THE REAL LINE IN ELEMENTARY SUBMODELS OF SET THEORY

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The use of elementary submodels has become a standard tool in set-theoretic topology and infinitary combinatorics. Thus, in studying some combinatorial objects, one embeds them in a set, M, which is an elementary submodel of the universe, V (that is, $(M; \in) \prec (V; \in)$). Applying the downward Löwenheim-Skolem Theorem, one can bound the cardinality of M. This tool enables one to capture various complicated closure arguments within the simple "$\prec$".

However, in this paper, as in the paper [JT], we study the tool for its own sake. [JT] discussed various general properties of topological spaces in elementary submodels. In this paper, we specialize this consideration to the space of real numbers, $\mathbb{R}$. Our models M are not in general transitive. We will always have $\mathbb{R} \in M$, but *not* usually $\mathbb{R} \subseteq M$. We plan to study properties of the $\mathbb{R} \cap M$'s. In particular, as M varies, we wish to study whether any two of these $\mathbb{R} \cap M$'s are isomorphic as topological spaces, linear orders, or fields.

As usual, it takes some sleight-of-hand to formalize these notions within the standard axioms of set theory (ZFC), since within ZFC, one cannot actually define the notion $(M; \in) \prec (V; \in)$. Instead, one proves theorems about M such that $(M; \in) \prec (H(\theta); \in)$, where θ is a "large enough" cardinal; here, $H(\theta)$ is the collection of all sets whose transitive closure has size less than θ.

For the rest of this paper, when we talk about a topological space $\langle X, \mathcal{T} \rangle$ in M, we really mean that $\langle X, \mathcal{T} \rangle \in M$ and $(M; \in) \prec (H(\theta); \in)$, where θ is large enough to make the argument at hand work. In practice, it is sufficient to choose $\theta > 2^{|X|}$, since that will guarantee that $H(\theta)$ contains all elements of X, all subsets of X, and all subsets of $\mathcal{P}(X)$. This is sufficient because the standard topological properties of X are expressed by quantifying over such objects. Of course, this estimate $\theta > 2^{|X|}$ presumes that the elements of X are chosen so that the transitive closure of X has size $|X|$.

Now, working in ZFC, we can apply the Löwenheim-Skolem Theorem to the *set* $H(\theta)$: Given any $S \subseteq H(\theta)$, there is an M such that $(M; \in) \prec (H(\theta); \in)$ and $|M| = \max(|S|, \aleph_0)$.

Given a topological space $\langle X, \mathcal{T} \rangle$ in M, we define X_M to be $X \cap M$ with topology generated by $\{U \cap M : U \in \mathcal{T} \cap M\}$ [JT]. In particular, taking $H(\theta)$ such that the real line $\mathbb{R}$ is a member of $H(\theta)$ (i.e., $\theta > 2^{\aleph_0}$), we can look at elementary submodels M of $H(\theta)$ containing $\mathbb{R}$. Let us emphasize that these models contain $\mathbb{R}$ as an element but need not include it as a subset. We came to the subject of this paper from [JT] via considering $\mathbb{R}_M$ – how does it depend on M, in particular the cardinality of M? These $\mathbb{R}_M$'s are the subject of our study. In fact, though, the topological space $\mathbb{R}_M$ is nothing other than the subspace topology $\mathbb{R} \cap M$ inherits from $\mathbb{R}$ (this is easy to prove directly, and follows more generally for first countable spaces [JT]). Thus, from now on we shall talk about $\mathbb{R} \cap M$ rather than $\mathbb{R}_M$. Now $\mathbb{R} \cap M$ also has an order and algebraic structure; $\mathbb{R} \cap M$ is topologically dense in $\mathbb{R}$ by elementarity, and so its linear order topology coincides with its subspace topology. Also by elementarity, it is easy to see that $\langle \mathbb{R} \cap M, +, \cdot, 0, 1 \rangle$ is a subfield of $\mathbb{R}$. The basic question we shall study in this paper is, *how many different types of $\mathbb{R} \cap M$'s can there be of the same cardinality?* The answers will vary according to the way we consider two $\mathbb{R} \cap M$'s to be the same—topologically, order-theoretically, or algebraically—and also according to what set-theoretic axioms we assume. The non-logician will be able to understand the statements of most of our results, but the proofs involve non-trivial set theory. In addition to standard forcing arguments, we will be using some basic large cardinal theory. We refer the reader to [K] for information about $0^{\#}$ and Ramsey cardinals.

Here are our principal results:

THEOREM.

(a) *As subfields of $\mathbb{R}$, $\mathbb{R} \cap M$ is isomorphic to $\mathbb{R} \cap N$ if and only if they are equal.*

(b) CH *implies that there are only two order-isomorphism (homeomorphism) types of $\mathbb{R} \cap M$'s, namely $\mathbb{Q}$ and $\mathbb{R}$.*

(c) *It is consistent that there are exactly three order-isomorphism (homeomorphism) types of $\mathbb{R} \cap M$'s.*

(d) *If it is consistent that there is a Ramsey cardinal, then it is consistent that there are $2^{2^{\aleph_0}}$ pairwise non-homeomorphic (non-order isomorphic) $\mathbb{R} \cap M$'s of power $2^{\aleph_0}$.*

(e) *It is consistent with $2^{\aleph_0} = \aleph_2$ that there are $2^{\aleph_1}$ pairwise non-homeomorphic (non-order isomorphic) $\mathbb{R} \cap M$'s of size $\aleph_1$.*

(f) *It is consistent that all $\mathbb{R} \cap M$'s of size $\aleph_1$ are homeomorphic but that there exist $\mathbb{R} \cap M$ and $\mathbb{R} \cap N$ of size $\aleph_1$ which are not order-isomorphic.*

It turns out that the algebraic types are trivial, and that most of the results about topological types follow from those about order types. Therefore we will quickly dispose of the algebra of $\mathbb{R} \cap M$, then consider the order, and then the topology. We will end by looking at $\mathbb{R}^n \cap M$, and in particular at $\mathbb{C} \cap M$.

THEOREM 1. *Considering $\mathbb{R} \cap M$ and $\mathbb{R} \cap N$ as subfields of $\mathbb{R}$, $\mathbb{R} \cap M$ is isomorphic to $\mathbb{R} \cap N$ if and only if $\mathbb{R} \cap M = \mathbb{R} \cap N$.*

PROOF. Suppose that f is a field isomorphism from $\mathbb{R} \cap M$ onto $\mathbb{R} \cap N$. Then f is the identity on the rationals, $\mathbb{Q}$ (every field isomorphism is), and f is order-preserving (these are real-closed fields, so the order is defined from the field operations). Since $\mathbb{Q}$ is dense in $\mathbb{R}$, f must be the identity map. ⊣

Actually, as is well-known, no two distinct real-closed subfields of the reals can be isomorphic.

Now we come to order. If $\mathbb{R} \cap M$ is countable, it is easy to see that by elementarity, $\mathbb{R} \cap M$ is a countable dense linear order without endpoints and so is isomorphic to $\mathbb{Q}$. Under $V = L$, there is little else to be said:

THEOREM 2. $V = L$ *implies that* $\mathbb{R} \cap M$ *is either equal to* $\mathbb{R}$ *or order-isomorphic to* $\mathbb{Q}$.

This follows immediately from

LEMMA 3. *If* $V = L$ *and* M *is uncountable then* $\mathbb{R} \subseteq M$.

PROOF. By a standard argument, there is an $\in$-isomorphism i from some L_α onto M. Since M is uncountable, $\alpha \geq \omega_1$, so that $\mathbb{R} \subseteq L_\alpha$. Now, $i(q) = q$ for each $q \in \mathbb{Q}$, since q has an absolute definition in set theory. Thus, since i is order preserving, $i(x) = x$ for each $x \in \mathbb{R}$, so that $\mathbb{R} \subseteq M$. $\dashv$

If $2^{\aleph_0} > \aleph_1$ it is easy to get M, N such that $|M \cap \mathbb{R}| = |N \cap \mathbb{R}| = \aleph_1$ but $M \cap \mathbb{R} \neq N \cap \mathbb{R}$ (start with M, choose $r \in \mathbb{R} \setminus M$, and let N be the Skolem hull of $M \cup \{r\}$). However, getting $|M \cap \mathbb{R}| = |N \cap \mathbb{R}| = \aleph_1$ but $M \cap \mathbb{R}, N \cap \mathbb{R}$ non-isomorphic is not so easy. In fact it's consistent with $2^{\aleph_0} > \aleph_1$ that it can't be done:

THEOREM 4. *It is consistent with ZFC that* $2^{\aleph_0} > \aleph_1$ *and for every* M *and* N *such that* $|\mathbb{R} \cap M| = |\mathbb{R} \cap N| = \aleph_1$, $\mathbb{R} \cap M$ *is order-isomorphic to* $\mathbb{R} \cap N$.

DEFINITION. Let κ an infinite cardinal, Y a topological space, and $X \subseteq Y$. X is κ-**dense** in Y if and only if every non-empty open subset of Y meets X in a set of size exactly κ. Y is κ-**dense** if and only if Y is κ-dense in Y.

PROOF OF THEOREM 4. If $|\mathbb{R} \cap M| = \aleph_1$ then $\mathbb{R} \cap M$ is $\aleph_1$-dense in $\mathbb{R}$ (since $\mathbb{R} \cap M$ is a subfield of $\mathbb{R}$). Baumgartner [B₁] forced to obtain a model in which every two sets of reals $\aleph_1$-dense in $\mathbb{R}$ are order-isomorphic and (hence) $2^{\aleph_0} > \aleph_1$, establishing Theorem 4. $\dashv$

We can improve Theorem 4 to:

THEOREM 5. *It is consistent with ZFC that* $2^{\aleph_0} = \aleph_2$ *and for every* M *and* N *of size* $\aleph_1$, $\mathbb{R} \cap M$ *and* $\mathbb{R} \cap N$ *are order-isomorphic.*

To help reduce Theorem 5 to Theorem 4, first observe:

LEMMA 6. *Suppose* $|\omega_1 \cap M| = \aleph_1 = |M|$. *Then* $|\mathbb{R} \cap M| = \aleph_1$.

$0^{\#}$ is a Δ_3^1 set of natural numbers, the existence of which has large cardinal strength. Its existence is equivalent to the existence of an elementary embedding $j : L_\alpha \to L_\beta$ for some α and β, such that some ordinal less than $|\alpha|$ is moved (Kunen, see e.g., [K, p. 277]).

LEMMA 7. *If* $0^{\#}$ *does not exist, then if* $|M| \geq \kappa$, *then* $M \supseteq \kappa$.

PROOF. Consider the Mostowski isomorphism, $i : M \cong T$, where T is transitive. Since $|M| \geq \kappa$, $T \supseteq \kappa$ and hence $T \supseteq L_\kappa$. Then $i^{-1} : T \cong M \subseteq H(\theta)$ is an elementary embedding and therefore so is $i^{-1}|L_\kappa : L_\kappa \to L_\theta$. But if $M \not\supseteq \kappa$, some $\alpha < \kappa$ gets moved. $\dashv$

PROOF OF THEOREM 5. Start with L and perform Baumgartner's forcing to get $2^{\aleph_0} = \aleph_2$ and all sets of reals $\aleph_1$-dense in $\mathbb{R}$ order-isomorphic. $0^{\#}$ cannot be added

by set forcing (see e.g., [K, p. 186]), so in the resulting model, any uncountable M will include uncountably many reals. Now apply the proof of Theorem 4.   ⊣

THEOREM 8. *If* $0^\#$ *does not exist and* $|\mathbb{R} \cap M| = 2^{\aleph_0}$, *then* $\mathbb{R} \subseteq M$.

PROOF. By elementarity and by taking θ sufficiently large, we may assume there is in M a bijection between the cardinal $2^{\aleph_0}$ and $\mathbb{R}$. But by Lemma 7, $2^{\aleph_0} \subseteq M$, so therefore $\mathbb{R}$ is.   ⊣

COROLLARY 9. *In the model of Theorem 5, there are only 3 isomorphism types of* $\mathbb{R} \cap M$'*s*.

It follows from Theorem 8 that if CH holds and $0^\#$ does not exist, then $\mathbb{R} \cap M$ is isomorphic to either $\mathbb{Q}$ or $\mathbb{R}$. Surprisingly, I. Farah has improved this to get the following result, which we include with his kind permission.

THEOREM 10. CH *implies that if* $\mathbb{R} \cap M$ *is uncountable, then* $\mathbb{R} \cap M = \mathbb{R}$ *and hence there are only 2 isomorphism types of* $\mathbb{R} \cap M$'*s*.

PROOF. By CH, there is a bijection $f : \omega_1 \to \mathbb{R}$. Hence there is such a bijection $f \in M$ and $f''(\omega_1 \cap M) = \mathbb{R} \cap M$. $\mathbb{R} \cap M$ is uncountable, so $\omega_1 \cap M$ is uncountable and hence $= \omega_1$. But then $\mathbb{R} = f''\omega_1 = \mathbb{R} \cap M$.   ⊣

This is quite a contrast to the situation under CH for order types of subsets of $\mathbb{R}$ which need not be of form $\mathbb{R} \cap M$—there are $2^{2^{\aleph_0}}$ different ones—see e.g., [B$_2$].

REMARK. Notice the difference between Lemma 3 and Theorem 10: in the former, we assume M is uncountable, in the latter $\mathbb{R} \cap M$ is uncountable. By Lemma 7, the nonexistence of $0^\#$ makes the difference: if M is uncountable, it includes ω_1 (by $\neg 0^\#$); since there is an injection from ω_1 into $\mathbb{R}$; the range of the injection in M is uncountable. However, assuming CH plus *Chang's Conjecture*, there are indeed uncountable models M for which $\mathbb{R} \cap M$ is countable; just apply Chang's Conjecture to $(H(\theta), \mathbb{R})$, which is a structure of type $(\geq \aleph_2, \aleph_1)$, to get an elementary submodel of type $(\aleph_1, \aleph_0)$.

We can get sharper, axiomatic versions of Theorems 4 and 5 and in the process obtain a theorem distinguishing consistently between the number of order types of $\aleph_1$-dense sets and the number of order-types of $\mathbb{R} \cap M$'s of size $\aleph_1$. First, some definitions from [ARS]:

DEFINITION. Let X be a second countable space of size $\aleph_1$. Let
$$D(X) = X \times X - \{\langle x, x \rangle : x \in X\}.$$
An open cover $\mathscr{U} = \{U_0, \dots, U_{n-1}\}$ of $D(X)$ consisting of symmetric sets is called an **open coloring** of X. $A \subseteq X$ is $\mathscr{U}$-**homogeneous** if for some $i < n$, $D(A) \subseteq U_i$.

[ARS] call the assertion that for every such X and $\mathscr{U}$, X can be partitioned into countably many $\mathscr{U}$-homogeneous sets "OCA"; the second author is grateful to S. Todorčević for informing him that this is not equivalent to what is now known as "OCA" (see [T]) and that the proof in [ARS] that MA plus their OCA (which we will call "OCA$_1$") implies $2^{\aleph_0} = \aleph_2$ is incorrect, although the conjunction of these three hypotheses is consistent.

DEFINITION. A set $A \subseteq \mathbb{R}$ of cardinality $\aleph_1$ is called an **increasing set** if for every $n \in \omega$ and any set $\{\langle a(\alpha, 0), \dots, a(\alpha, n-1) \rangle : \alpha < \omega_1\} \subseteq A^n$ of pairwise disjoint

n-tuples there are $\alpha, \beta < \omega_1$, such that for every $i < n$, $a(\alpha, i) < a(\beta, i)$. **ISA** is the assertion that an increasing set exists.

THEOREM 11. MA *plus* OCA$_1$ *implies that if* $|\mathbb{R} \cap M| = |\mathbb{R} \cap N| = \aleph_1$, *then* $\mathbb{R} \cap M$ *and* $\mathbb{R} \cap N$ *are isomorphic.*

COROLLARY 12. MA *plus* OCA$_1$ *plus* $2^{\aleph_0} = \aleph_2$ *plus* $0^{\#}$ *doesn't exist implies there are exactly* 3 *order-types of* $\mathbb{R} \cap M$'s.

In order to prove these, we first need three results from [ARS].

LEMMA 13. *Assume* MA *plus* OCA$_1$ *plus not* ISA. *Then every two* $\aleph_1$-*dense sets of reals are isomorphic.*

LEMMA 14. *Assume* MA *plus* OCA$_1$ *plus* ISA. *Then there is an increasing* $\aleph_1$-*dense set* A *such that* $A, A^*(= \{-a : a \in A\})$ *and* $A \cup A^*$ *are homogeneous, and every homogeneous* $\aleph_1$-*dense set is isomorphic to one of these three sets.*

LEMMA 15. *If* A *is increasing,* $A \not\cong A^*$.

PROOF OF THEOREM 11. By elementarity, any $\mathbb{R} \cap M$ is homogeneous, and

$$(\mathbb{R} \cap M)^* = \mathbb{R} \cap M.$$

Therefore, assuming ISA, every $\mathbb{R} \cap M$ of size $\aleph_1$ is isomorphic to $A \cup A^*$. On the other hand, by Lemma 13, we also have only one type of $R \cap M$ of size $\aleph_1$ if ISA fails. ⊣

Corollary 12 follows as usual. The conjunction of MA, OCA$_1$, and ISA is interesting in that it gives an example to show the structure of $\mathbb{R} \cap M$-types may differ from that of the $\aleph_1$-dense types, even without CH.

So far, we have been looking at models in which the number of isomorphism types of $\mathbb{R} \cap M$'s has been very small; however it is indeed consistent that it be as large as possible, i.e., $2^{2^{\aleph_0}}$, assuming the consistency of a Ramsey cardinal.

DEFINITION. κ is Ramsey if and only if $\kappa \to (\kappa)_2^{<\omega}$.

Comparing with our previous assumption, let us note that the existence of a Ramsey cardinal implies $0^{\#}$ exists, but not vice versa.

THEOREM 16. *If it is consistent that there is a Ramsey cardinal, then it is consistent that there are* $2^{2^{\aleph_0}}$ *pairwise non-homeomorphic and hence non-isomorphic* $\mathbb{R} \cap M$'s *of size* $2^{\aleph_0}$.

The proof will use the Ramsey cardinal to obtain a model in which there are $2^{2^{\aleph_0}}$ distinct $\mathbb{R} \cap M$'s; on the other hand, by Lavrentieff's Theorem (see e.g., [E, 4.3.21]) any homeomorphism from $S \subseteq \mathbb{R}$ to $T \subseteq \mathbb{R}$ can be extended to a homeomorphism from S' to T', where S' and T' are G_δ subsets of $\mathbb{R}$. There are only $2^{\aleph_0}$ many such homeomorphisms, so each $\mathbb{R} \cap M$ is homeomorphic to at most $2^{\aleph_0}$ many $\mathbb{R} \cap N$'s, so if there are $2^{2^{\aleph_0}}$ distinct ones, there are $2^{2^{\aleph_0}}$ pairwise non-homeomorphic ones.

Let κ be Ramsey. Let P be the partial order for adding κ Cohen reals. The idea of the proof is that the Ramsey cardinal gives us a set I of κ indiscernibles. The submodels M of the generic extension determined by independent subsets of I will yield distinct $\mathbb{R} \cap M$'s. Now for the details.

DEFINITION. For a model M and $I \subseteq \kappa \cap M$, κ an ordinal, I is a **set of indiscernibles** for M if and only if for every formula $\varphi(v_1, \dots, v_n)$ and $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$ all in I, $M \models \varphi[x_1, \dots, x_n]$ if and only if $M \models \varphi[y_1, \dots, y_n]$.

LEMMA 17 (Silver, see e.g., [K, p. 100]). κ *Ramsey implies that there is a set of indiscernibles* $I \in [\kappa]^\kappa$ *for any* $M \supseteq \kappa$.

Let $\widehat{H}(\theta)$ denote $H(\theta)$ with Skolem functions added; these are needed when discussing indiscernibles so that the notion of "Skolem hull" is defined.

PROOF OF THEOREM 16. Let θ be sufficiently large, and let $I \in [\kappa]^\kappa$ be a set of indiscernibles for $\widehat{H}(\theta)$. Let $\{I_\alpha\}_{\alpha < 2^\kappa}$ be an independent family of subsets of I. Let $\mathscr{H}(I_\alpha)$ be the Skolem hull of $I_\alpha \cup \{\kappa\}$ in $\widehat{H}(\theta)$. Since no indiscernible can be defined from the others, $\mathscr{H}(I_\alpha) \cap I = I_\alpha$. Let G be P-generic over V and hence over $H(\theta)$. In $V[G]$ take $M_\alpha = \{\tau_G : \tau \text{ is a } P\text{-name and } \tau \in \mathscr{H}(I_\alpha)\}$.

By a standard argument [S, 3.2.1], M_α is an elementary submodel of $H(\theta)^{V[G]}$. We now claim that $\alpha \neq \beta$ implies (in $V[G]$) that $\mathbb{R} \cap M_\alpha \neq \mathbb{R} \cap M_\beta$. Another standard argument [S, 3.2.13] shows $M_\alpha \cap I = \mathscr{H}(I_\alpha) \cap I = I_\alpha$. In $V[G]$ there is a bijection $f : \kappa \to \mathbb{R}$, which is named by a P-name τ which is in $\mathscr{H}(\emptyset)$ and hence in every $\mathscr{H}(I_\alpha)$. Then f is in M_α and in M_β. Take $i \in I_\alpha - I_\beta$. Then $f(i) \in \mathbb{R} \cap M_\alpha$, but $f(i) \notin \mathbb{R} \cap M_\beta$, else $i \in \mathscr{H}(I_\beta)$, which is impossible. $\dashv$

We do not know the precise consistency strength of the conclusion to Theorem 16, but it is at least "$0^\#$ exists" by Theorem 8. Without any large cardinals, we can still get a large number of types, but of size $\aleph_1$ rather than $2^{\aleph_0}$:

THEOREM 18. *Adjoin* $\aleph_2$ *Cohen reals to a model of GCH. Then there exist* $2^{\aleph_1} = \aleph_2$ *non-homeomorphic (and hence non-isomorphic)* $\mathbb{R} \cap M$'s *of size* $\aleph_1$.

PROOF. In the ground model, V, let $\{M_\alpha\}_{\alpha < \omega_2}$ be an increasing sequence of countably closed elementary submodels of $H(\theta)$, each of size $\aleph_1$, with $\omega_2 \subseteq \bigcup_{\alpha < \omega_2} M_\alpha$. Let $\gamma_\alpha = M_\alpha \cap \omega_2$; this is an ordinal and $\omega_1 < \gamma_\alpha < \omega_2$. Choose the M_α so that $\gamma_\alpha < \gamma_\beta$ (and hence $|\gamma_\beta \setminus \gamma_\alpha| = \aleph_1$) whenever $\alpha < \beta$. Let G be $\text{Fn}(\omega_2, 2)$-generic over V. If $S \subseteq \omega_2$, let $G|S = G \cap \text{Fn}(S, 2)$, which is then $\text{Fn}(S, 2)$-generic over V. Let $M'_\alpha = M_\alpha[G|\gamma_\alpha]$. Then, by a standard argument [S, 3.2.12], M'_α is an elementary submodel of $H(\theta)^{V[G]}$.

In $V[G]$, let $\mathbb{R}_\alpha = \mathbb{R} \cap M'_\alpha$. Note that since M_α is countably closed,

$$\mathbb{R}_\alpha = \mathbb{R} \cap V[G|\gamma_\alpha].$$

Fix $\alpha < \beta < \omega_2$; we shall show that $\mathbb{R}_\alpha$ and $\mathbb{R}_\beta$ cannot be homeomorphic. If they were homeomorphic, then, by Lavrentieff's Theorem, we would have a map h, such that $\text{dom}(h)$ and $\text{ran}(h)$ are G_δ sets of reals, h is a homeomorphism from $\text{dom}(h)$ onto $\text{ran}(h)$, and $h''\mathbb{R}_\alpha = \mathbb{R}_\beta$. Then h is coded by a real, in some $V[G|S]$, where $S \in V$ is a countable subset of ω_2. But then, $\mathbb{R}_\beta \subseteq V[G|(S \cup \gamma_\alpha)]$, which is impossible, since $|\gamma_\beta \setminus \gamma_\alpha| = \aleph_1$. $\dashv$

Note that we can have $0^\#$ not existing in such a model, so the number of types of $\mathbb{R} \cap M$'s of a certain size bears no linear relationship to that size.

In view of the above results, one might conjecture that $\mathbb{R} \cap M$'s are homeomorphic if and only they are isomorphic. For example, this is trivially true if $V = L$ by Theorem 2. However, this is also consistently false by:

THEOREM 19. *It is consistent that every two* $\mathbb{R} \cap M$'s *of cardinality* $\aleph_1$ *are homeomorphic, but there exist* $2^{\aleph_1}$ *many* $\mathbb{R} \cap M$'s *of cardinality* $\aleph_1$ *that are pairwise non-order-isomorphic.*

To prove this, we will use the fact σ-centered partial orders can force homeomorphisms (Lemma 20), but not isomorphisms (Lemma 22).

LEMMA 20. *Let A, B be κ-dense separable metric spaces. Then there is a σ-centered partial order which forces them to be homeomorphic.*

PROOF. We will be brief, since results of this type are folklore. First, set

$$A = \bigcup_{\alpha < \kappa} A_\alpha \qquad \text{and} \qquad B = \bigcup_{\alpha < \kappa} B_\alpha,$$

where the A_α are disjoint, the B_α are disjoint, $|A_\alpha| = |B_\alpha| = \aleph_0$, and each A_α, B_α is dense in A, B, respectively. Then, as in Lemma 3.1 of [BB], there is a σ-centered partial order which forces a homeomorphism $f : A \to B$ which takes each A_α to B_α. The proof in [BB] assumes that A, B are 0-dimensional, but this is not a problem, since A, B may be made 0-dimensional by a preliminary forcing which adds one Cohen real. ⊣

COROLLARY 21. *If $\mathrm{MA}(\sigma\text{-centered})$ holds and $\kappa < 2^{\aleph_0}$, then any two κ-dense separable metric spaces are homeomorphic.*

Of course, the $\kappa = \aleph_0$ case of this is true in ZFC, and is a classical result of Sierpiński [Si]. To prove Theorem 19, we also need:

LEMMA 22. *Let $V[F][H]$ be obtained by forcing over V with $\mathrm{Fn}(\omega_1 \times \omega, 2) * \dot{Q}$, where F is $\mathrm{Fn}(\omega_1 \times \omega, 2)$-generic, and $\dot{Q}$ names a σ-centered partial order in $V[F]$. Then in $V[F][H]$ there is no order-preserving injection from $\mathbb{R} \cap V[F]$ into $\mathbb{R} \cap V$.*

PROOF. In $V[F]$, think of F as coding a set of Cohen reals,

$$C = \{c_\alpha : \alpha < \omega_1\} \subseteq \mathbb{R}$$

in some standard way; say, we obtain c_α from $F \cap \mathrm{Fn}(\{\alpha\} \times \omega, 2)$.

We shall actually show that in $V[F][H]$, there is no order-preserving injection from C into $\mathbb{R} \cap V$: Suppose we had such an injection. Then, in $V[F]$, the fact that we could force such a map in a σ-centered (in fact, property (K)) extension implies that we have $g, D \in V[F]$ such that D is an uncountable subset of C and g is an order-preserving injection from D into $\mathbb{R} \cap V$. Since g is order-preserving, it can have only countably many discontinuities, so, as in the proof of Theorem 18, we could, in $V[F]$, extend g to a bijection $f : B_1 \to B_2$, where B_1 and B_2 are Borel sets and f is a Borel map. Let $\gamma < \omega_1$ be large enough so that the Borel codes for f, B_1, B_2 all lie in $V[F \cap \mathrm{Fn}(\gamma \times \omega, 2)]$. But then, applying f^{-1}, we see that all elements of D lie in $V[F \cap \mathrm{Fn}(\gamma \times \omega, 2)]$, which is impossible. ⊣

PROOF OF THEOREM 19. We start with a model V of GCH, and now let $V[G]$ be a ccc extension satisfying $MA(\sigma\text{-centered})$ and $2^{\aleph_0} = \aleph_2$. By Corollary 21, all $\mathbb{R} \cap M$'s of cardinality $\aleph_1$ are homeomorphic in $V[G]$. To produce the desired non-isomorphic $\mathbb{R} \cap M$'s, we construct $V[G]$ specifically as follows:

G will be P-generic over V. $P = P_{\omega_2}$ is obtained as a finite support iteration, $\langle P_\alpha : \alpha \leq \omega_2 \rangle$, where $|P_\alpha| = \aleph_1$ for each $\alpha < \omega_2$. As usual in finite support iterations, take unions at limits. $P_{\alpha+1}$ is $P_\alpha * \mathrm{Fn}(\omega_1 \times \omega, 2) * \dot{Q}_\alpha$, where $\dot{Q}_\alpha$ is a $P_\alpha * \mathrm{Fn}(\omega_1 \times \omega, 2)$-name for a σ-centered partial order. The $\dot{Q}_\alpha$ are chosen by the usual bookkeeping to make sure that $MA(\sigma\text{-centered})$ holds in $V[G]$. Let G_α abbreviate $G \cap P_\alpha$, and let $\mathbb{R}_\alpha$ denote the set of reals in $V[G_\alpha]$.

In V, let K be the set of all $\alpha < \omega_2$ such that there is a countably closed elementary submodel M of $H(\theta)$ such that $|M| = \aleph_1$, $M \cap \omega_2 = \alpha$, and $P \in M$. Call such an M *nice* for α. By standard arguments, if M is nice for $\alpha \in K$, then in $V[G]$, $M[G_\alpha]$ will be an elementary submodel of the $H(\theta)$ of $V[G]$, and $\mathbb{R} \cap M[G_\alpha] = \mathbb{R}_\alpha$.

Since $2^{\aleph_1} = \aleph_2 = |K|$ in $V[G]$, we will be done if we can show that in $V[G]$, these $\mathbb{R}_\alpha$, for $\alpha \in K$, are non-isomorphic. Suppose we had $\alpha < \beta < \omega_2$ and an isomorphism $f : \mathbb{R}_\beta \to \mathbb{R}_\alpha$ in $V[G]$. Fix γ with $f \in V[G_\gamma]$ and $\alpha < \beta < \gamma < \omega_2$. But then f contradicts Lemma 22, where we now view $V[G_\gamma]$ as obtained by forcing over $V[G_\alpha]$. Thus, $V[G_\gamma]$ is obtained by first adding a $\mathrm{Fn}(\omega_1 \times \omega, 2)$-generic filter, F, to $V[G_\alpha]$, and then doing a finite support iteration of $\leq \aleph_1$ σ-centered partial orders, and such an iteration is also σ-centered. But now, in $V[G_\gamma]$ f provides an order-preserving injection from $\mathbb{R} \cap V[G_\alpha][F]$ into $\mathbb{R} \cap V[G_\alpha]$, contradicting Lemma 22. ⊣

Now let us turn to two dimensions and consider $\mathbb{C} \cap M$, where $\mathbb{C}$ is the set of complex numbers. Again, we first consider algebra and regard $\mathbb{C} \cap M$ as a subfield of $\mathbb{C}$.

THEOREM 23. CH *is equivalent to the assertion that if* $\mathbb{R} \cap M$ *is uncountable,* $\langle \mathbb{C} \cap M, +, \cdot, 0, 1 \rangle \cong \langle \mathbb{C} \cap N, +, \cdot, 0, 1 \rangle$ *implies* $\langle \mathbb{R} \cap M, +, \cdot, 0, 1 \rangle \cong \langle \mathbb{R} \cap N, +, \cdot, 0, 1 \rangle$.

PROOF. We have seen that the conclusion merely says $\mathbb{R} \cap M = \mathbb{R} \cap N$; on the other hand, if $\mathbb{R} \cap M$ and hence $\mathbb{C} \cap M$ is uncountable, the hypothesis is equivalent to $|\mathbb{R} \cap M| = |\mathbb{R} \cap N|$. To see this, note that $\mathbb{C} \cap M$ is, by elementarity, an algebraically closed field of characteristic 0. Any two such fields of the same uncountable cardinality are isomorphic. Under CH, if $\mathbb{R} \cap M$ is uncountable then $\mathbb{R} \cap M = \mathbb{R}$, and the implication in the theorem is trivially true. On the other hand, if $2^{\aleph_0} > \aleph_1$ and we construct distinct $\mathbb{R} \cap M$ and $\mathbb{R} \cap N$ of cardinality $\aleph_1$, they will not be field-isomorphic but $\mathbb{C} \cap M$ and $\mathbb{C} \cap N$ will be. ⊣

For the countable case, any two countable $\mathbb{C} \cap M$'s are also isomorphic, since each has a countably infinite transcendence base over the algebraic numbers. On the other hand, we can construct distinct countable $\mathbb{R} \cap M$'s.

It is interesting to note that although $\mathbb{R}$ and $\mathbb{C}$ are not homeomorphic, $\mathbb{R} \cap M$ and $\mathbb{C} \cap M$ may be. This is trivially true if $\mathbb{R} \cap M$ and (hence) $\mathbb{C} \cap M$ are countable, since then, as countable metrizable spaces without isolated points, they are both homeomorphic to $\mathbb{Q}$ [Si]. For the uncountable case:

THEOREM 24. *ZFC does not decide whether there is an* M *such that* $\mathbb{R} \cap M$ *is uncountable and* $\mathbb{R} \cap M$ *and* $\mathbb{C} \cap M$ *are homeomorphic.*

PROOF. If CH and $\mathbb{R} \cap M$ is uncountable, then $\mathbb{R} \cap M = \mathbb{R}$ and $\mathbb{C} \cap M = \mathbb{C}$, so they are of course not homeomorphic. The consistency the other way is immediate from the next theorem. ⊣

THEOREM 25. *Assume* $\mathrm{MA}(\sigma\text{-centered})$ *plus* $0^\#$ *does not exist. Then* $\mathbb{R} \cap M$ *and* $\mathbb{C} \cap M$ *are homeomorphic if they have cardinality less than* $2^{\aleph_0}$, *but not if they have cardinality* $2^{\aleph_0}$.

PROOF. If $|\mathbb{R} \cap M| = 2^{\aleph_0}$ then $\mathbb{R} \cap M = \mathbb{R}$ and $\mathbb{C} \cap M = \mathbb{C}$ by Theorem 8, so they are not homeomorphic. However, if $|\mathbb{R} \cap M| = \kappa < 2^{\aleph_0}$ then both $\mathbb{R} \cap M$ and $\mathbb{C} \cap M$ are κ-dense separable metric spaces, so they are homeomorphic by Corollary 21. ⊣

Getting back to $\mathbb{R}$, a stronger assertion than that there is a homeomorphism between any two κ-dense sets A, B is the assertion that there is an autohomeomorphism h of $\mathbb{R}$ such that $h''A = B$. Steprāns and Watson [SW] show that this is equivalent to any such sets being order-isomorphic, and so for say $\kappa = \aleph_1$, requires more than MA. On the other hand, if one works with the Cantor set rather than $\mathbb{R}$, one only needs MA(σ-centered): the homeomorphism of Corollary 21 extends [BB]. Surprisingly, this also works for κ-dense subsets of $\mathbb{R}^n, n > 1$ [SW].

We close with a question. It is a long-standing open problem whether it is consistent with $2^{\aleph_0} > \aleph_2$ that all $\aleph_2$-dense sets of reals are order-isomorphic; a less demanding question is: Is it consistent with $2^{\aleph_0} > \aleph_2$ that all the $\mathbb{R} \cap M$'s of size $\aleph_2$ are order-isomorphic?

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