AN EXTREME DUODENARY FORM

H. S. M. COXETER AND J. A. TODD

1. Introduction. Let $f(x_1, \ldots, x_n)$ be a positive definite quadratic form of determinant Δ; let M be its minimum value for integers $x_1, \ldots, x_n$, not all zero; and let $2s$ be the number of times this minimum is attained, i.e., the number of solutions of the Diophantine equation

$$f(x_1, \ldots, x_n) = M.$$

The form is said to be *extreme* if, for all infinitesimal variations of the coefficients, Δ/M^n is minimum. A form for which this minimum is as small as possible is said to be *absolutely* extreme. For a list of the known extreme forms with $n < 12$ (including the absolutely extreme forms A_2, A_3, D_4, D_5, E_6, E_7, E_8), see Coxeter [**1**, pp. 394, 439]. Continuing this list, we may say that there are three known extreme forms for $n = 12$:

1.1
$$A_{12} = x_1^2 - x_1x_2 + \ldots + x_9^2 - x_9x_{10} + x_{10}^2 - x_{10}x_{11} + x_{11}^2 - x_{11}x_{12} + x_{12}^2,$$

1.2
$$D_{12} = x_1^2 - x_1x_2 + \ldots + x_9^2 - x_9x_{10} + x_{10}^2 - x_{10}x_{11} + x_{11}^2 - x_{10}x_{12} + x_{12}^2,$$

1.3
$$D_{12}^2 = x_1^2 - x_1x_2 + \ldots + x_9^2 - x_9x_{10} + x_{10}^2 - x_9x_{11} + x_{11}^2 - x_{11}x_{12} + \tfrac{3}{2}x_{12}^2.$$

Apart from a numerical factor, these are equivalent to the forms U_{12}, V_{12}, W_{12} of Korkine and Zolotareff [**5**, p. 367]. Since $2D_{12}^2$ has integral coefficients and determinant 1, but cannot take the value 1 (for integers $x_1, \ldots, x_{12}$), it must also be equivalent to the form f_{12} of Chao Ko [**4**, p. 85]. When giving the number of automorphs as $2^{10}\,12!$ instead of $2^{11}\,12!$ [**1**, p. 434], Ko was doubtless omitting the automorphs of negative determinant.

In a letter to one of us, dated January 14, 1947, T. W. Chaundy announced the form

1.4
$$\begin{aligned} J_{12} = {} & (\tfrac{1}{2}x_1 + x_2 + x_3)^2 + (\tfrac{1}{2}x_1 + x_2 + x_4)^2 \\ & + (\tfrac{1}{2}x_1 + x_3 + x_4 + x_5 + x_6)^2 + (\tfrac{1}{2}x_1 + x_7 + x_8 + x_9)^2 \\ & + \sum_5^8 (\tfrac{1}{2}x_1 + x_j + \tfrac{1}{2}x_{12})^2 + \sum_9^{11} (x_j + \tfrac{1}{2}x_{12})^2 \\ & + (x_{10} + x_{11} + \tfrac{1}{2}x_{12})^2 \end{aligned}$$

Received October 15, 1952.

https://doi.org/10.4153/CJM-1953-043-4 Published online by Cambridge University Press

as a possible candidate for the title "absolutely extreme." However, Δ/M^{12} is still smaller for the new form

$$1.5 \qquad K_{12} = \tfrac{1}{12}\left\{\sum_1^5 (6x_j + 3y_j + 2y_6)^2 + 3\left(6x_6 + 2\sum_1^5 y_j - y_6\right)^2 + 3\sum_1^5 y_j^2 + y_6^2\right\},$$

which we shall derive from a lattice in unitary 6-space, somewhat resembling the lattice in unitary 3-space that represents the senary form E_6 [**1**, p. 421]. We shall prove that K_{12} is extreme, but we still do not know whether it is *absolutely* extreme.

The following table shows how these five forms compare with one another:

	s	M	$2^{12}\Delta$	$(2/M)^{12}\Delta$
A_{12}	78	1	13	13
D_{12}	132	1	4	4
D_{12}^2	132	1	1	1
J_{12}	324	2	2^{10}	$\frac{1}{4}$
K_{12}	378	2	3^6	$\frac{729}{4096}$

The value of s for K_{12} indicates that equal "solid" spheres in Euclidean 12-space can be packed in such a way that each touches 756 others.

2. Eutactic stars. In Euclidean n-space, a set of $2s$ vectors $\pm\,\mathbf{a}_1, \ldots, \pm\,\mathbf{a}_s$ is called a *eutactic star* if the sum of the squares of the orthogonal projections of $\mathbf{a}_1, \ldots, \mathbf{a}_s$ on a line is the same in all directions [**1**, p. 401], i.e., if there is a constant σ such that

$$\sum_{l=1}^{s} (\mathbf{a}_l \cdot \mathbf{z})^2 = \sigma \mathbf{z}^2,$$

for every vector $\mathbf{z}$. In terms of rectangular Cartesian coordinates, we write

$$\mathbf{a}_l = (\mu_{1l}, \ldots, \mu_{nl}), \quad \mathbf{z} = (\zeta_1, \ldots, \zeta_n)$$

and the condition becomes

$$\sum \sum \sum \mu_{jl}\, \mu_{kl}\, \zeta_j\, \zeta_k = \sigma \sum \zeta_j^2,$$

i.e.,

$$\sum_{l=1}^{s} \mu_{jl}\, \mu_{kl} = \sigma \delta_{jk}$$

[**1**, p. 397, with $\rho_k = 1/\sigma$ and subscripts used instead of superscripts].

https://doi.org/10.4153/CJM-1953-043-4 Published online by Cambridge University Press

Let us now extend this notion to a *unitary* n-space (in which the distance between two points is the square root of the sum of the norms of the differences of their coordinates ζ_j). The above "orthogonality" relation suggests that it would be appropriate to define a eutactic star

$$\pm (\mu_{1l}, \ldots, \mu_{nl}),$$

in such a space, by the "Hermitian" relation

2.1
$$\sum_{l=1}^{s} \bar{\mu}_{jl} \mu_{kl} = \sigma \delta_{jk}.$$

If we write $\mu_{jl} = \xi_{jl} + \eta_{jl} i$, where the ξ and η are real, this becomes

$$\sum (\xi_{jl} - \eta_{jl} i)(\xi_{kl} + \eta_{kl} i) = \sigma \delta_{jk},$$

and the real part yields

$$\sum (\xi_{jl} \xi_{kl} + \eta_{jl} \eta_{kl}) = \sigma \delta_{jk}.$$

Hence

2.2 *When the real and imaginary parts of the coordinates in unitary n-space are interpreted as coordinates in real Euclidean $2n$-space, a eutactic star remains eutactic.*

3. The lattice L. As usual, we write $\omega = e^{2\pi i/3}$ and

3.1
$$\lambda = 1 - \omega = -2\omega - \omega^2.$$

In unitary space of six dimensions, let L denote the set of points whose coordinates $(\zeta_1, \ldots, \zeta_6)$ are integers of the Eisenstein field $R(\omega)$, mutually congruent modulo λ and adding up to a multiple of 3, so that

3.2
$$\zeta_j \equiv \zeta_k \pmod{\lambda} \qquad (j, k = 1, \ldots, 6),$$
$$\sum \zeta_j \equiv 0 \pmod{3}.$$

The corresponding vectors

$$(\zeta_1, \ldots, \zeta_6) = \sum \zeta_j \mathbf{p}_j$$

have the property that the difference of any two belongs to the set; hence L is a *lattice*.

Since the coordinates are Eisenstein integers, we may write

$$\zeta_j = A_j \omega + B_j \omega^2,$$

where A_j and B_j are rational integers. The conditions 3.2 are equivalent to

3.3
$$A_j + B_j \equiv A_k + B_k \pmod{3},$$
$$\sum A_j \equiv \sum B_j \equiv 0 \pmod{3}.$$

Analogy with E_6 [**1**, p. 421] suggests the following theorem:

3.4 *The twelve complex vectors*

$$\mathbf{t}_j = 3\mathbf{p}_j \ (j = 1, \ldots, 5), \qquad \mathbf{t}_6 = -3\lambda\mathbf{p}_6,$$
$$\mathbf{t}_k = \lambda(\mathbf{p}_{k-6} - \mathbf{p}_6) \qquad (k = 7, \ldots, 11),$$
$$\mathbf{t}_{12} = \mathbf{p}_1 + \ldots + \mathbf{p}_6$$

form a rational integral basis for the lattice L.

Proof. Consider the vector

$$\mathbf{z} = \sum_1^6 x_j \mathbf{t}_j + \sum_1^6 y_j \mathbf{t}_{j+6},$$

where x_j and y_j are rational integers. In terms of the orthogonal unit vectors $\mathbf{p}_1, \ldots, \mathbf{p}_6$,

$$\mathbf{z} = \sum_1^5 (3x_j + \lambda y_j + y_6)\mathbf{p}_j + \left(-3\lambda x_6 - \lambda \sum_1^5 y_j + y_6\right)\mathbf{p}_6.$$

If this is the same as $\sum (A_j\omega + B_j\omega^2)\mathbf{p}_j$, we have, by 3.1,

$$A_j\omega + B_j\omega^2 = -(2\omega + \omega^2)y_j - (\omega + \omega^2)(3x_j + y_6) \qquad (j = 1, \ldots, 5),$$
$$A_6\omega + B_6\omega^2 = (2\omega + \omega^2)(3x_6 + y_1 + \ldots + y_5) - (\omega + \omega^2)y_6,$$

whence

$$A_j = -2y_j - 3x_j - y_6,$$
$$B_j = -y_j - 3x_j - y_6 \qquad (j = 1, \ldots, 5),$$
$$A_6 = 6x_6 + 2(y_1 + \ldots + y_5) - y_6,$$
$$B_6 = 3x_6 + y_1 + \ldots + y_5 - y_6,$$

and

$$x_j = \tfrac{1}{3}(A_j - 2B_j - A_6 + 2B_6),$$
$$y_j = -A_j + B_j \qquad (j = 1, \ldots, 5),$$
$$x_6 = \tfrac{1}{3}\sum_1^6 (A_k - B_k),$$
$$y_6 = A_6 - 2B_6.$$

These are rational integers whenever A_k and B_k satisfy 3.3. Thus 3.4 is proved.

The corresponding quadratic form, being the norm of $\mathbf{z}$, is

$$\sum (3x_j + \lambda y_j + y_6)(3x_j + \bar{\lambda} y_j + y_6)$$
$$+ (3\lambda x_6 + \lambda \sum y_j - y_6)(3\bar{\lambda} x_6 + \bar{\lambda} \sum y_j - y_6),$$

https://doi.org/10.4153/CJM-1953-043-4 Published online by Cambridge University Press

where the $\sum$ indicates summation from 1 to 5. Since $\lambda + \bar{\lambda} = \lambda\bar{\lambda} = 3$, this is equal to

$$9\sum x_j^2 + 9\sum x_j y_j + 6\sum x_j y_6 + 3\sum y_j^2 + 3\sum y_j y_6 + 5y_6^2 + 27x_6^2$$
$$+ 18x_6\sum y_j - 9x_6 y_6 + 3(\sum y_j)^2 - 3\sum y_j y_6 + y_6^2$$
$$= 9\sum x_j^2 + 9\sum x_j y_j + 6\sum x_j y_6 + 27x_6^2 + 18\sum y_j x_6 - 9x_6 y_6 + 3\sum y_{j2}$$
$$+ 3(\sum y_j)^2 + 6y_6^2$$
$$= 9\sum (x_j + \tfrac{1}{2}y_j + \tfrac{1}{3}y_6)^2 + 27(x_6 + \tfrac{1}{3}\sum y_j - \tfrac{1}{6}y_6)^2 + \tfrac{3}{4}\sum y_j^2 + \tfrac{1}{4}y_6^2$$
$$= \tfrac{1}{4}\{\sum (6x_j + 3y_j + 2y_6)^2 + 3(6x_6 + 2\sum y_j - y_6)^2 + 3\sum y_j^2 + y_6^2\}$$
$$= 3K_{12}.$$

The form K_{12} has integral coefficients, and represents 2 but not 1; therefore $M = 2$. The determinant is

$$\Delta = \frac{36^5\ 108\ .\ 3^5}{12^{12}} = \left(\frac{3}{4}\right)^6 = \frac{729}{4096}.$$

4. Proof that the new form is eutactic. We know [**1**, pp. 397, 401] that a positive definite form is extreme if it is both eutactic and perfect (in the sense of Voronoï [**10**, p. 100]), and that it is eutactic if its minimal vectors constitute a eutactic star.

In the case of our form $3K_{12}$, where $M = 6$, the minimal vectors go from the origin to the lattice points at distance $\sqrt{6}$. In the notation of Shephard [**7**, p. 96], these 756 points are easily seen to consist of the 486 points

4.1 $$\pm (\omega^{m_1}, \ldots, \omega^{m_6}), \qquad m_1 + \ldots + m_6 \equiv 0 \pmod 3$$

and the 270 points

4.2 $$({}_3\lambda, -{}_3\lambda, 0, 0, 0, 0)',$$

where the pre-subscript 3 indicates multiplication by any power of ω, and the prime indicates all possible permutations.

From the symmetrical nature of the coordinates, we see at once that these minimal vectors

$$\pm (\mu_{1l}, \ldots, \mu_{6l})$$

satisfy the criterion 2.1 for a eutactic star in the unitary 6-space. By 2.2, the same 756 vectors form a eutactic star when regarded as belonging to the real 12-space. Hence

4.3 *The form K_{12} is eutactic.*

5. Proof that the new form is extreme. We know [**1**, p. 400] that a positive definite form is perfect if its minimal vectors (in the Euclidean space) do not lie on a quadric cone (with the origin as vertex). In the case of K_{12}, since we can show that the 135 lines joining the origin to the points 4.2 do not lie on such a cone, there is no need to examine 4.1.

The six points $({}_3\lambda)$ and $(-{}_3\lambda)$ in the unitary 1-space (or complex line) correspond to the vertices of two equilateral triangles in the Euclidean plane (or Argand diagram). These are similar to the triangles

$$(-1, 0, 1) \quad (0, 1, -1) \quad (1, -1, 0) \quad \text{and} \quad (1, 0, -1) \quad (0, -1, 1) \quad (-1, 1, 0)$$

in the plane $\zeta_1 + \zeta_2 + \zeta_3 = 0$ of a Euclidean 3-space. Hence the figure formed by the 270 points 4.2 is similar to

5.1 $$((-1, 0, 1)^\circ, (1, 0, -1)^\circ, (0, 0, 0), (0, 0, 0), (0, 0, 0), (0, 0, 0))'$$

in the 12-dimensional subspace

5.2 $$x_1 + x_2 + x_3 = x_4 + x_5 + x_6 = \ldots = x_{16} + x_{17} + x_{18} = 0$$

of a Euclidean 18-space. (The "degree" sign in 5.1 indicates cyclic permutation of the three numbers within the parantheses, and the prime indicates all possible permutations of the six triads of coordinates.)

Any 11-dimensional quadric cone (with the origin as vertex) containing the 270 points 5.1, could be regarded as a section of a 17-dimensional cone, say

5.3 $$\sum_1^{18} \sum_1^{18} b_{jk} \zeta_j \zeta_k = 0 \qquad (b_{jk} = b_{kj}).$$

Direct substitution yields 135 equations for the b_{jk}, such as

$$- e_{13} - e_{14} + e_{16} + e_{34} - e_{36} - e_{46} = 0,$$

where

$$e_{jk} = 2b_{jk} - b_{jj} - b_{kk}.$$

Adding the three equations

$$e_{13} + e_{14} - e_{16} - e_{34} + e_{36} + e_{64} = 0,$$
$$e_{13} + e_{15} - e_{14} - e_{35} + e_{34} + e_{45} = 0,$$
$$e_{13} + e_{16} - e_{15} - e_{36} + e_{35} + e_{56} = 0,$$

we obtain

$$3e_{13} + (e_{56} + e_{64} + e_{45}) = 0.$$

Since 123 and 456 could just as well have been any other two of the six triads, we deduce that all the e_{jk} are equal, say $e_{jk} = c$. Any one of the equations now yields $c = 0$, whence

$$2b_{jk} = b_{jj} + b_{kk},$$

and

$$\sum \sum b_{jk} \zeta_j \zeta_k = \tfrac{1}{2} \sum \sum (b_{jj} + b_{kk}) \zeta_j \zeta_k$$
$$= \sum \sum b_{kk} \zeta_i \zeta_k = \sum \zeta_j \sum b_{kk} \zeta_k.$$

It follows that *every* quadric cone 5.3 containing the 270 points consists of two 17-spaces, one of which is $\sum \zeta_j = 0$. Hence there is no such cone in the 12-space 5.2, and we deduce

5.4 *The form K_{12} is perfect.*

Combining this result with 4.3, we have

5.5 *The form K_{12} is extreme.*

6. The reciprocal form. When the real and imaginary parts of the coordinates $\zeta_j = \xi_j + \eta_j i$ are interpreted as coordinates in real $2n$-space, the scalar product

$$\sum \xi_j \xi'_j + \sum \eta_j \eta'_j$$

of two Euclidean vectors is the real part of

$$\sum (\xi_j - \eta_j i)(\xi'_j + \eta'_j i).$$

Accordingly, if we define the scalar product of two complex vectors

$$\mathbf{z} = (\zeta_1, \ldots, \zeta_n), \quad \mathbf{z}' = (\zeta'_1, \ldots, \zeta'_n)$$

to be

6.1 $$[\mathbf{z} \cdot \mathbf{z}'] = \sum \bar{\zeta}_j \zeta'_j$$

[**11**, p. 15 (24); cf. **12**, p. 16], we see that a necessary and sufficient condition for the corresponding Euclidean vectors to be orthogonal is that the real part of this scalar product should vanish.

It is easily verified that the twelve basic vectors 3.4 satisfy

$$\Re[i\mathbf{t}_j \cdot \mathbf{t}_k] = 0 \qquad (j = 1, \ldots, 12; \quad k \neq j \pm 6),$$
$$\Re[i\mathbf{t}_j \cdot \mathbf{t}_{j+6}] = -\tfrac{3}{2}\sqrt{3} \qquad (j = 1, \ldots, 6),$$
$$\Re[i\mathbf{t}_j \cdot \mathbf{t}_{j-6}] = \tfrac{3}{2}\sqrt{3} \qquad (j = 7, \ldots, 12).$$

Let L' denote the lattice generated by the twelve vectors

$$\mathbf{t}^1 = i\mathbf{t}_7, \ \ldots, \ \mathbf{t}^6 = i\mathbf{t}_{12},$$
$$\mathbf{t}^7 = -i\mathbf{t}_1, \ \ldots, \ \mathbf{t}^{12} = -i\mathbf{t}_6.$$

Then, since

$$\Re[\mathbf{t}^j \cdot \mathbf{t}_k] = \tfrac{3}{2}\sqrt{3}\, \delta^j_k \qquad (j, k = 1, \ldots, 12),$$

the lattices in Euclidean 12-space corresponding to L and L' are reciprocal [**1**, p. 399].

Hence the reciprocal (or "adjoint") form is a numerical multiple of the norm of

$$\sum_1^6 x_j \mathbf{t}^j + \sum_1^6 y_j \mathbf{t}^{j+6}.$$

This is derived from $3K_{12}$ itself by changing

$$x_1, \ldots, \quad x_6, \quad y_1, \ldots, y_6$$

into

$$-y_1, \ldots, -y_6, \quad x_1, \ldots, x_6.$$

Hence

6.2 *The form K_{12} is equivalent to its own reciprocal.*

In this respect, K_{12} resembles C_n, A_2, D_4, $A^r_{r^2-1}$ (which is extreme when $r \geqslant 3$) and D^2_{2r} (which is extreme when $r \geqslant 4$) [**1**, pp. 406, 423, 431, 434]. (A^3_8 and D^2_8 are absolutely extreme, both being equivalent to E_8.)

7. Remarks on the lattice. As we have observed before, the form K_{12} does not represent 1. The solutions of the Diophantine equation

$$K_{12} = N \qquad (N = 2, 3, 4, \ldots),$$

or $3K_{12} = 3N$, are represented geometrically by the points of the lattice L at distance $(3N)^{\frac{1}{2}}$ from the origin.

When $N = 2$, we find the 756 points 4.1, 4.2, which are the vertices of the complex uniform polytope $(2\,1;\, 3_3)^3$ of Shephard [**7a**, p. 380]. They lie by sixes on 126 lines through the origin, and the hyperplane at infinity meets these lines in 126 points whose homogeneous coordinates (in complex projective 5-space) are

$$(\omega^{m_1}, \ldots, \omega^{m_6}) \qquad (m_1 + \ldots + m_6 \equiv 0 \pmod 3)$$

and

$$(1, -\omega^m, 0, 0, 0, 0)'.$$

These 126 points are the centres of the homologies in Mitchell's primitive collineation group [**6**; **3**; **8**; **9**; see especially **2**, p. 402]. The simplex of reference is one of the so-called α-hexahedra [**2**, p. 407].

When $N = 3$, we find the 4032 points

$$_6(3, 0, 0, 0, 0, 0)', \qquad \pm\lambda(\omega^{l_1}, \omega^{l_2}, \omega^{l_3}, 0, 0, 0)'$$

and

$$_6(-2, \omega^{m_1}, \ldots, \omega^{m_5})' \qquad (m_1 + \ldots + m_5 \equiv 0 \pmod 3)).$$

which are the vertices of Shephard's polytope $(2_2\,1;\, 3)^3$. The corresponding

https://doi.org/10.4153/CJM-1953-043-4 Published online by Cambridge University Press

configuration at infinity consists of $6 + 180 + 486 = 672$ points, which are the vertices of the 112 α-hexahedra.

When $N = 4$, we find the 20412 points

$$\lambda(\omega^{l_1}, \omega^{l_2}, -\omega^{l_3}, -\omega^{l_4}, 0, 0)',$$

$$\pm(-2\omega^{m_1}, -2\omega^{m_2}, \omega^{m_3}, \ldots, \omega^{m_6})' \quad (m_1 + \ldots + m_6 \equiv 0 \pmod 3),$$

$$_6(2 - \omega^{n_0}, \omega^{n_1}, \ldots, \omega^{n_5})' \quad (n_0 = 1 \text{ or } 2, n_1 + \ldots + n_5 \equiv n_0 \pmod 3)).$$

The configuration at infinity consists of $1215 + 1215 + 972 = 3402$ points, which are the vertices of the 567 β-hexahedra [**2**, p. 408].

References

1. H. S. M. Coxeter, *Extreme forms*, Can. J. Math., *3* (1951), 391–441.

2. C. M. Hamill, *On a finite group of order* 6,531,840, Proc. London Math. Soc. (2), *52* (1951), 401–454.

3. E. M. Hartley, *A sextic primal in five dimensions*, Proc. Cambridge Phil. Soc., *46* (1950), 91–105.

4. Chao Ko, *On the positive definite quadratic forms with determinant unity*, Acta Arithmetica, *3* (1939), 79–85.

5. A. Korkine and G. Zolotareff, *Sur les formes quadratiques*, Math. Ann., *6* (1873), 366–389.

6. H. H. Mitchell, *Determination of all primitive collineation groups in more than four variables which contain homologies*, Amer. J. Math., *36* (1914), 1–12.

7. G. C. Shephard, *Regular complex polytopes*, Proc. London Math. Soc. (3), *2* (1952), 82–97.

7a. ———, *Unitary groups generated by reflections*, Can. J. Math., *5* (1953), 364–383.

8. J. A. Todd, *The invariants of a finite collineation group in five dimensions*, Proc. Cambridge Phil. Soc., *46* (1950), 73–90.

9. ———, *The characters of a collineation group in five dimensions*, Proc. Royal Soc. London, A, *200* (1950), 320–336.

10. G. Voronï, *Sur quelques propriétés des formes quadratiques positives parfaites*, J. reine angew. Math., *133* (1907), 97–178.

11. H. Weyl, *Gruppentheorie und Quantenmechanik* (Leipzig, 1928).

12. ———, *The theory of groups and quantum mechanics* (New York, 1931).

University of Toronto
University of Cambridge