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STABILITY OF LINEAR NEUTRAL DELAY-DIFFERENTIAL SYSTEMS

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Sufficient conditions are obtained for the stability of linear neutral delay-differential systems by using a delay-differential inequality.

1. INTRODUCTION

This paper extends the result [2] for linear delay-differential systems to the neutral case:

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) + C\dot{x}(t-\tau) \qquad (t \geq 0) \tag{1.1}$$

where, $x(t) = (x,(t), x_2(t), \ldots, x_n(t))^T$, A, B and C denote real constant $n \times n$ matrices with elements $a_{ij}.b_{ij}.c_{ij}(ij = 1, 2, \ldots, n)$ respectively, and $\tau > 0$ is a constant. We adopt the following norms for vectors $x = (x_1, x_2, \ldots, x_n)^T$ and matrices $A = (a_{ij})_{n\times n}$ respectively:

$$\|x\|_1 = \max_i |x_i|, \qquad \|x\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}, \qquad \|x\|_\infty = \sum_{i=1}^n |x_i|,$$

$$\|A\|_1 = \max_j \sum_{i=1}^n |a_ij|, \qquad \|A\|_2 = \{\lambda \max[A^T A]\}^{\frac{1}{2}}, \qquad \|A\|_\infty = \max_i \sum_{j=1}^n |a_ij|.$$

The measure $\mu(A)$ of a matrix A is defined by

$$\mu(A) = \lim_{h\to 0+} \frac{\|I + hA\| - 1}{h},$$

which for the corresponding norms reduces to

$$\mu_1(A) = \max_j [a_{ij} + \sum_{\substack{i=1 \\ i\neq j}}^n |a_{ij}|],$$

$$\mu(A) = \frac{1}{2}\lambda_{\max}[A^T + A],$$

$$\mu_\infty(A) = \max_i [a_{ii} + \sum_{\substack{j=1 \\ j\neq i}}^n |a_{ij}|],$$

where A^T denotes the transposed of A, $\lambda_{\max}[B]$ denotes the largest eignevalue of B, and I denotes the unit matrix.

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2. A Delay-Differential Inequality

A delay-differential inequality was discussed in [1] and [4]. Here an extension is given

THEOREM 2.1. *Let* $p_i(t) \geqslant 0$ $(i = 1, 2)$ *be continuously differentiable functions on* $[-\tau, +\infty)$ *which satisfy*

$$\text{(2.1)} \qquad \begin{cases} \dot{P}_1(t) \leqslant a_{11}p_1(t) + a_{12}p_2(t) + b_{11}\widetilde{p_1(t)} + b_{12}\widetilde{p_2(t)} \\ \qquad 0 \leqslant a_{21}p_1(t) + a_{22}p_2(t) + b_{21}\widetilde{p_1(t)} + b_{22}\widetilde{p_2(t)} \end{cases} (t \geqslant 0)$$

where, $a_{ij} \geqslant 0 (i \neq j)$, $b_{ij} \geqslant 0$, $\widetilde{p_i(t)} = \sup_{t-\tau \leqslant \theta \leqslant t} p_i(\theta)$ $(i, j = 1, 2)$. *If* $a_{ii} + b_{ii} < 0$ *and the real parts of all eigenvalues of the matrix* $(a_{ij} + b_{ij})_{2\times 2}$ *are negative. Then there exist constants* $M \geqslant 1$, $\alpha > 0$ *such that:*

$$\text{(2.2)} \qquad p_i(t) \leqslant M\left(\sum_{j=1}^{2} \widetilde{p_j(0)}\right) e^{\alpha t} \quad t \in [-\tau, \infty).$$

PROOF: From the properties of a stable Metzler - Matrix [3], there exist constants $\alpha_i > 0$ $(i = 1, 2)$ such that:

$$(a_{11} + b_{11})\alpha_1 + (a_{12} + b_{12})\alpha_2 < 0,$$
$$(a_{21} + b_{21})\alpha_1 + (a_{22} + b_{22})\alpha_2 < 0.$$

We choose two constants, $\alpha > 0$ and $k > 0$, such that

$$\text{(23)} \qquad \begin{aligned} \alpha\alpha_1 + a_{11}\alpha_1 + b_{11}\alpha_1 e^{\alpha\tau} + a_{12}\alpha_2 + b_{12}\alpha_2 e^{\alpha\tau} < 0, \\ a_{21}\alpha_1 + b_{21}\alpha_1 e^{\alpha\tau} + a_{22}\alpha_2 + b_{22}\alpha_2 e^{\alpha\tau} < 0, \end{aligned}$$

and

$$k\alpha_i e^{-\alpha\tau} > 1 \qquad (i = 1, 2).$$

For a sufficiently small real numble $\varepsilon > 0$, we define

$$w_i(t) = k\alpha_i\left(\sum_{j=1}^{2} \widetilde{p_j(0)} + \varepsilon\right) e^{-\alpha t} \qquad (i = 1, 2,\ t \geqslant -\tau).$$

It is easy to check that $p_i(t) < w_i(t)$ $(i = 1, 2,\ t \in [-\tau, 0])$. We want to prove that:

$$p_i(t) < w_i(t) \qquad (i = 1, 2,\ t \in [0, +\infty)).$$

If this inequality did not hold, then one of the following two situations would occur:

1. There exists a $t_1 > 0$, such that

$$p_1(t) = w_1(t_1) \text{ and } p_i(t) \leqslant w_i(t) \qquad (i = 1, 2, -\tau \leqslant t \leqslant t_1);$$

furthermore, $\dot{p}_1(t_1) \geqslant \dot{w}_1(t_1)$. Also, from (2.3), we have

$$\begin{aligned}\dot{p}_1(t_1) &\leqslant a_{11}w_1(t_1) + a_{12}w_2(t_1) + b_{11} \sup_{t_1-\tau\leqslant\theta\leqslant t_1} w_1(\theta) + b_{12} \sup_{t_1-\tau\leqslant\theta\leqslant t_1} w_2(\theta),\\ &= a_{11}w_1(t_1) + a_{12}w_2(t_1) + b_{11}w_1(t_1-\tau) + b_{12}w_2(t_1-\tau),\end{aligned}$$

so

$$\begin{aligned}\dot{w}_1(t_1) &= -k\alpha_1\alpha\left(\sum_{j=1}^{2}\widetilde{p_j(0)} + \varepsilon\right)e^{-\alpha t_1},\\ &> k(a_{11}\alpha_1 + b_{11}\alpha_1 e^{\alpha\tau} + a_{12}\alpha_2 + b_{12}\alpha_2 e^{\alpha\tau})\left(\sum_{j=1}^{2}\widetilde{p_j(0)} + \varepsilon\right)e^{-\alpha t_1},\\ &= a_{11}w_1(t_1) + a_{12}w_2(t_1) + b_{11}w_1(t_1-\tau) + b_{12}w_2(t_1-\tau).\end{aligned}$$

That is, $\dot{w}_1(t_1) > \dot{p}_1(t_1)$. This contradicts $\dot{p}_1(t_1) \geqslant \dot{w}_1(t_1)$.

2. There exists a $t_1 > 0$, such that

$$p_2(t_1) = w_2(t_1) \text{ and } p_i(t) \leqslant w_i(t) \qquad (i = 1, 2, -\tau \leqslant t \leqslant t_1).$$

Then from (2.3), we obtain

$$\begin{aligned}-a_{22}p_2(t_1) &\leqslant a_{21}w_1(t_1) + b_{21}w_1(t_1-\tau) + b_{22}w_2(t_1-\tau)\\ &= k\left(\sum_{j=1}^{2}\widetilde{p_j(0)} + \varepsilon\right)e^{-\alpha t_1}(a_{21}\alpha_1 + b_{21}\alpha_1 e^{\alpha\tau} + b_{22}\alpha_2 e^{\alpha\tau}),\\ &< k\left(\sum_{j=1}^{2}p_j(0) + \varepsilon\right)e^{-\alpha t}(-a_{22}\alpha_2),\\ &= -a_{22}w_2(t_1).\end{aligned}$$

Since $-a_{22} > 0$, we have $p_2(t_1) < w_2(t_1)$. This contradicts $p_2(t_1) = w_2(t_1)$.

Hence, we see that

$$p_i(t) < w_i(t) \qquad (i = 1, 2, t \geqslant 0).$$

Let $M = k(\alpha_1 + \alpha_2)$ and $\varepsilon \to 0^+$. Then (2.2) is satisfied and the proof is complete. ∎

Theorem 2.1 can be generalised immediately to the vector case:

THEOREM 2.2. *Let $p_i(t) \geqslant 0$ $(i = 1, 2, \dots, 2n)$ be continuously differentiable functions on $[-\tau, +\infty)$ such that $p^*(t) = (p_1(t), \dots, p_n(t), 0 \dots 0)^T$ and $p(t) = (p_1(t), \dots, p_n(t), p_{n+1}(t) \dots p_{2n}(t))^T$ satisfies*

$$p^*(t) \leqslant Ap(t) + B\widetilde{p(t)}$$

where, $\widetilde{p(t)} = \left(\sup_{t-\tau \leqslant \theta \leqslant t} p_1(\theta), \dots, \sup_{t-\tau \leqslant \theta \leqslant t} p_{2n}(t) \right)^T$, *and* $A = (a_{ij})_{2n \times 2n}$, $B = (b_{ij})_{2n \times 2n}$, *with* $a_{ij} \geqslant 0 (i \neq j) b_{ij} \geqslant 0$ $(i, j = 1, 2, \dots, 2n)$. *If* $a_{ii} + b_{ii} < 0$ *and* $Re\lambda(A + B) < 0$, *then there exist constants* $M \geqslant 1$, $\alpha > 0$ *such that:*

$$p_i(t) \leqslant M \left(\sum_{j=1}^{2n} \sup_{-\tau \leqslant \theta \leqslant 0} p_j(\theta) \right) e^{-\alpha t}, t \in [-\tau, +\infty) \quad i = 1, 2, \dots, 2n \tag{2.4}$$

3. A STABILITY THEOREM

By $C^{(1)}[-\tau, 0]$, we mean the Banach space of all functions $u(t)$ (u an n-vector) which are continuously differentiable on $[-\tau, 0]$ with norm:

$$\|u\|_\tau = \sup_{-\tau \leqslant \theta \leqslant 0} \|u(\theta)\| + \sup_{-\tau \leqslant \theta \leqslant 0} \|\dot{u}(\theta)\|$$

Consider the neutral delay-differential system

$$\dot{x}(t) = Ax(t) + Bx(t - \tau) + C\dot{x}(t - \tau). \tag{3.1}$$

Let us define the stability of the solution $x = 0$ for (3.1) as follows. Suppose that ϕ is a given continuously differentiable function on $[-\tau, 0]$ (that is $\phi \in C^{(1)}[-\tau, 0]$) and that $x(t) = x(t, \phi)$ on $[-\tau, +\infty)$ denotes the unique solution of (3.1) with $x(t) = \phi(t)$, $\dot{x}(t) = \dot{\phi}(t)$ for $t \in [-\tau, 0]$.

DEFINITION 1: The solution $x = 0$ of (3.1) is *stable* in $C^{(1)}[-\tau, 0]$ if for each $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $\|\phi\|_\tau < \delta$ implies that the solution $x(t, \phi)$ of (3.1) satisfies

$$\|x(t, \phi)\| + \|\dot{x}(t, \phi)\| < \varepsilon \qquad t \in [-\tau, +\infty).$$

DEFINITION 2: The solution $x = 0$ of (3.1) is *asymptotically stable* in $C^{(1)}[-\tau, 0]$ if it is stable in $C^{(1)}[-\tau, 0]$ and there exists $b_0 > 0$ such that $\|\phi\|_\tau \leqslant b_0$ implies

$$\lim_{t \to +\infty} (\|x(t), \phi\| + \|\dot{x}(t, \phi)\|) = 0.$$

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Theorem 3.1. *Suppose the coefficient matrices A, B, C of (3.1) satisfy the following*

$$\|c\| < 1 \text{ and } \mu(A) + \frac{\|B\| + \|A\| \cdot \|C\|}{1 - \|C\|} < 0. \tag{3.2}$$

Then the solution $x = 0$ of the system, (3.1) is asymptotically stable in $c^{(1)}[-\tau, 0]$ and there exist $M \geqslant 1$, $\alpha > 0$ such that

$$\|x(t, \phi)\| + \|\dot{x}(t, \phi)\| \leqslant 2M \, \|\phi\|_\tau \, e^{-\alpha t}$$

for every solution $x(t, \phi)$ of (3.1) with $x(t) = \phi(t)$, $\dot{x}(t) = \dot{\phi}(t)$ on $t \in [-\tau, 0]$.

Proof: From the definition of the measure $\mu(A)$ we have for $t \in [0, +\infty)$,

$$\begin{aligned} \frac{d\,\|x(t)\|}{dt} - \mu(A)\,\|x(t)\| &= \lim_{h \to 0^+} \frac{1}{h}[\|x(t+h)\| - \|(I + hA)\| \cdot \|x(t)\|] \\ &\leqslant \lim_{h \to 0^+} \frac{1}{h}[\|x(t+h) - (I + hA)X(t)\|] \\ &\leqslant \|B\|\,\|x(t-\tau)\| + \|C\| \cdot \|\dot{x}(t-\tau)\|, \end{aligned}$$

that is

$$\frac{d\,\|x(t)\|}{dt} \leqslant \mu(A)\,\|x(t)\| + \|B\| \sup_{t-\tau \leqslant \theta \leqslant t} \|x(\theta)\| + \|C\| \sup_{t-\tau \leqslant \theta \leqslant t} \|\dot{x}(\theta)\|. \tag{3.3}$$

From (3.1), we have directly

$$0 \leqslant \|\dot{x}(t)\| + \|A\|\,\|x(t)\| + \|B\| \sup_{t-\tau \leqslant \theta \leqslant t} \|x(\theta)\| + \|C\| \sup_{t-\tau \leqslant \theta \leqslant t} \|\dot{x}(\theta)\|. \tag{3.4}$$

Consider the functions $p_1(t)$, $p_2(t)$ in Theorem 2.1 defined by

$$p_1(t) = \|x(t)\|, \; p_2(t) = \|\dot{x}(t)\| \qquad t \in [-\tau, +\infty).$$

It follows from (3.3) and (3.4) that

$$\begin{cases} \dot{p}_1(t) \leqslant \mu(A) p_1(t) + \|B\|\,\widetilde{p_1(t)} + \|C\|\,\widetilde{p_2(t)}, \\ \quad 0 \leqslant \|A\|\,p_1(t) - p_2(t) + \|B\|\,\widetilde{p_1(t)} + \|C\|\,\widetilde{p_2(t)}. \end{cases}$$

We know from hypothesis (3.2) that the real parts of all eigenvalues of the matrix

$$\begin{pmatrix} \mu(A) + \|B\| & \|C\| \\ \|A\| + \|B\| & -1 + \|C\| \end{pmatrix}_{2\times 2}$$

are negative and $\mu(A)+\|B\|<0$, $-1+\|C\|<0$. Therefore, there exist $M \geqslant 1$, $\alpha > 0$ such that

$$p_i(t) \leqslant M\left(\sum_{j=1}^{2} \widetilde{p_j(0)}\right)e^{-\alpha t} \qquad t \geqslant -\tau.$$

By Theorem 2.1, we have

$$\|x(t,\,\phi)\| + \|\dot{x}(t,\,\phi)\| \leqslant 2M\,\|\phi\|_\tau\, e^{-\alpha t}, \quad t \geqslant -\tau$$

and the proof is complete. ∎

Consider the following system in component form

$$\dot{x}_i(t) = a_{ii}x_i(t) + \sum_{\substack{j\neq i\\ j=1}}^{n} b_{ij}x_j(t-\tau) + \sum_{j=1}^{n} C_{ij}\dot{x}_j(t-\tau) \tag{3.5}$$

where, a_{ii}, $b_{ij}(i \neq j)$, C_{ij} $(i = 1,2,\ldots n)$ are constants and $a_{ii} < 0$ $(i = 1,2,\ldots,n)$. Imitating the proof of Theorem 3.1 by using the delay-differential inequality in vector form (Theorem 2.2), we can easily obtain the following theorem.

THEOREM 3.2. *Suppose that the coefficients of (3.5) satisfy the following*

$$a_{ii} + \sum_{\substack{j\neq i\\ j=1}}^{n} |b_{ij}| + \sum_{j=1}^{n} |C_{ij}| < 0 \text{ and } |a_{ij}| + \sum_{\substack{j\neq i\\ j=1}}^{n} |b_{ij}| + \sum_{j=1}^{n} |C_{ij}| < 1$$

$(i = 1,2,\ldots,n)$. *Then the solution* $x_i = 0$ *of (3.5) is asymptotically stable in* $C^{(1)}[-\tau,\,0]$.

Remark. Using Theorem 2.2, we can discuss the stability of the trivial solution for more complex neutral delay-differential systems.

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