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HÖLDER CONDITIONS AND THE TOPOLOGY OF SIMPLY CONNECTED DOMAINS*

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ABSTRACT. Let f be regular univalent and normalized in the unit disc U (i.e. $f \in S$) and continuous on $U \cup T$, where T denotes the boundary of U.

Recently Essén proved [5] a conjecture of Piranian [7] stating that if the derivative of $f \in S$ is bounded in U and $f(z_1) = f(z_2) = \cdots = f(z_n)$ for $z_j \in T$, $1 \le j \le n$, then $n \le 2$. In fact, Essén proved a more general result, using a deep result on harmonic functions. The aim of the following article is to replace Essén's proof by a completely different proof which is based only on Goluzin's inequalities and is much more elementary.

Let U be the unit disc and T its boundary. Let S denote the class of regular univalent functions f in U such that $f(0) = 0$ and $f'(0) = 1$.

THEOREM. *Let $f \in S$ be continuous in $U \cup T$, and suppose that for some $n \ge 2$ there exist n distinct points z_j $(j = 1, 2, \ldots, n)$ on T and a complex number λ such that $f(z_j) = \lambda$ for $j = 1, 2, \ldots, n$. Then, for each r $(0 < r < 1)$*

$$\sup_{1 \le j \le n} |f(rz_j) - f(z_j)| > C(1-r)^{2/n}, \tag{1}$$

and for some sequence $\{r_p\}$, $r_p \to 1^-$,

$$\sup_{|z| = r_p} |f'(z)| > C_1(1-r_p)^{2/n-1}, \tag{2}$$

where C and C_1 are positive constants depending only on n and the distribution of $\{z_j\}_{j=1}^n$ on the boundary T.

M. Essén [5] established a slightly weaker form of this result, and he thus proved a conjecture of Piranian [7].

In our proof, which is completely different, we use Goluzin's inequalities (see [6, pp. 119–126] and Pommerenke [8, pp. 346–347]).

We also refer the reader to [3], where a deeper connection between Essén's approach and ours led to a striking new proof of Denjoy's conjecture; the new

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proof makes no use of harmonic measure. Further results along this line will be given in [4]. We would like also to mention [1] and [2], where the authors investigated conditions satisfied by extremal functions for minimal-area problems with side conditions. It turns out that these extremal functions satisfy certain growth conditions (such as boundness of the derivative in U). In fact, Piranian made his conjecture in connection with these investigations.

Proof of the Theorem. From Goluzin inequalities one gets ([6, p. 119] and [8, p. 346]) for $0<r<1$:

$$\prod_{\substack{j=1\\ j\neq k}}^{n}\prod_{k=1}^{n}\left|\frac{f(rz_j)-f(rz_k)}{(rz_j)-(rz_k)}\right|\prod_{j=1}^{n}\prod_{k=1}^{n}\left|\frac{(rz_j)(rz_k)}{f(rz_j)f(rz_k)}\right|\geq\prod_{j=1}^{n}\prod_{k=1}^{n}|1-(rz_j)(r\bar{z}_k)| \tag{3}$$

From this we get:

$$\left(\sup_{1\leq j\leq n}|f'(rz_j)|\right)^n\sup_{j,k}\left|\underset{j\neq k}{f(rz_j)-f(rz_k)}\right|^{n^2-n}>C_0(1-r)^n \tag{4}$$

for some positive constant C_0, which may depend on the location of $\{z_j\}_{j=1}^n$ and n, but not on r. Indeed, to get (4) from (3) we use the $\frac{1}{4}$-theorem for $f(z)/z$, (i.e. $|f(z)/z|\geq\frac{1}{4}$, $z\in U$) and the (easily proved) existence of positive lower bounds of the products

$$\prod_{j=1}^{n}\prod_{k=1}^{n}\left|\underset{j\neq k}{1-(rz_j)(r\bar{z}_k)}\right|,\qquad \prod_{j=1}^{n}\prod_{k=1}^{n}\left|\underset{j\neq k}{(rz_j)-(rz_k)}\right|,$$

provided r is bounded away from zero. (Note that for each fixed disc $|z|\leq r_0$ $(r_0<1)$, (1) and (2) are obviously true, and that it is therefore sufficient to consider values of r in an interval $0<r_0\leq r\leq 1$.)

We further use the well-known fact

$$|f'(rz_j)|\leq\frac{4}{1-r^2}|f(z_j)-f(rz_j)|,\qquad 1\leq j\leq n. \tag{5}$$

(Indeed, (5) is proved at once by applying the $\frac{1}{4}$-theorem to the function

$$\frac{f\left(\dfrac{z+\zeta}{1+\bar{\zeta}z}\right)-f(\zeta)}{(1-|\zeta|^2)f'(\zeta)}$$

and putting $\zeta=rz_j$, $z+\zeta/1+\bar{\zeta}z=z_j$.)

The hypothesis $f(z_j)=f(z_k)$ implies that

$$|f(rz_j)-f(rz_k)|\leq|f(rz_j)-f(z_j)|+|f(z_k)-f(rz_k)|\leq 2\sup_{1\leq j\leq n}|f(rz_j)-f(z_j)|.$$

Combining this with (5), we get from (4):

$$\left(\frac{4}{1-r^2}\right)^n 2^{n^2-n}\sup_{1\leq j\leq n}|f(rz_j)-f(z_j)|^{(n^2-n)+n}>C_0(1-r)^n.$$

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But this leads at once to (1). Since (2) is an obvious consequence of (1), the theorem is proved.

It is very easy to show the sharpness of the theorem checking Keobe function and its transformations [5].

As a final remark we mention that similar results hold for functions univalent outside $\bar{U}$. The proof is almost identical.

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