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RIEMANN SURFACES OVER REGULAR MAPS

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Introduction. The idea of one surface covering another is a useful one in the study of regular maps. Coxeter and Moser discuss a particular instance [**3**, p. 115] and the maps produced by Sherk [**6**] and Garbe [**4**] are formed by such coverings, though neither paper mentions that fact. In [**5**], Sherk explicitly constructs coverings of regular maps on the sphere.

In this paper, we will discuss the phenomenon of one regular map covering another, and we will describe an algorithm for finding all such coverings of a given map where the projection is one-to-one at each vertex or face-center. Examining the catalogue of regular maps produced in [**7**], we see that more than 97% of all regular maps can be generated in this way.

Definitions. By a *map* we mean a division of a compact, two-dimensional manifold into simply-connected open regions by arcs, called the *edges* of the map; the regions are called the *faces* of the map. Edges meet only at their endpoints, the *vertices* of the map; the ends of an edge need not be distinct vertices. We fix a point in the interior of each face and the relative interior of each edge, and call them *face-centers* and *edge-centers* respectively. We then subdivide the map by drawing an arc from each face-center to each vertex and edge-center surrounding it. The triangular regions of this subreticulation are called *flags*. A *symmetry* or *automorphism* of the map is a homeomorphism of the surface onto itself which preserves flags and edges (and in consequence preserves vertices, faces, face-centers and edge-centers). The map is called *regular* provided that for any two flags A and B, there is a symmetry of the map which takes A onto B.

The meaning of the word "regular" in this paper agrees with that commonly in use for polytopes and complexes, as in [**2**], for instance. When applied to maps, "regular" usually has a less restricted meaning, due primarily to Brahana [**1**]. We will consider such maps in the section titled "Rotary Maps".

While there will be many topologically different symmetries with the same combinatorial effect, it is clear that we can choose one from each combinatorial class to form a group, and that the structure of the group is independent of the choice of representatives. This group we call $G(M)$, the *group of the map* M.

To be more precise about the choice of elements of G, let D be a right triangle in the plane, with right angle E and other vertices F and V. Set the flags of M in order $A_1, A_2, A_3, \ldots, A_{4E}$ so that for $i > 1$, A_i borders some earlier A_j. Let $f_1 = F_1$ be any homeomorphism of A_1 onto D which sends the incident edge-center, face-center and vertex onto E, F, V respectively. Inductively, let

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f_i be any homemorphism of A_i onto D which sends the incident edge-center to E, the face-center to F and the vertex to V, and which agrees with F_{i-1} on the boundary of A_i. Let F_i be the union of F_{i-1} and f_i. Let ψ be F_{4E}.

Then the set of all symmetries W such that $\psi(xW) = \psi(x)$ for all x on the surface forms a group under composition and contains exactly one representative from each combinatorial class. This is the set we choose to be our group $G(M)$.

It is clear that for any flag, the group $G(M)$ must contain reflections which exchange that flag with each of its three neighbors. In fact, these three reflections generate $G(M)$.

We can consider the map to represent the group in the following way: choose any flag to represent I, the identity in $G(M)$, and let each other element of G be represented by the flag that is the image of I under that element. In Figure 1, for instance, the three reflections we mentioned are called α, β, and X, and the corresponding flags are so labeled. Thus, α is a reflection about the leg FE of I, β reflects about VE, and X about FV.

Of the remaining symmetries in Figure 1, γ is a rotation by 180° around the edge-center at the "right angle" of I; R is a rotation by one step counterclockwise around the face whose center is a vertex of I and S is a rotation by one step clockwise around the vertex of M which is also a vertex of I. T is a glide reflection along the "line" joining the midpoint of the edges adjacent to I and R. (We will write the symmetries to the right of their arguments, so the image of a set L under the symmetry R, for instance, is LR. Thus, $R = \alpha X$, $S = \beta X$, and $T = \gamma X$.) These symmetries, then, clearly satisfy the following relations:

$$(1) \quad I = \alpha^2 = \beta^2 = \gamma^2 = \alpha\beta\gamma = X^2 = RX\alpha = SX\beta = TX\gamma.$$

This $1 - 1$ correspondence between flag and symmetry, map and group says that specifying M and specifying $G(M)$ are essentially the same. Though only α, β and X are necessary to generate G, it will be convenient to regard G as a factor group of the group $\mathscr{F}$ with seven generators α, β, γ, X, R, S, T and defining relations (1). We will then specify G by giving only the additional necessary relations. For instance, the group of the cube is defined in this way by: $I = R^4 = S^3$. Our choice of generators makes it possible always to give the definining relations of a map by setting equal to identity I a number of words, each of which is a product of positive powers of R, S, and T, and we will regard this as a standard form of the defining relations.

We will rely heavily on this correspondence between flags and group elements, often referring to a flag as if it were a group element, and vice versa. We will call the vertex (edge, face) incident to I the *central* vertex (edge, face).

Conventions. Henceforth, all maps are to be regular unless otherwise specified. We will abbreviate "clockwise" and "counterclockwise" by "CW" and "CCW" respectively, and we will use the following notation for various important numbers associated with the map M:

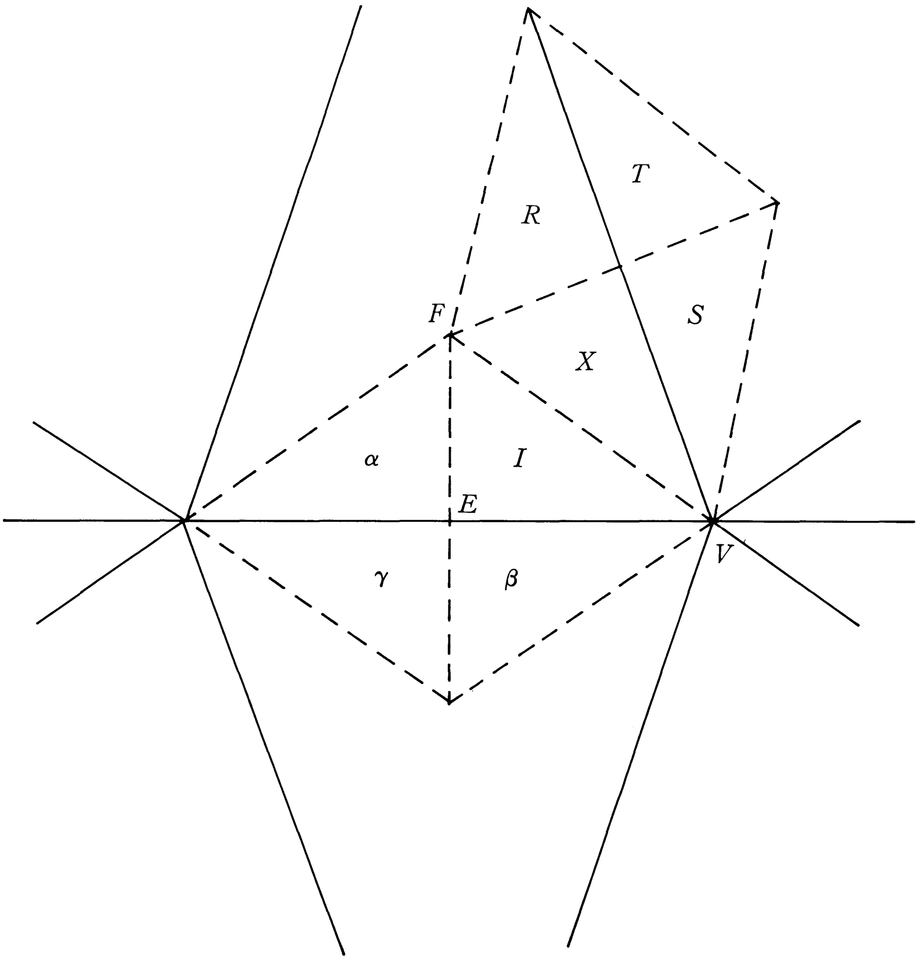

FIGURE 1. The generators of $G(M)$ identified with their flags

E = the number of edges;
F = the number of faces;
V = the number of vertices;
p = the number of edges appearing around a face;
q = the number of edges appearing around a vertex.

In other words, p is the order of R in the group, and q is the order of S. An easy counting argument yields $2E = pF = qV$.

Definition of a covering. Suppose N and M are regular maps, and that there is a function $\varphi: N \to M$ which satisfies the following properties:

1. φ is continuous on all of N.

2. φ is a local homeomorphism everywhere except perhaps at the vertices and face-centers.
3. φ preserves flags and edges (and in consequence preserves vertices, faces, face-centers and edge-centers).

Then N is called a *covering* of M, and φ the corresponding *projection*. Let $K_{e,f,v}(M)$ be the set of all such N for which the corresponding projection is f-to-one on face centers, v-to-one on vertices, and e-to-one on edges and all other points besides the face-centers and vertices. Since it is clear that $q(N)$ must be a multiple of $q(M)$, and since $q(N)vV(M) = q(N)V(N) = 2E(N) = 2eE(M) = eq(M)V(M)$, we see that v must divide e; similarly f must divide e. Further $q(N) = (e/v)q(M)$ and $p(N) = (e/f)p(M)$.

If $e = f = v$, the covering is called *smooth*; otherwise it is *branched* or *ramified*. We will be mostly concerned with coverings in which φ is one-to-one at each vertex, i.e., maps in $K_{ab,a,1}(M)$ for some positive integers a, b. Such a covering will be called *totally ramified.*

Some examples. Figure 2 shows some examples which should illustrate the variety possible among totally ramified coverings. Map B, for instance, is

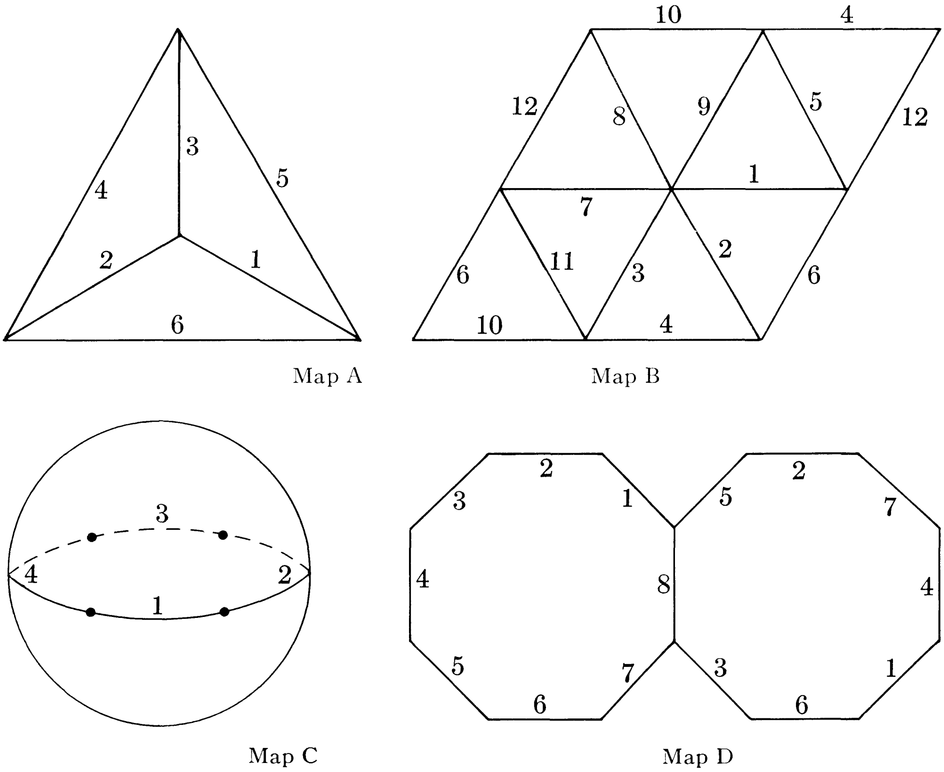

FIGURE 2

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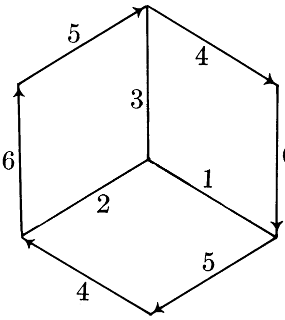
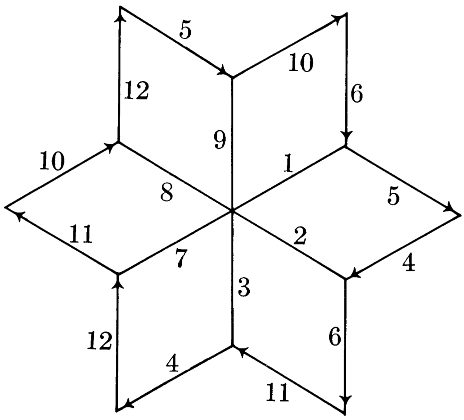
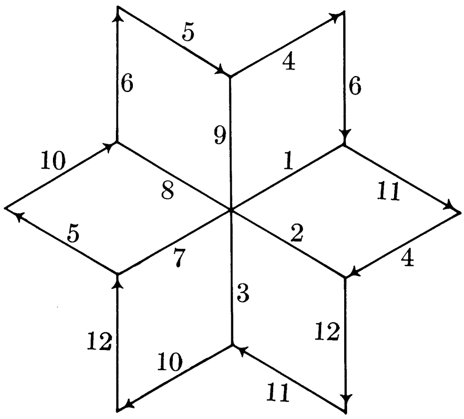


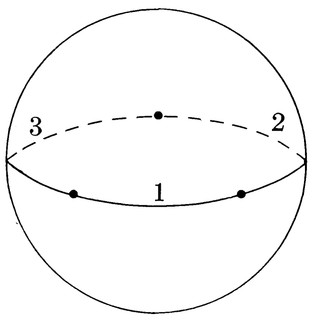
Map E



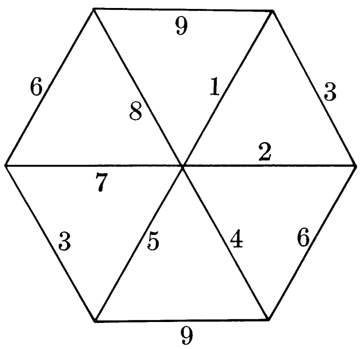
Map F



Map G



Map H



Map I

FIGURE 2—(Continued)

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in $K_{2,2,1}(A)$, the projection being reduction mod 6 on the numbers labelling the edges. Map D is in $K_{2,1,1}(C)$, projection being reduction mod 4. Note that the covering is ramified at the four vertices and at the two face-centers. Map E is nonorientable, as are F and G, each of which is in $K_{2,2,1}(E)$. Thus, this process of covering a map with a branched surface does not lead to a unique result, even in the twofold case. To see that other multiplicities of covering are possible, consider map I; it is in $K_{3,3,1}(H)$.

Facts about coverings. Suppose N and M are regular maps such that M satisfies all the defining relations for N, and suppose that B is a set of relations (i.e., words in R, S, T to be set equal to the identity) which together with some set of defining relations for N form a set of defining relations for M. In this case we will write $M = N/B$. This is appropriate notation, since $G(M) = G(N)/H$, where H is the normal closure of B in $G(N)$.

Conversely, if B is a set of words whose normal closure H in N contains none of α, β, γ, X, then $G(N)/H$ is the group of some regular map M, and then $M = N/B$. Call such a set B *allowable* in N.

LEMMA 1. *Let N be a regular map, B an allowable set of words in R, S, T and let $M = N/B$. Then N is a covering of M.*

Proof. Let D be a right triangle in the plane, let ψ_M be the function from M onto D defined earlier, and let ψ_N be a similar function from N onto D. Let H be the normal closure of B in $G(N)$. By the correspondence between flags and symmetries, the canonical homomorphism of $G(N)$ onto $G(N)/H = G(M)$ induces a function g from the set of flags of N to that of M. This in turn induces a projection of N onto M as follows: For each x in N, say in flag A, let $\varphi(x)$ be that point t in $g(A)$ for which $\psi_N(x) = \psi_M(t)$. The only places where φ might fail to be a local homeomorphism are the vertices and face-centers, so φ is a projection.

COROLLARY. *If N is a covering of M and B is a set of words each of which is the identity in M, then B is allowable in N.*

LEMMA 2. *If $N \in K_{ab,a,1}(M)$, then $M = N/S^q$.*

Proof. Let $L = N/S^q$. Since M satisfies every relation that N does, and also satisfies $I = S^q$, L is a covering of M. Since $L = N/S^q$, $q(L) \leqq q$, and since L is a covering of M, $q(L) \geqq q$; so $q(L) = q$. Thus both M and L have $qV/2$ edges, and so must be the same map.

***$K_a(M) = K_{a,a,1}(M)$*.** We will develop our algorithm by first applying ourselves to the problem of finding, for a given *orientable* map M all maps N in $K_{a,a,1}(M)$ for all positive integers a; i.e., all those coverings which are totally ramified at the vertices and smooth at the face-centers. Abbreviate $K_{a,a,1}$ by K_a. This is the case where considerations are the simplest. Later we will show

how to extend the procedure to handle the nonorientable case, and then to general $K_{ab,a,1}$.

Sheets. Let A_1 be the interior of any face of M and let $B_1 = A_1$; inductively, let A_i be the interior of any face adjoining B_{i-1}, E_i be the relative interior of any edge joining A_i to B_{i-1}. Let $B_i = B_{i-1} \cup A_i \cup E_i$. Let $Z = B_F$; then Z is simply-connected, contains the interior of every face, but contains none of the vertices. Suppose N is in $K_a(M)$ with the projection φ. Then $\varphi^{-1}(Z)$ consists of a number of separated homeomorphic copies of Z. These we call *sheets* of the covering, and the manipulation of these sheets is our primary technique. Z itself we call the *under-sheet* and $M \backslash Z$ is, irresistibly, called the *blanket*.

Suppose Q is a word in R, S, T which is the identity in M. First of all, since $Z = ZQ$, $\varphi^{-1}(Z)Q = \varphi^{-1}(ZQ) = \varphi^{-1}(Z)$, such a word must simply permute the sheets. Now pick a flag I in N to represent the identity and pick $I = \varphi(I)$ in M to represent the identity there. Since φ is one-to-one on vertices, the flag Q in N must be incident with the same vertex as I, and so must be a power of S, in fact a power of S^q. On the other hand, consider the powers of S^q, a in number. Since each flag in M has exactly a pre-images under φ, these must be all the pre-images of I, which makes them the kernel of the factor homeomorphism of $G(N)$ onto $G(M)$. Therefore, $\langle S^q \rangle$, the group generated by S^q, is normal in $G(N)$. Moreover, each flag S^{iq} lies on a different sheet, so S^q permutes the sheets *cyclically*. Let Z_0 be the sheet containing I, and let $Z_i = Z_0 S^{iq}$, $1 \leqq i < a$; for any face W of N, let $Z(W)$ be the number of the sheet to which its interior belongs.

The parameter ρ. Now consider the symmetry $\gamma S^q \gamma$, which is rotation by q steps around the vertex incident to the central vertex along the central edge. By the normality of $\langle S^q \rangle$, $\gamma S^q \gamma = S^{\rho q}$ for some ρ. Then $S^q = \gamma\gamma S^q \gamma\gamma = \gamma(S^{\rho q})\gamma = S^{\rho^2 q}$, so $\rho^2 \equiv 1 \pmod{a}$. By the symmetries of the map, then, in general, rotation by q steps CW about a vertex is rotation by ρq steps around each of its neighbors, which is in turn rotation by $\rho^2 = 1$ times q steps around each of *their* neighbors, etc. If the underlying graph of the map is bipartite, then rotation by q steps CW around a vertex will be the same around each vertex in the same bipartite class, and will be rotation by ρq steps CW around each vertex in the other. If the graph is not bipartite, that will force ρ to be 1.

The *e* values. Now let E be any edge in M, E_0 any edge in $\varphi^{-1}(E)$, X_0 and Y_0 the faces it separates, $E_i = E_0 S^{iq}$, $X_i = X_0 S^{iq}$, $Y_i = Y_0 S^{iq}$. If $Z(X_0) = j$ and $Z(Y_0) = j + e$, then $Z(X_i) = i + j$ and $Z(Y_i) = i + j + e$; in other words, at each E_i, the sheet on the Y side is numbered e more than the sheet on the X side. Thus, to each edge E of M there is a signed number $e(E)$ mod a, such that at each pre-image of E, the sheet on one side is numbered $e(E)$ more

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than the sheet on the other, the direction being the same throughout. Naturally, $e(E)$ is 0 in both directions for each edge E in the under-sheet.

On the other hand, if we simply assign e values *ad lib* to the edges in the blanket, then take a copies of M slit along the blanket edges and glue them together according to the given e values, we will obtain a map of some sort on a manifold. We need to find necessary and sufficient conditions on the e's so that the result will be regular. The conditions that we will provide will be that the e's satisfy a certain system of linear equations mod a. There are three classes of equations in this system, and we will illustrate their derivation with an example, $M = \{4, 4\}_{2,0}$, Figure 3. After the demonstration, we will summarize the equations.

This map is bipartite, and we have chosen the identity flag and the labelling of the vertices so that rotation by 4 steps CW around a black vertex is S^4

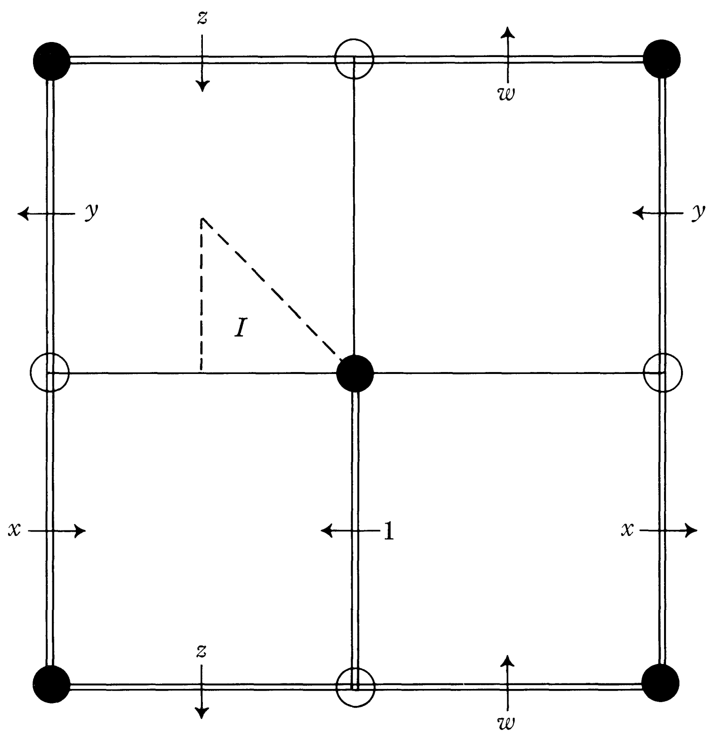

FIGURE 3: e variables on edges of $\{4,4\}_{2,0}$

while rotation 4 steps CW around a white vertex is $S^{4\rho}$. One possible choice of blanket is indicated by the labelled edges, e of each blanket edge being indicated by a variable and an arrow; across the edge marked x, for instance, the sheet at the head of the arrow is numbered x more than the one at the tail, mod a.

Class I equations. Our first class of equations stems from our observations on the nature of q-step CW rotations; these say that the sum of the e values CW around each vertex is either 1 or ρ, depending on its bipartite class. This is what tells us that the edge marked "1" in Figure 3 really does have e value 1 in that direction. The remaining equations in this class are:

$$\begin{aligned}&w + x + y + z \equiv 1\\&-x - y \equiv \rho\\&-w - z - 1 \equiv \rho\end{aligned}$$

(here and throughout we will write "$\equiv$" for "$\equiv$ (mod a)"). Adding these together we get $2(\rho + 1) \equiv 0$. In general, we will get $V(\rho + 1)/2 \equiv 0$; if ρ is 1, as it must be if M is not bipartite, $V \equiv 0$, i.e., a must divide V. In Sherk's work, for instance, the relation 4.1 on p. 458 of [**5**] is essentially the requirement that $\rho = 1$, and the results listed in table I, p. 460 (the maps there are really the duals of the ones we are discussing here) give $DK_j(M)$ where M is a map on the sphere and j divides V.

The numbers *a*, *b*, *c*, *d* of a face. Now let U be a face of M and let $U_0 = Z_0 \cap \varphi^{-1}(U)$; let

$$\begin{aligned}a(U) &= Z(U_0\alpha) & c(U) &= Z(U_0\gamma)\\ b(U) &= Z(U_0\beta) & d(U) &= Z(U_0X).\end{aligned}$$

The remaining two classes of equations concern these numbers, which are easily determined from the diagram in the following way: let P be any path from the center of face U to the flag I which lies entirely in Z. To compute $a(U)$, for instance, make sure P terminates on the reflection axis of the symmetry α. Let P^* be $P \cup P\alpha$. Then a pre-image of P in N will lead from U_0 to I, and the corresponding P^* will lead to $U_0\alpha$. Its sheet number can be determined by adding up the e values along P^*. Figure 4 shows our sample map with the numbers a, b, c, d in that order indicated in each face.

Class II equations. Suppose W is some pre-image of U, say $W = Z_i \cap \varphi^{-1}(U) = U_0S^{iq}$, so that $Z(W) = i$. Then $Z(W\alpha) = Z(U_0S^{iq}\alpha) = Z(U_0\alpha\alpha S^{iq}\alpha) = Z(U_0\alpha S^{-i\rho q}) \equiv Z(U_0\alpha) - i\rho = \alpha(U) - \rho Z(W)$. Thus if W is any face in N and $U = \varphi(W)$, then:

$$\begin{aligned}Z(W\alpha) &\equiv a(U) - \rho Z(W)\\ Z(W\beta) &\equiv b(U) - Z(W)\\ Z(W\gamma) &\equiv c(U) + \rho Z(W)\\ Z(WX) &\equiv d(U) - Z(W)\end{aligned} \tag{2}$$

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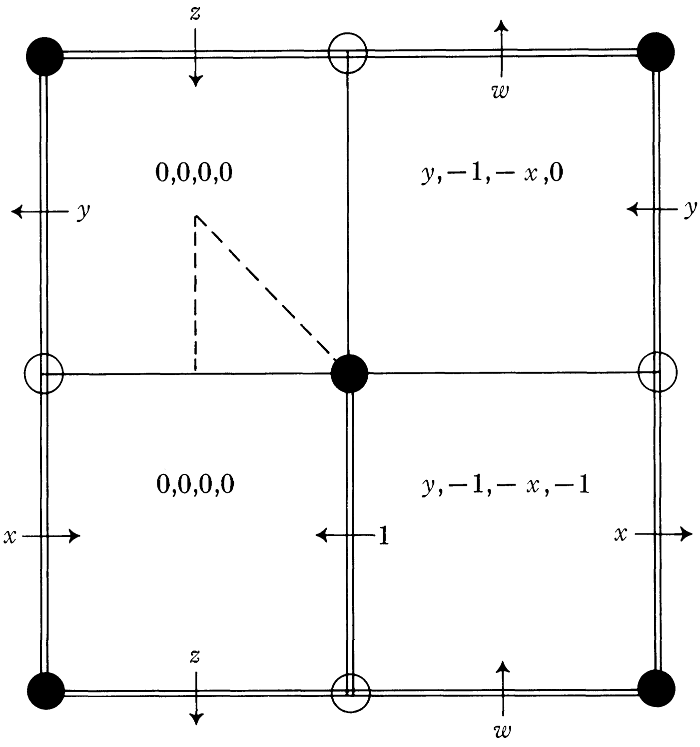

FIGURE 4: a, b, c, d of each face in $\{4,4\}_{2,0}$.

If in these equations, we let $W\alpha$, $W\beta$, $W\gamma$, WX respectively play the role of W and solve, we get our second class of equations:

$$(3)\qquad \begin{aligned} a(U\alpha) &\equiv \rho a(U) \\ b(U\beta) &\equiv b(U) \\ c(U\gamma) &\equiv -\rho c(U) \\ d(UX) &\equiv d(U). \end{aligned}$$

In our example, this yields only two new equations:

$$\begin{aligned} y &\equiv \rho y \\ x &\equiv -\rho x. \end{aligned}$$

Class III equations. Now suppose the edge $\bar{E}$ in N separates faces W_1 and W_2, and let $E = \varphi(\bar{E})$, $U_i = \varphi(W_i)$. Suppose $e(E\alpha)$ is chosen so that it is $Z(W_2\alpha) - Z(W_1\alpha)$, instead of vice versa. Then

$$\begin{aligned} e(E\alpha) &\equiv Z(W_2\alpha) - Z(W_1\alpha) \\ &\equiv a(U_2) - \rho Z(W_2) - (a(U_1) - \rho Z(W_1)) \quad \text{by (2)} \\ &\equiv a(U_2) - a(U_1) - \rho(Z(W_2) - Z(W_1)) \\ &\equiv a(U_2) - a(U_1) - \rho e(E). \end{aligned}$$

We obtain similar results from the other symmetries, and we can rewrite those equations as follows:

$$
(4)\qquad
\begin{aligned}
e(E\alpha) + \rho e(E) &\equiv a(U_2) - a(U_1)\\
e(E\beta) + e(E) &\equiv b(U_2) - b(U_1)\\
e(E\gamma) - \rho e(E) &\equiv c(U_2) - c(U_1)\\
e(EX) + e(E) &\equiv d(U_2) - d(U_1).
\end{aligned}
$$

There are two things to note about this third class of equations. The first and most important is the assumption about the direction in which $e(E\alpha)$ is evaluated: if our assignment of variable and arrow has the arrow pointing in the other direction, then $e(E\alpha)$ in our equation is the negative of the variable. For instance, in our example, the symmetry X carries the edge marked x onto the one marked w, but with arrows reversed. Therefore the fourth equation of (4) with $E = x$ reads: $-w + x \equiv 0 - (-1)$; i.e., $x \equiv w + 1$. The second thing is that though the equations are asymmetric in E and its image under the respective symmetries, and so theoretically one set is required for each edge, the equations (3) guarantee that the first equation will be the same for E as for $E\alpha$. The new equations in our example are:

$$
\begin{aligned}
x &\equiv w + 1\\
y &\equiv z\\
z(\rho + 1) &\equiv 0\\
w(\rho + 1) &\equiv 0\\
x &\equiv y + 1.
\end{aligned}
$$

If we let $y \equiv t$, then $w \equiv y \equiv z \equiv t$, $x \equiv t + 1$, and the system boils down to:

$$
\begin{aligned}
4t &\equiv 0\\
2t &\equiv -(\rho + 1)\\
t(\rho - 1) &\equiv 0\\
t(\rho + 1) &\equiv 0.
\end{aligned}
$$

But the last two imply that $2t \equiv 0$, and with the second, that implies that $\rho \equiv -1$; t must either be 0 or half of a if a is even. Thus, there are two kinds of solutions:

I. $\rho \equiv -1,\quad x \equiv 1,\quad w \equiv y \equiv z \equiv 0.$

II. $\rho \equiv -1,\quad a \equiv 2t,\quad x \equiv t + 1,\quad w \equiv y \equiv z \equiv t.$

Sufficiency of the equations. We have shown that each map N in $K_a(M)$ must satisfy our three classes of linear equations mod a. Conversely, suppose we have a solution to the system of equations and have made a map N from a copies of M, numbered 1 to a, slit along the blanket edges and glued according to the e values; we wish to show that this map is regular. We can label each

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face of N by an ordered pair (U, i), where U is the corresponding face of M, and i is the number of the sheet (copy of M) to which it belongs. Define α, β, γ and X by these formulae (suggested by (2)):

$$(U, i)\alpha = (U\alpha, a(U) - \rho i)$$
$$(U, i)\beta = (U\beta, b(U) - i)$$
$$(U, i)\gamma = (U\gamma, c(U) + \rho i)$$
$$(U, i)X = (UX, d(U) - i).$$

To show that these are symmetries of N, we need to show that they preserve adjacency. Suppose (U, i) and (W, j) are adjacent in N and that E separates U and W in M and that $e(E)$ is directed so that it is $j - i$ instead of vice versa. Likewise, suppose $e(E\alpha)$ is directed to match E under α. Then from the third class of equations, $e(E\alpha) \equiv a(W) - a(U) - \rho e(E) \equiv a(W) - a(U) - \rho j + \rho i \equiv a(W) - \rho j - (a(U) - \rho i)$; from the definition of α, $(U, i)\alpha = (U\alpha, a(U) - \rho i)$, $(W, j)\alpha = (W\alpha, a(W) - \rho j)$. Since the difference of the sheet numbers is exactly $e(E\alpha)$, these faces are adjacent in N. Similar arguments show that β, γ, and X are symmetries of N. Moreover, they send the flag at $(I, 0)$ to the proper nearby flags, so they are precisely the symmetries we need to display to demonstrate that N is regular.

Defining relations. A bonus of this algorithm is that it provides a set of defining relations for N. Let A_i, $1 \leqq i \leqq n$ be a set of words in R, S, T so that $I = R^p = S^q = A_1 = A_2 = \ldots = A_n$ are defining relations for M. Then each A_i must be a power of S^q in N, and the power can be determined by tracing out the symmetry in the diagram or by applying the equations (2). If $A_i = S^{t_i q}$ in N, let $B_i = A_i S^{-t_i q}$. Then N clearly satisfies

$$I = R^p = S^{aq} = B_1 = B_2 = \ldots = B_n, \quad \gamma S^q \gamma = S^{\rho q}.$$

This last relation in standard form is $TS^{q-1}TS^{\rho q-1}$. Conversely, by examining the Coxeter-Todd coset enumeration of the cosets of $\langle S \rangle$, we see that the map determined by these relations is no larger than N and so must be N. In our example, M has defining relations $I = R^4 = S^4 = (RS)^2$. From the diagram, $(RS)^2$, which is translation two steps to the right along the central horizontal axis, subtracts y from the number of the sheet. From our solutions $y \equiv t$, and $\rho \equiv -1$, so N has defining relations:

Case I: $I = R^4 = S^{4a} = (RS)^2 = (RS^3)^2$
Case II: $I = R^4 = S^{8t} = (RS)^2 S^{4t} = (RS^3)^2$.

K_a of non-orientable maps. In discussing the application of this algorithm to the non-orientable case, the notion of *pseudo-orientability* will be useful. One way to say that a map is orientable is to require that the vertices can be labelled with direction arrows so that at each edge, the arrows at the vertices

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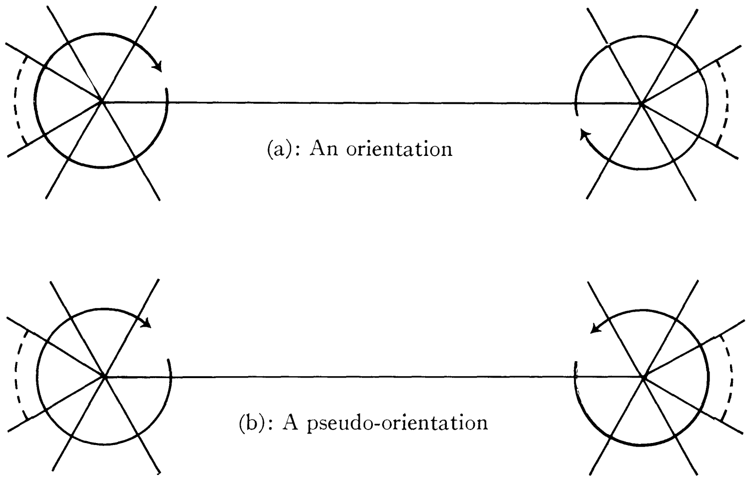
at each end cross the edge in *opposite* directions (see Figure 5a). We will call the map *pseudo-orientable* (*PSO*) if the vertices can be labelled with direction arrows so that the arrows at the ends of an edge cross it in the *same* direction (See Figure 5b).



(a): An orientation

(b): A pseudo-orientation

FIGURE 5.

LEMMA. (1) *M is PSO; if and only if*

(2) *travel along a cycle of even length preserves orientation, and travel along a cycle of odd length reverses orientation.*

Proof. Since the arrows around vertices alternate direction along any path in a *PSO* map, it is clear that (1) implies (2). On the other hand, if (2) holds, we can pseudo-orient M by choosing the arrow for one vertex arbitrarily, then extending to neighboring vertices according to the definition, and continuing until all vertices are labelled. (2) guarantees that no contradictions will occur.

Now let M be a non-orientable map, $N \in K_a(M)$, φ the projection. Then, as before, $\langle S^q \rangle$ is normal in $G(N)$, and $\gamma S^q \gamma = S^{\rho q}$ for some ρ such that $\rho^2 \equiv 1 \pmod{a}$. This says that a.ong any path beginning with the central vertex, S^q is rotation around each vertex "CW" by q and ρq steps, alternately, the orientation of each vertex being chosen to agree locally with the previous one. We consider cases:

1. *M is not PSO:* By the lemma, either some even cycle reverses orientation, or some odd cycle preserves it. If some even cycle reverses, then S^q, rotation q steps CW at the beginning of the cycle, is rotation by q steps "CW" at the end, and by the reversal of orientation, "CW" is really CCW, so $q \equiv -q$; i.e., a must be 2. On the other hand, if all the even cycles preserve orientation, then some odd cycle must preserve and some odd cycle must reverse (after all, some

cycle must reverse, and none of the even ones do). By symmetry, we can choose such cycles both to contain the central vertex, and then the cycle formed by tracing one and then the other is even and reverses. Thus, in either case, if M is not PSO, a must be 2.

2. *M is PSO:* Then along an odd cycle, which must reverse orientation, q steps CW at the beginning is ρq steps "CW" = CCW at the end, and so ρ must be $-1 \bmod a$. Pictorially, if we choose a pseudo-orientation on M so that the central vertex has an arrow in the direction of the rotation S, then in N, S^q is rotation around each vertex by q steps in the direction of its arrow.

With these restrictions on a and ρ, the same technique for constructing maps in $K_a(M)$, i.e., solving the same three classes of linear equations mod a, works as well for the non-orientable case as for the orientable.

Summary.

Class I Equations. In an orientable bipartite map, the sum CW around each vertex of the e values of adjacent edges is either ρ or 1; 1 for vertices in the same bipartite class as the central vertex, ρ for those in the other. In an orientable non-bipartite map, $\rho \equiv 1$, and the sum CW around every vertex is 1.

In a non-orientable PSO map, choose the pseudo-orientation so that S is rotation by one step in the direction of the arrow around the central vertex. Then $\rho \equiv -1$ and the sum around each vertex in the direction of its arrow is 1.

In a non-orientable non-PSO map the sum around every vertex in either direction is $1 \equiv -1$, and $\rho \equiv 1 \equiv -1$.

Class II equations. If U is any face, then:

$$\begin{aligned} a(U\alpha) &\equiv \rho a(U) \\ b(U\beta) &\equiv b(U) \\ c(U\gamma) &\equiv -\rho c(U) \\ d(UX) &\equiv d(U). \end{aligned}$$

Class III equations. If the edge E separates the faces U_1 and U_2, and the e values of E and its images under α, β, γ and X are measured in compatible directions, then:

$$\begin{aligned} e(E\alpha) + \rho e(E) &\equiv a(U_2) - a(U_1) \\ e(E\beta) + e(E) &\equiv b(U_2) - b(U_1) \\ e(E\gamma) - \rho e(E) &\equiv c(U_2) - c(U_1) \\ e(EX) + e(E) &\equiv d(U_2) - d(U_1). \end{aligned}$$

The general case: $K_{ab,a,1}$ of a map. We now wish to extend the existing procedure to the more general case of constructing all maps in $K_{ab,a,1}(M)$ for all a, b. To that end, we need to redefine the word "sheet".

Consider the subreticulation of M into flags, and call the arcs of that subdivision *pseudo-edges*. Let $A_1 = B_1$ be the interior of any flag; inductively, let A_i be the interior of any flag bordering B_{i-1} and let E_i be the relative interior of any pseudo-edge joining them. Let $B_i = B_{i-1} \cup A_i \cup E_i$; finally, let $Z = B_{4E}$. Then Z is simply connected and contains none of the possible branch points of the covering, namely the vertices and face-centers. As before, Z is the undersheet, connected components of $\varphi^{-1}(Z)$ are the sheets of the covering, and $M \backslash Z$ is the blanket.

It will make life simpler if we insist on a certain canonical kind of undersheet. It is clear that we can make the choices in the construction of Z so that all the pseudo-edges in the blanket are in fact edges except for one pseudo-edge in each face joining its center to an incident vertex.

As before, $\langle S^q \rangle$ is normal in $G(N)$, being the kernel of the group homomorphism induced by φ; so again, $\gamma S^q \gamma = S^{\rho q}$ for some ρ such that $\rho^2 \equiv 1 \pmod{ab}$. Further, R^p is the identity in M and must be a power of S^q in N, say $R^p = S^{\tau q}$ for some τ. Now, R has order bp in N, since $E(N) = abE$, $F(N) = aF$, and $p(N)F(N) = 2E(N)$. So $|\tau|_{ab} = b$, and so $(\tau, ab) = a$, say $\tau = ma$, where $(m, b) = 1$. Moreover, we have $RS^qR^{-1} = \gamma SS^qS^{-1}\gamma = \gamma S^q \gamma = S^{\rho q}$, so that $R^p = RR^pR^{-1} = RS^{\tau q}R^{-1} = (RS^qR^{-1})^\tau = S^{\rho\tau q} = (S^{\tau q})^\rho = (R^p)^\rho$. Thus $\rho \equiv 1 \pmod b$, say $\rho - 1 = nb$. Then $(\rho - 1)\tau = (ma)(nb)$ so that $\rho^\tau \equiv \tau \pmod{ab}$. In the orientable case, then, rotation by p steps CW around one face is rotation by p steps CW around every face.

In the non-orientable case, the same is true locally, and if we traverse an orientation-reversing cycle of faces, we see that $R^p = R^{-p}$, i.e., b must be 1 or 2. Moreover, as before, if M is not PSO, ab must be 1 or 2, while if M is PSO, ρ must be -1.

With these restrictions in mind, if we now make a canonical choice of undersheet and blanket, the same considerations as before apply here. We can then assign variables to all the pseudo-edges in the blanket. The ones from a face-center to an adjacent vertex will be labelled τ pointing CCW if M is orientable; if M is not orientable, $\tau = -\tau$, so the direction is arbitrary. We then set up the same three classes of equations, except that now we must have a, b, c, d for each flag. By our choice of blanket, however, two flags which share an edge and a face have the same a, b, c, d values, so for ease of computation, we may combine each two such flags into one region called a banner. Figure 6 shows $\{4, 4\}_{2,0}$ separated into banners and ready for the algorithm.

The same summary as before may be used, with the word "banner" substituted for "face", and these two equations added:

$$(\tau, ab) = \tau; \quad \rho \equiv 1 \pmod b.$$

The three classes of equations in this case are:

$$\text{I.}\quad \begin{aligned} &w + z + 1 \equiv -\rho \\ &x + y \equiv -\rho \\ &x + y + z + w \equiv 1 - 4\tau \end{aligned}$$

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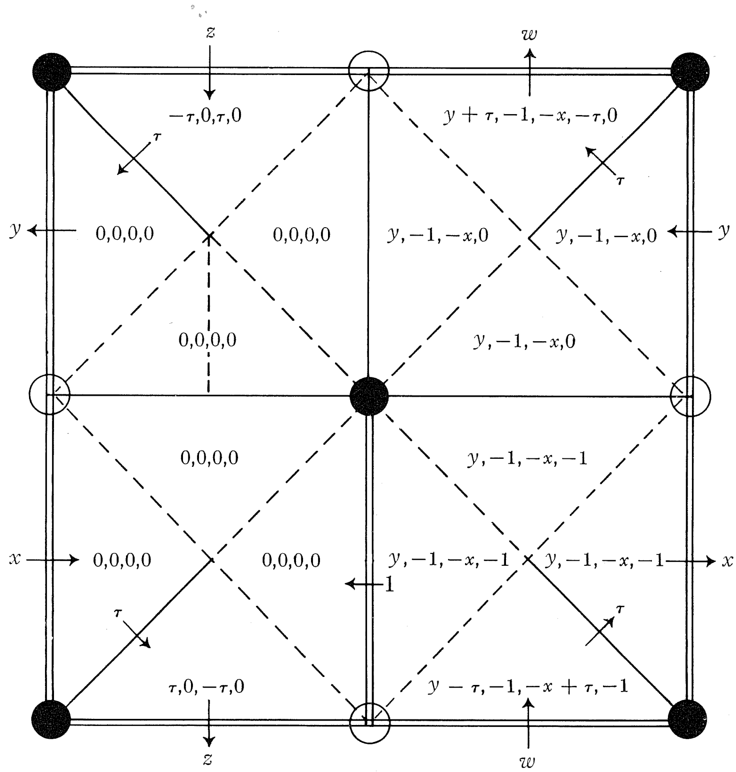

FIGURE 6. a, b, c, d values of banners in $\{4,4\}_{2,0}$

II. $y \equiv \rho y$

$x \equiv -\rho x$

III. $z(\rho + 1) \equiv 2\tau$

$w(\rho + 1) \equiv 2\tau$

$x - y \equiv 1$

$y \equiv z$

$x \equiv w + 1$

where all equivalences are mod ab. If we let $t \equiv y$, this system boils down to:

$$w \equiv y \equiv z \equiv t; \quad x \equiv t + 1; \quad (\tau, ab) \equiv a; \quad \rho \equiv 1 \pmod b$$

$$8\tau \equiv 0$$

$$\rho \equiv -1 - 2\tau$$

$$2t \equiv 2\tau$$

$$2t(t + 1) \equiv 0$$

The first few solutions are:

$ab =$	2	2	3	4	4	4	4	5	6	6	7	8	8	8	8
$a =$	2	1	3	4	1	2	1	5	6	3	7	8	1	4	1
$b =$	1	2	1	1	4	2	4	1	1	2	1	1	8	2	8
$\tau =$	0	1	0	0	1	2	3	0	0	3	0	0	3	4	7
$\rho =$	1	1	2	3	1	3	1	4	5	5	6	7	1	7	1
$t =$	0	0	0	0	1	0	1	0	0	0	0	0	3	0	3
or $t =$	1	1		2	3	2	3		3	3		4	7	4	7

The corresponding defining relations are:

$$I = R^4S^{-4\tau} = S^{4ab} = RS^3RS^{4t+1} = TS^3TS^{4\rho-1}.$$

Note that when $\tau \equiv 0 \equiv ab$, then $b = 1$ and $N \in K_a(M)$. Correspondingly, the solutions and defining relations for the case $\tau = 0$ agree with our previous results.

Of course, since this map is in $K_2(\epsilon_4)$, every map in $K_{ab,a,1}(M)$ is in $K_{2ab,2a,1}(\epsilon_4)$, and so is constructible in that way also. (The map $\epsilon_4 = \{4, 2\}$ consists of 4 vertices on the equator of a sphere.)

Rotary maps. Brahana [**1**] called a map "regular" provided that it had symmetries of types R, S, and γ; we will use the word *rotary* for such a map, since R, S, and γ may be thought of rotations, and generate the rotation group of M. These generators satisfy

$$I = R^p = S^q = \gamma^2, \quad \gamma R = S.$$

A map which is rotary but not regular is called *chiral*, or *irreflexible*. (Note also that the generator R in this paper plays the role of R^{-1} in [**1**; **2**; **3**; and **4**]). Sherk [**4**] and Garbe [**2**] construct maps N in $K_{2,1,2}(M)$, where M is a rotary map on the torus. Since relatively few such maps M are regular, the corresponding N's must also be chiral.

An algorithm for finding all rotary N in $K_{ab,a,1}(M)$ for rotary M may be constructed by analogy to the one for regular maps. This second algorithm proceeds exactly as the first through the derivation of the Class I equations. Then we let

$$c(U) \equiv Z(U_0\gamma)$$
$$r(U) \equiv Z(U_0R)$$
$$s(U) \equiv Z(U_0S).$$

Analogous to equations (2), we have

$$Z(W\gamma) \equiv c(U) + \rho Z(W)$$
$$Z(WR) \equiv r(U) + \rho Z(W)$$
$$Z(WS) \equiv s(U) + Z(W).$$

To get the class II equations, we replace W by $W\gamma$, and solve, getting $c(U) \equiv -\rho c(U)$, as before; we replace W by WR, WR^2, WR^3, etc., and solve to get

$$\sum_{i=1}^{p} \rho^i r(UR^i) \equiv \tau.$$

Similarly,

$$\sum_{i=1}^{q} s(US^i) \equiv 1.$$

To get the class III equations, we make precisely the same substitutions as before, and we get

$$\begin{aligned} e(E\gamma) - \rho e(E) &\equiv c(U_2) - c(U_1) \\ e(ER) - \rho e(E) &\equiv r(U_2) - r(U_1) \\ e(ES) - e(E) &\equiv s(U_2) - s(U_1). \end{aligned}$$

Again, these three classes of equations are both necessary and sufficient for the corresponding N to be rotary. An interesting thing to observe is that the algorithm might produce a chiral map N even when the base map M is regular. Consider, for instance, the regular map $M = \{6, 3\}_{1,1}$.

The equations yield four solutions for $ab = 3$, one with $\tau = 2$ and three with $\tau = 0$. The solution with $\tau = 2$ and one of the ones with $\tau = 0$ correspond to regular maps; the remaining two correspond to enantiamorphic copies of the same chiral map N shown in Figure 7. The existence of a rotation around the central vertex is clear and the symmetry

$$\begin{aligned} R = (1\ \ 2\ \ 18\ \ 16\ \ 23\ \ 27)(3\ \ 25\ \ 14\ \ 6\ \ 22\ \ 17) \\ (4\ \ 8\ \ 12\ \ 13\ \ 26\ \ 24)(5\ \ 15\ \ 10\ \ 20\ \ 21\ \ 7)(9\ \ 19\ \ 11) \end{aligned}$$

is a rotation around the face 1, expressed as a permutation on the edges. As a permutation on the faces, $R = (2\ 6\ 5\ 3\ 8\ 9)(4\ 7)$. The existence of these symmetries shows that the map is in fact rotary. To see that it is not regular, note that it is impossible to use edge 3, for instance, as an axis of reflection: such a reflection would try to interchange edges 10 and 20 near edge 3 at the top, but also switch edges 13 and 20 at the bottom of the diagram.

Composition. We close with two theorems concerning the composition of the K's. By $K_{a,b,c}(K_{d,e,f}(M))$, we mean the set of all regular (rotary) maps N for which there is a regular (rotary) map L in $K_{d,e,f}(M)$ such that N is in $K_{a,b,c}(L)$. Since the composition of the projection of N onto L and L onto M is the projection of N onto M, it is clear that $K_{a,b,c}(K_{d,e,f}(M))$ is a subset of $K_{ad,be,cf}(M)$.

THEOREM. *For any regular (rotary) map M, $K_{ab}(M) = K_a(K_b(M))$.*

Proof. From our comments above $K_a(K_b(M)) \subseteq K_{ab}(M)$, so we only need show the opposite inclusion; let $N \in K_{ab}(M)$. Then the cyclic subgroup

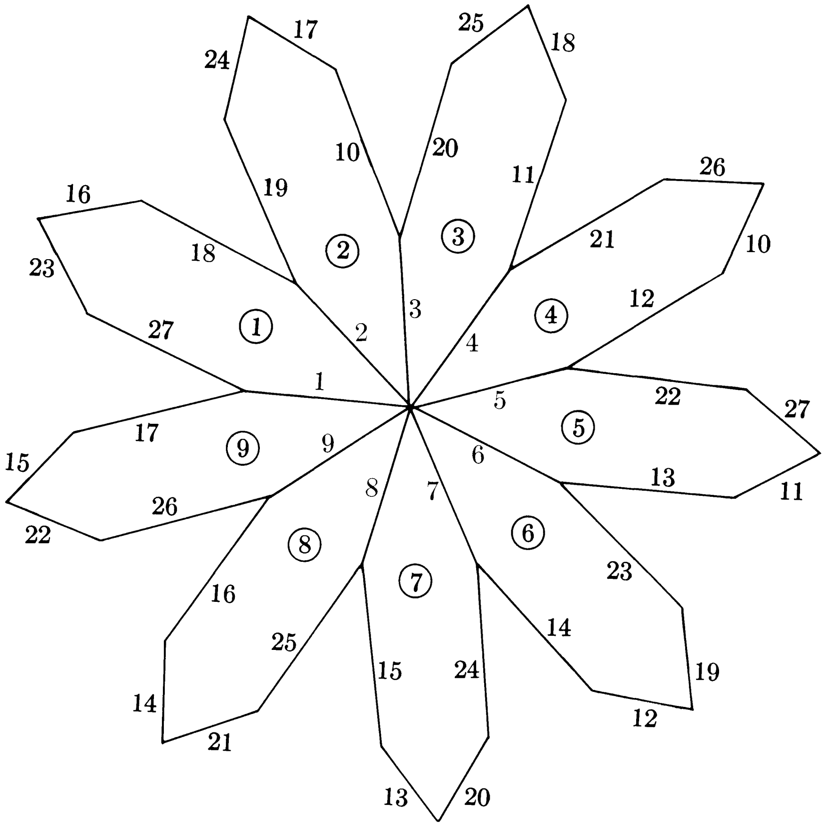

FIGURE 7. A chiral cover of a regular map.

generated by S^q in $G(N)$ is normal there, and $M = N/S^q$. Since every subgroup of a cyclic group is characteristic in that group, the group generated by S^{bq} is also normal in $G(N)$; let $L = N/S^{bq}$. By the normalcy of $\langle S^{bq}\rangle$, a vertex in L has valence exactly bq; since $L/S^q = N/S^q = M$, L is a Riemann lifting of M, and since M and N both have V vertices, so does L. Thus, $L \in K_b(M)$ and $N \in K_a(L)$, as required.

THEOREM. *For any regular (rotary) map M, and any positive integers a and b, let $d = (a, b)$, $a = \bar{a}d$, $b = \bar{b}d$. Then,*

$$K_{ab,a,1}(M) = K_{b,1,1}(K_a(M)) = K_{a,\bar{a},1}(K_{b,d,1}(M)).$$

Proof. Again, it is clear that $K_{ab,a,1}(M)$ contains both the other two sets in the statement of the theorem, and we only need show that opposite inclusion; let $N \in K_{ab,a,1}(M)$. Again, the cyclic group generated by S^q is normal in $G(N)$. Now, in N, $R^p = S^{\tau q}$, where $(\tau, ab) = a$, say, $\tau = ma$, where $(m, b) = 1$. Let $L = N/\{R^p, S^{aq}\} = N/S^{aq}$. Then $q(L) = aq$, $p(L) = p$, $V(L) = V$, so

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$E(L) = aE$, and so $F(L) = aF$. Thus L is in $K_{a,a,1}(M)$, and N is in $K_{b,1,1}(L)$, as required for the first equality.

Now let $\bar{L} = N/S^{bq}$. Then $q(\bar{L}) = bq$, $V(\bar{L}) = V$, so $E(\bar{L}) = bE$. To compute $p(\bar{L})$, we need to determine the order of R^p in $G(\bar{L}) = G(N)/S^{bq}$. In $G(N)$, $(R^p)^{\bar{b}} = (S^{maq})^{\bar{b}} = S^{\bar{a}\bar{b}dmq} = S^{bqm\bar{a}}$, which is a power of S^{bq}, so the order of R^p in $\bar{L}$ is a factor of $\bar{b}$; on the other hand, $(R^p)^x = S^{maqx}$ is a power of S^{bq} if and only if max is a multiple of b, and since $(m, b) = 1$, this happens if and only if ax is a multiple of b, i.e., if and only if x is a multiple of $\bar{b}$. Thus, the order of R^p in $\bar{L}$ is exactly $\bar{b}$, and so $p(\bar{L})$ is $\bar{b}p$. From this we get that $F(\bar{L})$ is dF, so $L \in K_{b,d,1}(M)$, and $N \in K_{a,\bar{a},1}(\bar{L})$ as required for the second equality.

COROLLARY. *If* $(a, b) \neq 1$, *then* $K_a(K_{b,1,1}(M)) = \emptyset$.

Proof. If $N \in K_a(K_{b,1,1}(M))$ then N/S^{bq} is in $K_{b,1,1}(M)$. On the other hand, such an N must be in $K_{ab,a,1}(M)$, and we saw in the proof of the theorem above that for any map N in $K_{ab,a,1}(M)$, N/S^{bq} is in $K_{b,d,1}(M)$, which is disjoint from $K_{b,1,1}(M)$ unless $d = 1$.

Added in proof. The map of Figure 7 was originally discovered by Peter Bergau [**4**, p. 293].

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