence[74] the inverse limit of $N^*$ (over the integers)[75] on $[F/\mathrm{Top}\colon \mathrm{Top}/\mathrm{O}]$ is trivial. Consequently, we can find the $M(d)$ that induces 0, as was our goal. □

**Remark 3.29** It is a consequence of the work of Kervaire and Milnor on differentiable structures on the sphere and smoothing theory that the map $S^{\mathrm{Diff}}(M) \to S^{\mathrm{Top}}(M)$ has finite kernel (whose order is also bounded by $M(d)^{\mathrm{rank} H^*(M;\mathbb{Z})}$). The above proposition shows that, although the image is not a subgroup, the cokernel has a similar bound.

As a result, we have that, for $\pi$–$\pi$ manifolds, $S^{\mathrm{Diff}}(M) \to \bigoplus H^{4i}(M;\mathbb{Z})$ is finite-to-one and has image that contains a lattice in the target (with even some information on the torsion, if we are so inclined).

We now give a general converse to the rigidity that holds[76] in $\mathbb{Q}$-rank $\leq 2$, but only for the topological category.

**Theorem 3.30** *If* $\mathbb{Q}$-rank$(\Gamma) > 2$*, then there is a finite index subgroup* $\Gamma'$ *of* $\Gamma$ *for which* $S^{p,\mathrm{Top}}(K\backslash G/\Gamma')$ *is nontrivial.*

**Remark 3.31** Indeed, we can make this an elementary abelian 2-group of arbitrarily large size by pushing strong approximation slightly harder than we do in the discussion below.

**Remark 3.32** If $\Gamma$ is arithmetic then the $\mathbb{Q}$-rank$(\Gamma)$ is defined as usual in terms of $\mathbb{Q}$-split tori. If it is reducible, then we add on to the arithmetic pieces the number of non-compact manifold factors it has. These non-arithmetic factors are all negatively curved, and they have the same general shape as $\mathbb{R}$-rank $-$ 1 non-compact symmetric spaces: they have cusps that can be compactified, and these boundaries are aspherical, with cusp subgroups that are of infinite index.

*Proof* We shall use Sullivan's decomposition of $F/\mathrm{Top}$ at the prime 2: $F/\mathrm{Top}$ has a $K(\mathbb{Z}/2, 2)$ factor, so we need to produce $\Gamma'$ with large $H^2(;\mathbb{Z}/2)$. Let us assume that we are in the arithmetic case, leaving the modifications for the reducible case to the reader.

Recall that, according to Lubotzky's theorem, $\Gamma$ has a subgroup of even index

[74] As the homotopy groups are all finite, there is no issue of $\lim^1$; it is also true in our case for the reason that we can work with a fixed finite skeleton.

[75] Note that $F/\mathrm{Top}_{(0)}$ can be thought of as an infinite mapping telescope of self-maps $F/\mathrm{Top} \to F/\mathrm{Top}$ induced by multiplications by the integers (no matter what $H$-space structure is used).

[76] We will explain this in a later chapter.

https://doi.org/10.1017/9781316529645.004 Published online by Cambridge University Press

– hence a normal subgroup of even index. Hence there's a finite group of even order $H$ that is a quotient of $\Gamma$. Let $\Gamma'$ be the inverse image of some involution in $H$.

If the Lie group $G$ has no rank-1 factors, then it has Property (T), and $H_1(\Gamma')$ is necessarily finite. If there are rank-1 factors, but $\Gamma$ is irreducible, we can deduce the same thing from superrigidity. In any case, we then see that $H_1(\Gamma')$ has an even-order cyclic summand. Consequently, we have $\mathrm{Ext}(H_1(\Gamma'), \mathbb{Z}/2) \neq 0$; by the universal coefficient theorem, there is an injection $0 \to \mathrm{Ext}(H_1(\Gamma'), \mathbb{Z}/2) \to H^2(\Gamma'; \mathbb{Z}/2)$, giving us a nontrivial element in the structure set as desired. □

The first remark is proved by producing quotients making use of many primes, and then having a large elementary abelian subgroup of which to take the inverse image.

**Problem 3.33** The above reasoning shows that we can make rank $H_1(\Gamma', \mathbb{Z}/2)$ large by taking a deep lattice. This rank is necessarily $O(\mathrm{vol}\,(K\backslash B/\Gamma'))$ and in the rank-1 case it can actually grow linearly (although this doesn't produce any exotic structures). However, if one takes a descending chain[77] or assumes that we are irreducible in a semisimple group of rank $\geq 2$, is it the case that this rank is $o(\mathrm{vol}\,(K\backslash B/\Gamma'))$?

As a converse to the low-rank proper rigidity, this theorem has a couple of weaknesses: these elements (at least in the irreducible case) die on passing to further covers. Also, it would be nice to know that some particular structure sets (groups, in fact) are infinite – and we would be interested in knowing whether we can say anything about how these groups grow in size as we move up a tower. We now address these questions.

**Remark 3.34** $H^i(X; \mathbb{Q}) \to H^i(Y; \mathbb{Q})$ is one-to-one for any finite cover $Y$ of any space $X$. So, if we detect a structure set using a *rational* Pontrjagin class, then these survive forever.

First of all, we note a case where we can prove that the proper structure set is infinite for a simple reason of this sort.

**Proposition 3.35** *If* $K\backslash G/\Gamma$ *is a Hermitian symmetric space with* $\mathbb{Q}$-rank$(\Gamma) > 2$*, then* $S^{p,\mathrm{Top}}(K\backslash G/\Gamma) \otimes \mathbb{Q} \neq 0$.

*Proof* We argue as above, but we will need to see that $H^4(K\backslash G/\Gamma; \mathbb{Q}) \neq 0$; the proposition will be proved by $p_1$. The obvious cohomology class to use is the square of the Kähler class. However, one needs to check that this class is nontrivial.

[77] And we are not in the case of a surface!

Let's now be a bit more explicit. Given a projective embedding, one can pullback the generator of $H^2(\mathbb{CP}^N)$: this is the Kähler class. The way it evaluates on homology is by intersecting with any linear hyperplane $\mathbb{CP}^{N-1}$. Using a projective embedding of the Baily–Borel compactification[78] of $K\backslash G/\Gamma$, it would then suffice to find a codimension-2 linear subprojective space that does not intersect the singularity set of the Baily–Borel compactification, since its intersection with $K\backslash G/\Gamma$ will be a subsurface on which the square of the Kähler class is nontrivial. This merely requires that the codimension of the singularities of the Baily–Borel compactification to be larger than 2. This can be seen by inspection, as noted in Jost and Yau (1987). □

At the cost of weakening the hypothesis on $\mathbb{Q}$-rank to one that is *not* necessary for rigidity, one can prove a much stronger theorem.

**Theorem 3.36** *Suppose $M = K\backslash G/\Gamma$ is a locally symmetric manifold with* $\mathbb{Q}$-rank$(\Gamma) > 3$*, then* $\lim S^{p,\mathrm{Top}}(K\backslash G/\Gamma') \otimes \mathbb{Q}$ *is of infinite rank (where the limit is taken with respect to arbitrary finite covers $K\backslash G/\Gamma'$ of $K\backslash G/\Gamma$).*[79]

We shall denote the limit, $\lim S^{p,\mathrm{Top}}(K\backslash G/\Gamma')$ by $S^{\mathrm{virtual}}(K\backslash G/\Gamma)$.

We shall, for simplicity, only deal with the case of $\Gamma = \mathrm{SL}_n(O_F)$ with $n > 3$ (note that the $\mathbb{Q}$-rank of such is $n - 1$) – which includes a very interesting $\mathbb{Q}$-rank $= 3$ example, and tells the complete story for this important class of lattices.

In light of our previous remarks and the theory of generalized modular symbols, all that we need to do is find proper $\mathbb{Q}$-parabolic subgroups for $\mathrm{SL}_n(O_F)$ whose unipotent radicals have dimension $\equiv 0 \bmod 4$.

Parabolics are associated to, perhaps incomplete, flags in $F^n$. If we use the flag $F^k \subset F^n$, then the dimension of the associated unipotent subgroup (of automorphism inducing the identity on the associated graded to this flag) is $dk(n-k)$, where $d = [F : \mathbb{Q}]$.[80] If $n$ is even, we can use $k = 2$, and if $n = 1 \bmod 4$, we can use $k = 1$. If $n = 3$, then the $\mathbb{Q}$-rank is 2, so we can assume that $n > 4$, so we can set $k = 4$.

The general case in the theorem is a similar case-by-case analysis.[81]

[78] See Baily and Borel (1966) for the completion of Hermitian locally symmetric spaces as projective varieties.

[79] One can form this limit with respect to various families of covers, and the limits can change. For example, one can show that if $\mathbb{Q}$-rank$(\Gamma) > 5$, then if one takes the sequence of squarefree congruence covers, the limit has an infinitely generated torsion subgroup that frequently dies when included in the limit over all finite index subgroups.

[80] The presence of this factor $d$ implies that if the lattice in the theorem were obtained by restriction of scalars from a number field of even degree, we would obtain the same growth of $S^{\mathrm{virtual}}$ in the problematic $\mathbb{Q}$-rank $= 3$ case not covered in the theorem.

[81] I am deeply appreciative of the help that Dave Witte-Morris gave us with these calculations that we had done incorrectly at first.

Finally, given the infinite rank of $S^{\text{virtual}}(K\backslash G/\Gamma)\otimes\mathbb{Q}$, it becomes reasonable to ask what is the growth rate of rank $S^{p,\text{Top}}(K\backslash G/\Gamma')$ as one moves up a tower. In general, it seems like the rank grows like some power of the volume $[\Gamma:\Gamma']^{\alpha}$ for some $\alpha \leq 1$.

The question of (approximately linear) growth follows easily from Lück's theorem combined with the results of Cheeger and Gromov and of Gaboriau explained in §3.6.

**Theorem 3.37** *Assuming that* $\dim(G/K) > 4$*, and* $\mathbb{Q}$-rank$(\Gamma) > 2$*, the ranks* $S^{p,\text{Top}}(K\backslash G/\Gamma') = o([\Gamma:\Gamma'])$ *iff* $\text{rank}_{\mathbb{C}}(G) > \text{rank}_{\mathbb{C}}(K)$ *or* $\dim(G/K)$ *is* not *divisible by* 8.

Indeed for $G$ semisimple with no rank-1 factors, one can prove that

$$\text{rank}\, S^{p,\text{Top}}(K\backslash G/\Gamma')\otimes\mathbb{Q} = o(\text{vol}\, K\backslash G/\Gamma')$$

(i.e. we do not have to assume that they are part of a tower). Presumably, this also holds in that case as well for irreducible lattices. And it is interesting to speculate on the nature of the torsion for these lattices (both in a tower and those that are not).

## 3.8 Morals

What do we learn from this discussion? Certainly that in large $\mathbb{Q}$-rank, the proper Borel conjecture fails.

But that's a summary, not a moral.

The reason that the proper Borel conjecture fails is interesting. It turns out that only in $\mathbb{Q}$-rank $\leq 2$ are the symmetric spaces "aspherical" in the "relevant sense," i.e. the sense relevant to *proper* rigidity. We observed that these low $\mathbb{Q}$-rank lattices are the only ones where the space at $\infty$, i.e. the Borel–Serre boundary, is aspherical.

What is a good way of thinking about "aspherical in the relevant sense"? We need to lose some geometry and move towards a categorical answer.

For proper maps, we are working in the proper category, and it makes sense to look for a properly aspherical space.

What should proper "aspherical" mean? This space should be defined to be a terminal object in the category of spaces and maps that are "1-equivalences,"[82]

[82] That this is the right thing to look at is suggested by the functoriality properties that we will see that $S^{\text{Top}}$ (i.e. surgery theory) is blessed with. More primitively, the $\pi$–$\pi$ theorem – which our discussion crucially depended on – suggests the very special role that the fundamental group (which is equivalent to 1-equivalence classes of connected spaces) will play.

https://doi.org/10.1017/9781316529645.004 Published online by Cambridge University Press

i.e. where one can solve all one-dimensional lifting problems in a way that is unique up to homotopy (in the category). If $\mathbb{Q}$-rank $\geq 3$, the terminal object should be "the core of $K\backslash G/\Gamma'' \times [0, \infty)$ – which is *not* $K\backslash G/\Gamma$. (There's a pretty straightforward proper map from latter to the former, but no proper section to this map.)

If we give up on doing *any* new geometry at infinity, might we be able to survive this lack of proper asphericity, and get some rigidity theorem for all $K\backslash G/\Gamma$?

One way we can do this is by insisting that our maps are homeomorphisms outside some compact set (and that we allow homotopies to be relative to the complement of a somewhat larger compact set). In this case, the symmetric space is a terminal object and we will see in Chapter 4 that the ordinary Borel conjecture for a closed aspherical manifold constructed by a Davis construction applied to the Borel–Serre compactification implies this is relative to infinity rigidity (i.e. relative to the complement of some large, unspecified compact set), so it is a consequence of the Borel conjecture.

And, indeed this case is, essentially, a theorem of Bartels *et al.* (2014b).[83]

In any case, this discussion enables (and forces) us to expand our attention to all aspherical manifolds with boundary, with the boundary aspherical or not, provided we work relative to the boundary. (Or rel$\infty$ in the noncompact case.)

There's another sense in which $K\backslash G/\Gamma$ is aspherical, if we make a category where maps are Lipschitz and not allowed to move any point too far. In that case, the large-scale geometry discussed in the first section comes to bear, and one can indeed prove that the $K\backslash G/\Gamma'$ are "boundedly rigid."[84] This bounded category (and other "controlled analogues") will play a large role when we discuss the Novikov conjecture in the upcoming chapter.

## 3.9 Notes

This chapter covered a lot of ground, and all of the topics discussed need more systematic treatments. Happily, many exist for them. A very good general reference for arithmetic manifolds is Witte-Morris (2015).

The subject of compactifications of $K\backslash G/\Gamma$ is an important one, and two of these played a role in our discussions, the Borel–Serre and the Baily–Borel (see

When we study groups with torsion, it will turn out that the terminal object is not rigid, and we will be led back to geometry and orbifolds, i.e. to enlarging the category.

83 Their paper covers the case of arithmetic lattices. Non-arithmetic lattices can be handled by the same idea of reduction to the arithmetic case used above in §3.2 when we defined the $\mathbb{Q}$-rank for the non-arithmetic case.

84 See Chang and Weinberger (2007).

Borel and Serre, 1973 and Baily and Borel, 1966). Both of these are extraordinarily important. The Borel–Serre compactification gives finite generation of group homology, the calculation of their cohomological dimension, and that these groups are duality groups in the sense of Bieri and Eckmann[85] (see Bieri and Eckmann, 1973). The Baily–Borel compactification shows finite generation of the spaces of modular forms via projective embedding. The literature on these and many others is surveyed and explained in Borel and Ji (2005).

We shall sometimes have need for Tits buildings defined for Lie groups over other fields. For example, if one wants to study $\mathrm{SL}_n(Z[1/p])$, it acts ergodically on $\mathrm{SL}_n(\mathbb{R})$ and we need to supplement $\mathrm{SL}_n(\mathbb{R})$ with $\mathrm{SL}_n(\mathbb{Q}_p)$ to get discreteness. Tits buildings give a structure that replaces the symmetric space $K\backslash G$. The group $\mathrm{SL}_n(Z[1/p])$ acts properly on the product of the real symmetric space and the building. An immediate consequence of this theory is that the virtual cohomological dimension of such groups is finite. There are many references for the theory of buildings, each with a different emphasis; for our purposes, Tits (1974) and Abramenko and Brown (2008) are especially recommended. I also highly recommend the paper Alperin and Shalen (1982) which is a model of this type of application.

Atiyah's theorem (about how much of the tangent bundle is homotopy invariant) is better phrased in terms of stable normal bundles (for an embedding in a very high-dimensional Euclidean space), rather than tangent bundles. In that case, the conceptual explanation, due to Spivak (1967), is that *as a spherical fibration*, this stable normal bundle is definable for any Poincaré complex; that is, for any finite complex that satisfies Poincaré duality.[86] The idea of this fibration is quite simple: Poincaré complexes can be characterized as those complexes $X$ for which a regular neighborhood of $X$, when polyhedrally embedded in Euclidean space looks like, in a homotopy-theoretic sense, the situation that arises for tubular neighgborhoods of smooth manifolds. This means specifically that the homotopy fiber[87] of the inclusion of the boundary of this neighborhood into the neighborhood is a homotopy sphere (like the epsilon-sphere bundle mapping to the smooth manifold, to which a tubular neighborhood deform retracts). Spivak also gives a homotopy-theoretic characterization of this fibration.[88]

[85] Actually they motivated the definition of Bieri–Eckmann duality by being a first nontrivial class of examples of this.

[86] See Spivak (1967) and Wall (1968) for what this notion means in detail: it generalizes the fact that the homology and cohomology groups must be isomorphic, but it also demands that it be implemented via a fundamental class, and also hold with arbitrary local coefficient systems – in particular any finite cover of a Poincaré complex is a Poincaré complex.

[87] Recall that any map can be replaced (at the cost of replacing the spaces involved by homotopy equivalent ones) by a fibration, as observed by Serre in his thesis.

[88] It is the unique stable spherical fibration whose top homology class is spherical (i.e. lies in the image of the Hurewicz homomorphism).

The surgery classification of manifolds was begun by Kervaire and Milnor (1963) in the case of smooth manifolds that are homotopy-equivalent (and therefore homeomorphic) to the sphere, and then extended to the simply connected case by Browder and Novikov and reformulated using classifying spaces by Sullivan. (References for surgery theory include Wall (1968), Browder (1972), Ranicki (1992, 2002), Weinberger (1994), Lück (2002a), and Chang and Weinberger (2020).) Although we have not yet dealt with the classification of closed manifolds (see Chapter 4!) the simply connected case can be deduced from the discussion given here: if $h\colon M' \to M$ is a homotopy equivalence, one can always deform it so that it will be transverse to a point $p$, and with $h^{-1}(p)$ a single point. In that case, there is a neighborhood isomorphic to a ball, whose inverse image is a ball. Deleting the interiors of these balls, we get a structure on the complement. On the other hand, any structure on the complement restricts to a homotopy sphere on the boundary, and, thanks to the Poincaré conjecture (in the PL and Topological categories), it can be completed to be a structure on the closed manifold. Thus $S^{\text{cat}}(M) \cong S^{p,\text{cat}}(M - p)$ for $M$ simply connected and where cat is equal to Top or PL.

The classifying space $F/\text{Top}$ is its own fourth loop space (more correctly, it is $\mathbb{Z} \times F/\text{Top}$ that is its own loopspace[89]) as can be seen from the description given in the text and using Bott periodicity at the odd primes. It turns out that this is the first step towards a functorial view of surgery theory, which cannot at all be explained in "without obstructions" terms, as our first pass went: the structure space[90] $S$ measures the difference between completely analogous local and global obstructions, i.e. $\mathbb{Z}\times F/\text{Top}$ is a cohomology theory associated to a spectrum whose homotopy groups are surgery obstruction groups.

Surgery theory is nicest in the topological category. The canonical reference for the foundational theorems in this setting is Kirby and Siebenmann (1977) – which is the original source for them. Unlike the smooth category where the foundations are built on Sard's theorem, Morse's lemma, and the fundamental existence theorem for ordinary differential equations (with smooth coefficients), the topological category is distinctly more difficult to get off the ground. The proofs of the basic theorems, essentially theorems about the topology of $\mathbb{R}^n$, require deep global results in the smooth or PL categories either about non-simply connected manifolds (*à la* Novikov and Kirby) or about manifolds

[89] This extra $\mathbb{Z}$ has significant geometric implications, hinting to an amazing world of non-resolvable homology manifolds.

[90] Indeed, the structures that we considered are promoted to being homotopy groups of a space rather than merely a set. This idea first arose in work of Casson (1967) and was developed and advocated by Quinn in his thesis.

"controlled over a metric space" – introduced by Quinn (1979, 1982b, 1982c, 1986) – a major theme in the coming chapters.

However, the theory ends up having an *even nicer* formulation than the topological category when one includes homology manifolds, but here the local issues currently seem even more difficult and the global theory is in much better shape than the local (see Bryant *et al.*, 1996). I will discuss this a bit in Chapter 4 discussing the functoriality of surgery and also in our discussion of the Wall conjecture (the "existence Borel conjecture").

That there is a lot of homology in congruence subgroups is something that I learnt from Ruth Charney (1984). Torsion in homology of arithmetic groups is quite mysterious. For $\mathrm{SL}(\mathbb{Z})$, in the limit, this is determined by the solution to Quillen–Lichtenbaum conjecture by Rost and Voevodsky[91] (by the work of Dwyer and Friedlander, 1986), but for congruence groups and other arithmetic groups the picture is still obscure.[92] Bergeron and Venkatesh (2010) and Calegari and Venkatesh (2019) have suggested that the analogue of the $L^2$-Betti story holds – something hard to tell in general because of issues involving regulators. (Test question:[93] in the stable range of Borel's theorem, how do the images of the cohomology lattices corresponding to different lattices in the same group relate to one another?) The stabilization by going up the congruence tower has been studied by Calegari and Emerton (2012) introducing a notion of completed cohomology making a connection to $p$-adic Lie groups).

A problem whose solution would seem to be illuminating in this direction is the following: Can one estimate the ratio of $b_i(X; \mathbb{Z}/p)/\mathrm{vol}\,(X)$ (where $\mathrm{vol}\,(X)$ is *some* simplicial notion of volume, say the number of simplices) for a simplicial complex by random sampling. That this is possible for rational Betti number is the idea of the Lück approximation theorem; see also Farber (1998), (where this point is clearer – his condition for Lück's theorem to hold for non-normal covers is precisely that the relative volume of the set of points where the covers do not look "universal" goes to 0), and Abert *et al.* (2017) and Elek (2010), where it is explicit.[94] This would then suggest that in the situation where $\mathrm{rank}_{\mathbb{C}} G - \mathrm{rank}_{\mathbb{C}} K = 1$ there would be growing torsion in covers, but not because of large elementary abelian subgroups.

We refer to Lubotzky and Segal (2003) for a survey of the strong approximation theorem. We note that, although the original proof (by Weisfeiler (1984) – see also Matthews *et al.* (1984) and Vasserstein) used the classification of

[91] See the survey Weibel (2005).
[92] But see Calegari (2015).
[93] It could be someone knows the answer to this and will email me!
[94] In this context, the paper of Clair (2003) where Lück's theorem is made more quantitative expressly in terms of injectivity radius of the covers.

finite simple groups, this is no longer necessary (as pointed out there) thanks to work of Nori (1987), Larsen and Pink (2011), and Hrushovski and Pillay (1995) (using algebraic geometric and/or model-theoretic ideas.)

I remember learning about amenability from Bob Brooks. When I was a student, he told me about Kesten's work and explained that, although there are no $L^2$ harmonic functions on a universal cover, amenability controls whether 0 is in the spectrum. This material appeared in Brooks (1981) and is a manifold version of the statement asserted for the discrete group. Other papers by Brooks, Sunada, and others compared the spectral geometry of the manifolds to the spectral geometry of the associated finite graphs. This can all be viewed part of the $L^2$-cohomology story (including an appropriate de Rham theorem for comparison of smooth and simplicial models) when one jazzes up the story to include foliated spaces rather than just covers (see Bergeron and Gaboriau, 2004). There are a number of excellent sources on amenability, the Banach–Tarski paradox, and its connection to random walks and to operator algebras (see e.g. Lubotzky, 1984; Paterson, 1988; Wagon, 1993).

The geometric group theory of amenability and non-amenability has led to the consideration of some remarkable groups. Non-amenable groups that don't contain free groups (the von Neumann conjecture) were first constructed by Ol'shanskii and Ju (1980) – the torsion groups satisfying Property (T) produced by the method of adding large relations are also examples. On the other hand, Whyte's thesis (see Whyte, 1999) gives a "true" analogue of the von Neumann conjecture that can be used, for instance, to extrapolate between the characterization of nonamenability in terms of random walks (i.e. vanishing of 0th $L^2$-homology) and that in terms of the existence of "Ponzi schemes"[95] (Block and Weinberger, 1992; 1997)) to all other $L^p$-homology with $p > 1$.

Grigorchuk's (1984) group of intermediate growth (i.e., so that the number of group elements that can be expressed as the product of $n$ generators grows more than a polynomial, but less than exponentially) was the first non-solvable amenable group. Bartholdi and Virag (2005) actually proved the amenability of some related groups by consideration of their random walk. Both of these are examples of automata groups – see the survey by Zuk (2003). Recently, Juschenko and Monod (2013) have given (uncountably many) *simple* amenable groups.

Property (T) is the subject of a very useful book (Bekka *et al.*, 2008) and is at the center of numerous problems.

It is now liberated from its original representation-theoretic roots by the

[95] The Cayley graph of a non-amenable group always supports a scheme wherein each vertex exchanges a uniformly bounded amount of money with its neighbors, so that each vertex ends up net positive. This is impossible on Cayley graphs of amenable groups.

method of Garland (1973), Ballman and Swiatkowski (1977), and Zuk (2003), and also by Shalom's work using bounded generation and algebraic methods (related to $K$-theory) to show that a number of interesting groups (like linear groups over Laurent series rings) have Property (T): see his ICM talk (Shalom, 2006) for information. The analytic method is useful in studying strengthenings of Property (T), e.g. to include group actions on other Banach spaces, or on other general spaces with curvature conditions. Stronger forms of Property (T) can be invaluable to extensions of the rigidity program in other directions, such as the Zimmer program, which, broadly defined, tries to study nonlinear actions of large groups (e.g. lattices) on manifolds – perhaps, but not necessarily, preserving some geometric structure (such as a volume form).[96]

I will leave the reader to consult Kowalski (2008) and Lubotzky (2012) for recent applications of expanders to discrete groups and to number theory, as well as to references on new proofs of Selberg's 3/16 theorem (at least $> 0$ theorem!) and the connections to additive combinatorics that enable all this. There have also been other constructions of explicit families of expanders, such as the zig-zag product of Alon and Wigderson (explained very nicely in Hoory *et al.* (2006)) and the new Ramanujan graphs[97] constructed by Marcus et al. (2015).

That both amenability and Property (T) have characterizations in terms of $L^2$-homology/cohomology should have made it possible to make a segue between this section and the one on $L^2$ and growth of Betti numbers, but this seemed forced, so I chose not to push this.

Lück's (2002b) book gives a good overview of how $L^2$ interacts with groups and compact manifolds. Entirely missing (and not relevant to our concerns in this essay) are relations of $L^2$-cohomology to intersection cohomology of compactifications and other stratified applications. There has been much work since that book was written, both internal to the subject (such as interesting examples of transcendental $L^2$-Betti numbers (see Grabowski, 2014, and references therein)[98] and of connections to other parts of topology.

Atiyah (1974) introduced $L^2$-Betti numbers and that $L^2$ indices to deal with the kernels of elliptic operators on universal covers and to get finite quantities measuring the sizes of these typically infinite-dimensional spaces. Connes

[96] In the notes to Chapter 8, we will mention a bit more about this. Here we content ourself with a citation of the Bourbaki talk (Cantat, 2017 on the theorem of Brown–Fisher–Hurtado showing that certain lattices don't have any effective $C^2$-actions on low dimensional manifolds. $\mathrm{SL}_n(\mathbb{Z})$ does not act effectively and $C^2$ on any manifold of dimension less than $n - 1$.

[97] Ramanujan graphs are graphs that have optimal spectral gaps for their Laplacians. The first examples were contructed by Lubotzky *et al.* (1988), using Deligne's solution of the Ramanujan conjecture – the circumstance that led to their name.

[98] On the other hand, even now, there is no known example of a torsion-free group where $L^2$-Betti numbers are not integers.

later proved an index theorem for foliations – see Connes (1982) and Moore and Schochet (2006) for a version closely related to the one most relevant here – that gives something like an average version of the indices one sees over the leaves (see Connes, 1994). If one views both theorems in the situation of limits of coverings (the Benjamini–Schramm limit of a sequence of finite covers being a transversely measured foliated space), then the cohomology related to Connes's theorem is exactly the one occurring in the Bergeron–Gaboriau theorem mentioned above.

Another interesting convergence arises when one thinks about the information given in the $L^2$-theory of symmetric spaces. Results about the limits of normalized Betti numbers (via thinking about the Matsushima formula and multiplicities) were first derived by DeGeorge and Wallach (1978, 1979) using the Selberg trace formula.

That symmetric spaces tend to be concentrated around the middle from the $L^2$ perspective has been well known for a while. I cannot track it down. Clearly Singer was aware of this when he conjectured in 1977 that the same might be true for the universal covers of arbitrary aspherical manifolds as an approach to the Hopf conjecture (discussed in the text). A very useful exposition (which does not make excessive demands on the reader's knowledge for harmonic analysis and which goes further and explains what occurs for additional invariants related to the spectrum near 0 for forms as well as functions) is Olbrich (2002). Atiyah and Schmid (1977) connect the use of the $L^2$ index theorem to representation theory and use this connection to re-prove some of the main results of Harish-Chandra.

Cheeger and Gromov were led to $L^2$ methods for an opposite reason than they arose for us: they wanted a substitute tool to use when there are not enough finite covers (see e.g. Cheeger and Gromov, 1985b) – for example, if one is studying the geometry of a manifold whose fundamental group is not residually finite. However, in some sense these are two sides of the same coin: the individual manifold (or lattice) might be hard to understand, but this limiting object is more transparent and brings order to the finite world. Their paper (Cheeger and Gromov, 1986) builds foundations for the theory and proves the generalization of Rosset's theorem on vanishing of Euler characteristic. Cheeger and Gromov (1985a) gives a direct but very delicate proof that the proportionality of Betti number to volume from locally symmetric manifolds remains true when one moves from the uniform to the nonuniform case. I personally prefer the method (Gaboriau, 2002) mentioned in the text.

The remarks about the rate of growth of $S^{p,\mathrm{Top}}(K\backslash G/\Gamma)$ are clearly essentially the same as questions about the rate of growth of Betti numbers. Besides the linear case, more remains at the level of conjecture. It seems reasonable to believe that the elements that don't come from the compact dual grow at a rate that's a power of volume, and that there is power upper bound – see Sarnak and Xue (1991), Xue (1992), Abert *et al.* (2017) – (and of course note that this is exactly the situation for the generalized modular forms, or, indeed, every use of geometric cycles I am aware of). The torsion story is harder to be sure of. In any case, if one climbs up the congruence tower of squarefree numbers, then the method based on comparison to the Lie algebra will give (at least if $\mathbb{Q}$-rank > 5) an infinitely generated torsion group in the limit. I suspect that quite generally $S^{\text{virtual}}(K\backslash G/\Gamma)$ will have infinitely generated torsion and infinitely divisible elements, but I do not have anything to show in justification of this suspicion.[99]

Finally, regarding the morals of the story told in this chapter: most stories are not improved by having their morals stated explicitly, and, further, most morals seem fairly obvious when just said outright,[100] and the tales told to illustrate them often seem more interesting than they are. And, perhaps this is true in our special case as well. In any case, as we proceed, we will now feel the need to be more and more functorial (and the geometric ideas that gave life to the problem, like good parents, will still be there, but in the background, giving guidance and perhaps providing inspirations, but never overwhelming independent development).

And the bounded rigidity of $K\backslash G/\Gamma$ mentioned there is proved in Chang and Weinberger (2007).

[99] But I do hope to hear more about this in coming years.

[100] Yet, I note, that they are frequently deep truths in the sense of Fermi, truths whose negations are also true.