ON THE SEMISIMPLICITY OF MODULAR GROUP ALGEBRAS. II

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Let G be a discrete group, let K be an algebraically closed field of characteristic $p > 0$ and let KG denote the group algebra of G over K. In a previous paper (**2**) I studied the Jacobson radical JKG of KG for groups G with big abelian subgroups or quotient groups. It is therefore natural to next consider metabelian groups, and I do this here. The main result is as follows.

THEOREM 1. *Let K be an algebraically closed field of characteristic p and let a group G have a normal abelian subgroup A with G/A abelian. Then $JKG \neq \{0\}$ if and only if G has an element g of order p such that the A-conjugacy class g^A is finite and such that the group $G/\mathcal{N}_G(g^A)$ is periodic.*

Note that since $\mathcal{N}_G(g^A) \supseteq A$ and G/A is abelian, we do in fact have $\mathcal{N}_G(g^A) \triangle G$.

The proof is basically simple but there are a number of technical details involved. We start with a series of lemmas.

LEMMA 2. *Let F be a field and let the polynomial ring $F[x]$ act on a vector space V. Let W be a finite-dimensional subspace of V. Suppose that S is an infinite set of positive integers and that for each $s \in S$ the sum*

$$W + x^s W + x^{2s} W + x^{3s} W + \ldots$$

is not direct. Then there exists non-zero $f(x) \in F[x]$ and non-zero $w \in W$ with $f(x)w = 0$.

Proof. We can assume that $V = F[x]W$. Since W is finite-dimensional, V is a finitely generated $F[x]$ module. Thus, since $F[x]$ is a principal ideal domain, there exists $v_1, v_2, \ldots, v_n \in V$ and $f_1(x), f_2(x), \ldots, f_n(x) \in F[x]$ such that every element $v \in V$ can be written as

$$v = g_1(v; x)v_1 + g_2(v; x)v_2 + \ldots + g_n(v; x)v_n,$$

and the polynomials $g_i(v; x)$ are determined modulo $f_i(x)$.

Since W is a finite-dimensional F-subspace, it follows that every element $v \in W$ can be written as in the above with $\deg g_i(v; x) \leqq t$ for all $i = 1, 2, \ldots, n$ and some fixed integer t. Choose $s \in S$ with $s > t$. The hypothesis then implies that there exists $w_0, w_1, \ldots, w_k \in W$, not all zero, with

$$w_0 + x^s w_1 + \ldots + x^{ks} w_k = 0.$$

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If we write w_i as above with $\deg g_j(w_i; x) < s$, then we have

$$0 = \sum_{i=0}^{k} \sum_{j=1}^{n} x^{is} g_j(w_i; x) v_j.$$

Thus, $f_j(x)$ divides $\sum_{i=0}^{k} x^{is} g_j(w_i; x)$.

Suppose that the ordering of $v_1, \ldots, v_n$ is so chosen that $f_1(x), \ldots, f_r(x)$ are not zero while the remaining $f_{r+1}(x), \ldots, f_n(x)$ are zero. Thus we see that for each $j = r + 1, \ldots, n$ we have

$$0 = \sum_{i=0}^{k} x^{is} g_j(w_i; x)$$

and since $\deg g_j(w_i; x) < s$, this yields $g_j(w_i; x) = 0$. Thus, if

$$f(x) = f_1(x) f_2(x) \ldots f_r(x),$$

then $f(x) \neq 0$ and $f(x) w_i = 0$ for all i. Since some w_i is non-zero, the result follows.

Let Z denote the ring of integers. A polynomial $f(x) \in Z[x]$ is primitive if the greatest common divisor of the coefficients of $f(x)$ is 1. By Gauss' lemma, a product of primitive polynomials is primitive.

LEMMA 3. *Let A be an additive abelian group acted upon by $Z[x]$ and let B be a finitely generated subgroup. Let S be an infinite set of positive integers and suppose that for each $s \in S$ the sum*

$$B + x^s B + x^{2s} B + x^{3s} B + \ldots$$

is not direct. Then there exists a non-zero element $b \in B$ and a primitive polynomial $f(x) \in Z[x]$ with $f(x) b = 0$.

Proof. For each prime p let $A_p = \{a \in A \mid pa = 0\}$ and let

$$A_0 = \{a \in A \mid ma = 0 \text{ for some } m \in Z, m \neq 0\}.$$

Thus A_p is a vector space over $\mathrm{GF}(p)$, A_0 is the torsion subgroup of A, and A/A_0 is a torsion-free Z module. Of course, each of the A_i is fully invariant, and hence $Z[x]$-invariant.

Set $B_p = B \cap A_p$ and $B_0 = B \cap A_0$. Since B is finitely generated, B_0 is finite say of order m and $B = B_0 + B_1$, where B_1 is a finitely generated torsion-free abelian group. Clearly $mB \subseteq B_1$. For each $i = 1$ or p with $p|m$, let S_i denote the set of $s \in S$ with the sum

$$B_i + x^s B_i + x^{2s} B_i + x^{3s} B_i + \ldots$$

not direct. We show that $\bigcup S_i = S$.

Let $s \in S$. Since

$$B + x^s B + x^{2s} B + x^{3s} B + \ldots$$

is not direct, there exists $b_0, b_1, \ldots, b_k \in B$ not all zero with

$$b_0 + x^s b_1 + x^{2s} b_2 + \ldots + x^{ks} b_k = 0.$$

If some b_j has infinite order, then $mb_j \neq 0$. Multiplying the above relation through by m and using $mB \subseteq B_1$ we see that $s \in S_1$. Now suppose that all b_i have finite order. There exist a prime $p|m$ and an integer n with $nb_j \neq 0$ for some j and $pnb_i = 0$ for all i. Multiplying the above relation through by n we see that $s \in S_p$.

Since $S = \bigcup S_i$ is infinite and the union is finite, it follows that S_i is infinite for some i. Suppose first that $i = p$ is a prime. Then $\mathrm{GF}(p)[x]$ acts on A_p. By Lemma 2 applied with $F = \mathrm{GF}(p)$, $V = A_p$, $W = B_p$, $S = S_p$, there exists a monic polynomial $\bar{f}(x) \in \mathrm{GF}(p)[x]$ and non-zero $b \in B_p$ with $\bar{f}(x)b = 0$. Let $f(x) \in Z[x]$ be monic with $\bar{f}(x) = f(x) \bmod p$. Then since $pb = 0$, we have $f(x)b = 0$. Since $f(x)$ is clearly primitive, the result follows in this case.

Now let S_1 be infinite and let Q denote the field of rationals. Then $Q[x]$ acts on $V = (A/A_0) \otimes_Z Q$. Let $W = (B/B_0) \otimes_Z Q$. Then W is a finite-dimensional subspace of V. By Lemma 2 applied to this situation with $S = S_1$ there exists non-zero $g(x) \in Q[x]$ and non-zero $w \in W$ with $g(x)w = 0$. We can clearly assume that $g(x) \in Z[x]$. Since B/B_0 is a torsion-free Z module, it is naturally contained in W and the quotient is periodic. Thus for some integer $n \neq 0$, $nw \in B/B_0$. Then $g(x)nw = 0$. Let $b_1 \in B_1$ with $b_1 + B_0 = nw$. Then $g(x)b_1 \in B_0$, and hence for some integer $k \neq 0$, $kg(x)b_1 = 0$. Write $kg(x) = cf(x)$, where $f(x)$ is a primitive polynomial and $c \in Z$. Then $f(x)cb_1 = 0$. Since $b_1 \in B_1$, $b_1 \neq 0$, we see that $b = cb_1 \neq 0$, and the result follows.

Lemma 4. *Let $f(x) \in Z[x]$ be a primitive polynomial and let $m \in Z$ with $m \neq 0$. If I is the ideal generated by m and $f(x)$, then $Z[x]/I$ is finite.*

Proof. We can assume that $m > 0$ and write $m = p_1p_2 \ldots p_r$ as a product of r not necessarily distinct primes. Let $d = \deg f(x)$. We will show that $|Z[x]/I| \leqq md$.

If p is a prime, then since $f(x)$ is primitive, it follows that

$$\bar{f}(x) = f(x) \bmod pZ[x]$$

is not zero. Since Z/pZ is a field, this implies that for any polynomial $g(x) \in Z[x]$ there exist $h(x), k(x) \in Z[x]$ with

$$g(x) \equiv h(x) + pk(x) \quad \bmod f(x)Z[x]$$

and $\deg h(x) < d$.

Let $g(x) \in Z[x]$. We set $k_0(x) = g(x)$ and we define $h_i(x), k_i(x) \in Z[x]$ for $i = 1, 2, \ldots, r$ inductively by

$$k_{i-1}(x) \equiv h_i(x) + p_ik_i(x) \quad \bmod f(x)Z[x]$$

with $\deg h_i(x) < d$. Setting

$$h(x) = h_1(x) + p_1h_2(x) + p_1p_2h_3(x) + \ldots + p_1p_2 \ldots p_{r-1}h_r(x),$$

we then have

$$g(x) \equiv h(x) + mk_r(x) \quad \bmod f(x)Z[x]$$

and deg $h(x) < d$. Thus

$$g(x) \equiv h(x) \mod I.$$

Since we may further restrict the coefficients of $h(x)$ to be between 0 and $m - 1$, we see that $|Z[x]/I| \leqq md$. This completes the proof.

If $\alpha \in KG$, then the supporting subgroup of α is the minimal subgroup H of G with $\alpha \in KH$. Clearly, H is the subgroup generated by all elements $g \in G$ which occur with non-zero coefficient in α. We let $\alpha(1)$ denote the coefficient of the identity in α.

LEMMA 5. *Let Q be a finite normal abelian p'-subgroup of a group E and let K be an algebraically closed field of characteristic p.*

(i) *Let e be a primitive idempotent of KQ and let $\alpha \in KE$. If $\alpha = e\alpha e$, then $\alpha \in KC$, where $C = \mathscr{C}_E(e)$.*

(ii) *Let $\alpha \in KE$ with $\alpha(1) \neq 0$. Then there exists a primitive idempotent e of KQ with $e\alpha e(1) \neq 0$.*

(iii) *Suppose that $\bar{E} = E/Q$ has a finite normal p-subgroup $\bar{P}$ with $\bar{E}/\bar{P}$ a torsion-free abelian group. Let e be a primitive idempotent of KQ and let $\gamma_1, \gamma_2, \ldots, \gamma_n \in KE$ with $\gamma_i = e\gamma_i e \neq 0$. If the supporting subgroup of each of the γ_i is a p'-group, then we have $\gamma_1\gamma_2 \ldots \gamma_n \neq 0$.*

Proof. (i) Since $Q \triangle E$, E permutes by conjugation the primitive idempotents of KQ. As is well known, if e_1 and e_2 are primitive idempotents of KQ, then either $e_1e_2 = 0$ or $e_1 = e_2$. Let e and α be given and write $\alpha = \sum k_h h$, where $h \in E$ and $k_h \in K$. Then

$$e\alpha e = \sum k_h ehe = \sum k_h ee^h h.$$

Since $ee^h = 0$ if $h \notin C$ and $e \in KQ \subseteq KC$, this yields $\alpha = e\alpha e \in KC$.

(ii) Write $\alpha = \sum_1^r \alpha_i h_i$, where $\alpha_i \in KQ$ and $h_1, h_2, \ldots, h_r$ are in distinct cosets of Q with $h_1 = 1$. Since $\alpha(1) \neq 0$, we have $\alpha_1 \neq 0$. Now Q is an abelian p'-group and K is algebraically closed of characteristic p. Hence in KQ we have $1 = \sum e_j$, a sum of primitive idempotents. Now $\alpha_1 = \sum e_j\alpha_1$ and

$$\alpha_1(1) = \alpha(1) \neq 0;$$

thus for some $e = e_j$ we have $e\alpha_1 e(1) = e\alpha_1(1) \neq 0$. Since

$$e\alpha e = \sum_1^r e\alpha_i h_i e = \sum_1^r (e\alpha_i e^{h_i})h_i \quad \text{and} \quad e\alpha_i e^{h_i} \in KQ,$$

we see that $e\alpha e(1) = e\alpha_1 e(1) \neq 0$.

(iii) By (i), each $\gamma_i \in KC$. Furthermore, $C \supseteq Q$ and if $\bar{C} = C/Q$, then $\bar{C}/(\bar{C} \cap \bar{P}) \simeq \bar{C}\bar{P}/\bar{P}$ is a torsion-free abelian group. Thus it suffices to assume that $E = C$, and hence that e is central in KE. For all $g \in Q$ we have $ge = \lambda(g)e$, where λ is a linear character of Q. Moreover, since e is central if $h \in E$, $g \in Q$, then $\lambda(g^h) = \lambda(g)$. Let Q_0 be the kernel of λ. Then $Q_0 \triangle E$ and Q/Q_0 is a cyclic p'-group central in E.

Let $D \supseteq Q$ with $D/Q = \bar{P}$. If P is a Sylow p-subgroup of D, then $D = QP$ and D/Q_0 is nilpotent. This easily yields $Q_0P \triangle E$. Let I denote the kernel of the natural epimorphism $KE \to K(E/Q_0P)$. Then I is the annihilator of $\widehat{Q_0P} = \hat{Q}_0\hat{P}$ in KE, where for any finite subset $S \subseteq E$, $\hat{S}$ denotes the sum of the elements of S in KE. We will show the following two facts which will yield the result. First, for all i, $\gamma_i \notin eIe$, and second that $eKEe/eIe$ has no zero divisors. Note that since e is central, $eKEe = eKE$ and $eIe = eI$.

If $\gamma_i \in eI$, then $\gamma_i\widehat{Q_0P} \in eI\widehat{Q_0P} = \{0\}$, and hence $\gamma_i\widehat{Q_0P} = 0$. Now $\gamma_i = \gamma_i e$ and $e\hat{Q}_0 = e|Q_0|$; thus

$$0 = \gamma_i\widehat{Q_0P} = |Q_0|\gamma_i e\hat{P} = |Q_0|\gamma_i\hat{P}.$$

Since Q_0 is a p'-group, $|Q_0| \neq 0$ in K, and hence $\gamma_i\hat{P} = 0$. Finally, since the supporting group of γ_i is a p'-group and $\gamma_i \neq 0$, this is clearly absurd. Thus $\gamma_i \notin eI$.

Set $\tilde{E} = E/Q_0P$, $\tilde{D} = D/Q_0P$, and let $\tilde{e}$ be the image of e under the map $KE \to K\tilde{E}$. Since e is an idempotent,

$$eI \subseteq eKE \cap I = e(eKE \cap I) = eI.$$

This implies that $eKE/eI \simeq \tilde{e}K\tilde{E}$. Moreover, since $D = Q(Q_0P)$, we see that $\tilde{e}K\tilde{D} = \tilde{e}K$. Thus $\tilde{e}K\tilde{E} \simeq K^t(\tilde{E}/\tilde{D})$, where the latter is a twisted group algebra for the torsion-free abelian group $\tilde{E}/\tilde{D}$. Now it is well known (and easily verified using a degree argument) that K^tA has no zero divisors if A is a torsion-free abelian group. Thus as we remarked above, this suffices to prove the result.

THEOREM 6. *Let K be an algebraically closed field of characteristic p, let G be a group and let H be a normal subgroup of G. Suppose that H' is a finite p'-group and that every finite p'-subgroup of H is abelian. Let $\alpha \in (JKG) \cap (KH)$ with $\alpha(1) \neq 0$ and let $L \subseteq H$ be the supporting subgroup of α. If $x \in G$, then some power x^j of x with $j \neq 0$ normalizes the coset $H'g$ for some element $g \in L$ of order p.*

We remark that the assumptions on H are certainly satisfied if H is abelian and in this case the proof is slightly simpler. However, as will be apparent later, we need this stronger result.

Proof. Let $G_1 = \langle H, x\rangle$. Then by (**2**, Lemma 1), $\alpha \in (JKG_1) \cap (KH)$ and also $\alpha \in JKL$. Thus it clearly suffices to assume that $G = G_1$. Suppose first that Hx has finite order in G/H. Then for some $j \neq 0$, $x^j \in H$, and hence x^j normalizes $H'g$ for all $g \in L$. Now L' is a finite p'-group since $L' \subseteq H'$, thus $JKL' = \{0\}$. Hence, since $JKL \neq \{0\}$, (**2**, Theorem 6) implies that L/L' is not a p'-group. Thus L contains an element of order p and the result follows in this case.

We assume now that Hx has infinite order. Since $G/H = \langle Hx\rangle$ is infinite-cyclic, (**2**, Theorem 6(iii)) implies that for each integer $s > 0$ and for each $\beta \in (JKG) \cap (KH)$ there exists $r > 0$ with

$$(*) \qquad \beta\beta^{x^s}\beta^{x^{2s}}\ldots\beta^{x^{rs}} = 0.$$

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The polynomial ring $Z[x]$ acts on the abelian group $A = H/H'$ and we use the notation $a^{f(x)}$ to denote the image of a under $f(x)$. Set $B = LH'/H'$. Since L is finitely generated, so is B. Let $D = \{b \in B | b^{f(x)} = 1$ for some primitive polynomial $f(x) \in Z[x]\}$. By Gauss' lemma, D is a subgroup of B, and hence D is also finitely generated. Set $E = D^{Z[x]}$. Then E is a finitely generated $Z[x]$ module and by Gauss' lemma, for each $a \in E$ there exists a primitive polynomial $f(x)$ with $a^{f(x)} = 1$. Hence $E \cap B = D$.

Let E_0 be the torsion subgroup of E. Since E_0 is a $Z[x]$ submodule of E, E is finitely generated as a $Z[x]$ module, and $Z[x]$ is a Noetherian ring, it follows that E_0 is a finitely generated $Z[x]$ module. Let $a \in E_0$ be one such generator, say of order $m > 0$, and let $f(x)$ be primitive with $a^{f(x)} = 1$. If

$$I = f(x)Z[x] + mZ[x],$$

then $\langle a \rangle^I = 1$. Hence $\langle a \rangle^{Z[x]}$ is a homomorphic image of the additive abelian group $Z[x]/I$, and thus is finite by Lemma 4. Since E_0 is finitely generated, it follows that E_0 is finite.

Write $E_0 = PQ$, where P is its Sylow p-subgroup and Q is its Sylow p'-subgroup. Let $\tilde{E}$, $\tilde{E}_0$, and $\tilde{Q}$ be subgroups of H containing H' with $\tilde{E}/H' = E$, $\tilde{E}_0/H' = E_0$ and $\tilde{Q}/H' = Q$. Then $\tilde{E}_0$ is finite and $\tilde{Q}$ is a finite p'-subgroup of H since H' is a finite p'-subgroup of H. Thus by assumption, $\tilde{Q}$ is abelian. Now x acts as an endomorphism on $\tilde{E}$, $\tilde{E}_0$, and $\tilde{Q}$ with trivial kernel, and thus x acts as an automorphism on finite $\tilde{E}_0$ and $\tilde{Q}$. Therefore for some integer $j > 0$, x^j centralizes $\tilde{E}_0$.

Suppose that $D \cap P \neq \langle 1 \rangle$. Since $D \subseteq LH'/H'$ and H' is a finite p'-group, it follows easily that there exists $g \in L$ with g of order p such that $H'g \in D \cap P$. Thus, since x^j centralizes $\tilde{E}_0/H' = E_0$, it follows that x^j normalizes $H'g$ and the result follows. We assume now that $D \cap P = \langle 1 \rangle$ and therefore that

$$DQ \cap P = \langle 1 \rangle$$

and derive a contradiction.

By Lemma 5(ii) (with $E = H$, $Q = \tilde{Q}$), there exists a primitive idempotent $e \in K\tilde{Q}$ with $\beta = e\alpha e$ and $\beta(1) \neq 0$. Then the supporting subgroup of β is clearly contained in $\tilde{Q}L$ and β belongs to the KH ideal $(JKG) \cap (KH)$. Moreover, by Lemma 5(i), $\beta \in K\tilde{C}$, where $\tilde{C} = \mathscr{C}_H(e)$. Write

$$\beta = \beta_1 h_1 + \beta_2 h_2 + \ldots + \beta_m h_m,$$

where $\beta_i \in K(\tilde{Q}L \cap \tilde{E})$ and $h_1, h_2, \ldots, h_m$ are elements of $\tilde{Q}L$ in distinct cosets of $\tilde{Q}L \cap \tilde{E}$ with $h_1 = 1$. Furthermore, we can assume that $\beta_i, h_i \in K\tilde{C}$.

Since $e\beta e = \beta$, $\tilde{Q} \subseteq \tilde{Q}L \cap \tilde{E}$, and $h_i \in \tilde{C}$, we see that $e\beta_i e = \beta_i$. Since $\beta(1) \neq 0$ we have $\beta_1 \neq 0$. Now $\beta_1 \in K(\tilde{Q}L \cap \tilde{E})$ and

$$(\tilde{Q}L \cap \tilde{E})/H' = QB \cap E = Q(B \cap E) = QD.$$

Since $QD \subseteq E$ and $QD \cap P = \langle 1 \rangle$, we see that QD is a p'-group, and hence so is $\tilde{Q}L \cap \tilde{E}$. Thus the supporting group of β_1, and in fact of each β_i, is a p'-group.

Let S denote the set of all positive integer multiples of j and let $s \in S$. Then by $(*)$ there exists $r > 0$ with the ordered product

$$(**) \qquad \prod_{i=0}^{r} \{\beta_1^{x^{is}} h_1^{x^{is}} + \beta_2^{x^{is}} h_2^{x^{is}} + \ldots + \beta_m^{x^{is}} h_m^{x^{is}}\} = 0.$$

Now $j|s$, thus x^s centralizes e. Hence, since x^s leaves $\tilde{E}$ invariant, we see that $\beta_1^{x^{is}} \in K\tilde{E}$ and $\beta_1^{x^{is}} = e\beta_1^{x^{is}}e \neq 0$. Moreover, since the supporting subgroup of β_1 is a p'-group, the same is true for the supporting subgroup of $\beta_1^{x^{is}}$. Hence by Lemma 5(iii) (with $E = \tilde{E}$, $Q = \tilde{Q}$),

$$(***) \qquad \beta_1 \beta_1^{x^s} \beta_1^{x^{2s}} \ldots \beta_1^{x^{rs}} \neq 0.$$

Set $\bar{A} = A/E = H/\tilde{E}$ and set $\bar{B} = BE/E = L\tilde{E}/\tilde{E}$. Then $Z[x]$ acts on abelian $\bar{A}$, and $\bar{B}$ is a finitely generated subgroup. Since $h_1, h_2, \ldots, h_m \in L\tilde{Q}$ and they are in distinct cosets of $L\tilde{Q} \cap \tilde{E}$, it follows that they are in distinct cosets of $\tilde{E}$. Hence $\tilde{E}h_1, \tilde{E}h_2, \ldots, \tilde{E}h_m$ are distinct elements of $\bar{B}$. Therefore, since $h_1 = 1$, $(**)$ and $(***)$ easily imply that the product

$$\bar{B} \times \bar{B}^{x^s} \times \bar{B}^{x^{2s}} \times \ldots \times \bar{B}^{x^{rs}}$$

is not direct. Since this assertion holds for all $s \in S$, the multiplicative analogue of Lemma 3 implies that there exists $\bar{b} \in \bar{B}$, $\bar{b} \neq 1$, and a primitive polynomial $f(x) \in Z[x]$ with $\bar{b}^{f(x)} = 1$. Now $\bar{B} = BE/E$, therefore $\bar{b} = Eb$ for some $b \in B$. Thus $b^{f(x)} \in E$ and hence there exists a primitive polynomial $g(x) \in Z[x]$ with $b^{f(x)g(x)} = 1$. By Gauss' lemma, $f(x)g(x)$ is primitive and hence $b \in D \subseteq E$. Thus $Eb = \bar{b} = 1$, a contradiction, and the result follows.

The following lemma can be used to give an alternate proof of (**2**, Theorem 3).

LEMMA 7. *Let G be a group with a normal abelian subgroup A of finite index and let $H = \{g \in G | \, [A: \mathscr{C}_A(g)] < \infty\}$. Then H is a subgroup of G containing A. Let K be a field and let I be a non-zero ideal of KG. Then $I \cap KH \neq \{0\}$.*

Proof. H is clearly a subgroup of G. Let $g_1, g_2, \ldots, g_n$ be a complete set of coset representatives for A in G. Suppose that $g_1 = 1$ and that the numbering is so chosen that $g_1, g_2, \ldots, g_r \in H$ while $g_{r+1}, \ldots, g_n \in G - H$. Then any element $\alpha \in KG$ can be written uniquely as $\alpha = \sum_1^n \alpha_i g_i$ with $\alpha_i \in KA$. For convenience, set $N(\alpha)$ equal to the number of $j = r + 1, \ldots, n$ with $\alpha_j \neq 0$. Thus $N(\alpha) = 0$ if and only if $\alpha \in KH$.

Since $I \neq \{0\}$, there exists $\alpha \in I$ with $\alpha_1 \neq 0$. Among all such elements, choose α so that $N(\alpha)$ is minimal. Suppose that $N(\alpha) \neq 0$. Then we have $\alpha_1 \neq 0$ and $\alpha_j \neq 0$ for some $j > r$. Now there is only a finite number of $b \in A$ with $\alpha_1 b = \alpha_1$, and there is an infinite number of distinct commutators $(a, g_j) = ag_j a^{-1} g_j^{-1}$ for $a \in A$ since $g_j \in G - H$. Hence we can choose $a \in A$ so that if $b = (a, g_j)$, then $\alpha_1 b \neq \alpha_1$. Set

$$\beta = b\alpha - a\alpha a^{-1} = \sum \beta_i g_i.$$

Then since A is abelian, $\beta_i = \alpha_i\{b - (a, g_i)\}$. Thus $\beta_j = 0, \beta_1 = \alpha_1(b-1) \neq 0$, and $\alpha_i = 0$ implies $\beta_i = 0$. This yields $\beta \in I, \beta_1 \neq 0$ and $N(\beta) < N(\alpha)$, a contradiction. This implies that we must have had $N(\alpha) = 0$ so that $\alpha \in I \cap KH$ and thus $I \cap KH \neq \{0\}$.

Proof of Theorem 1. Suppose first that $JKG \neq \{0\}$. Let H_1 be the subgroup of G such that $G \supseteq H_1 \supseteq A$ and H_1/A is the Sylow p-subgroup of abelian G/A. Since G/H_1 is an abelian p'-group, (**2**, Theorem 6) implies that

$$I_1 = (JKG) \cap (KH_1) \neq \{0\}.$$

Now H_1/A is a locally finite group, thus there clearly exists a group H_2 with $H_1 \supseteq H_2 \supseteq A$, $[H_2:A] < \infty$ and $I_2 = I_1 \cap (KH_2) \neq \{0\}$. Let

$$H = \{g \in H_2 | \ [A:\mathscr{C}_A(g)] < \infty\}.$$

By Lemma 7, H is a subgroup of H_2 and $I = I_2 \cap (KH) \neq \{0\}$. Clearly $I = (JKG) \cap (KH)$.

We consider some properties of H. First, since $H \supseteq A$, we have $H \triangle G$. Second, since H/A is a p-group, we see that every finite p'-subgroup of H is contained in A and hence is abelian. Finally, let $g_1, g_2, \ldots, g_n$ be a complete set of coset representatives of A in H. By definition of H, $[A:\mathscr{C}_A(g_i)] < \infty$, therefore $[A: \bigcap_1^n \mathscr{C}_A(g_i)] < \infty$. Hence $[H:\mathscr{Z}(H)] < \infty$, and thus $|H'| < \infty$ by (**3**, Theorem 15.1.13). Note that $H' \triangle G$.

There are two cases to consider. Suppose first that $p \| H'|$ and let $g \in H'$ be an element of order p. Since all conjugates of g are contained in H', we have g^G finite. Thus g^A is finite and $G/\mathscr{N}_G(g^A)$ is finite and hence periodic.

Now let H' be a p'-group. Since $I \neq \{0\}$, choose $\alpha \in I$ with $\alpha(1) \neq 0$ and let L be the supporting subgroup of α. Then L is finitely generated, and since LH'/H' is abelian we see that L contains only a finite number of elements of order p and say that these are $h_1, h_2, \ldots, h_m$. Let $C_i = \{x \in G | \ x^j$ normalizes h_i^A for some $j \neq 0\}$. Since $\mathscr{N}_G(h_i^A) \supseteq A$ and G/A is abelian, we see that C_i is a subgroup of G.

Let $x \in G$. By Theorem 6 there exists an integer $j' \neq 0$ such that $x^{j'}$ normalizes the coset $H'h_i$ for some i, and then since H' is finite, there exists a suitably larger integer j such that x^j centralizes $H'h_i$. Now $H \supseteq A, \{h_i\}$, thus clearly $h_i^A \subseteq H'h_i$. Therefore x^j centralizes, and hence normalizes, h_i^A; thus $x \in C_i$. Thus $G = \bigcup_1^m C_i$, and so by (**1**, Lemma 7) we have $[G:C_k] < \infty$ for some k. Clearly G/C_k is torsion-free, therefore this yields $G = C_k$ and hence if $g = h_k$, then g has order p, g^A is finite, and $G/\mathscr{N}_G(g^A)$ is periodic. This completes the necessity part of the proof.

Conversely, let $g \in G$ be given with g of order p, g^A finite, and $G/\mathscr{N}_G(g^A)$ periodic. Let $H = \langle A, g\rangle$; thus, since $H \supseteq A$, we have $H \triangle G$. The map $a \to (a, g)$ is easily seen to be an endomorphism of A with kernel $\mathscr{C}_A(g)$ and image (A, g). Thus (A, g) is a group which is clearly normal in H. Since $H/(A, g)$ is abelian we have $(A, g) = H' \triangle G$. Note that

$$H'g = (A, g)g = g^A,$$

hence H' is finite. Let $\alpha = \widehat{H'}(1 - g) \in KG$. We note that $\alpha \neq 0$ since if $g \in A$, then $H' = \langle 1 \rangle$ and if $g \notin A$, then $H' \subseteq A$. We show now that αKG is a nil ideal.

Let $\beta \in KG$. Then there exists a subgroup L of G with $G \supseteq L \supseteq H$, L/A finitely generated and $\beta \in KL$. Thus $\alpha\beta \in \alpha KL$ and we show that this latter ideal is nilpotent. Now $L/\mathcal{N}_L(g^A)$ is a finitely generated periodic abelian group, and hence is finite. Since g^A is finite, this implies that $[L:\mathscr{C}_L(g)]$ is finite. Let $g_1, g_2, \ldots, g_n$ denote the distinct conjugates of g in L. These are of course all contained in H since $H \triangle L$. Let $\alpha_i = \widehat{H'}(1 - g_i)$ and let $x_1, x_2, \ldots, x_r \in L$. Since the α_i are all central in KH, we have

$$(\alpha x_1)(\alpha x_2) \ldots (\alpha x_r) = \alpha_1^{a_1}\alpha_2^{a_2} \ldots \alpha_n^{a_n} x_1 x_2 \ldots x_r$$

for some integers $a_i \geqslant 0$ satisfying $a_1 + a_2 + \ldots + a_n = r$. Moreover, $\widehat{H'}$ is central in KH, thus $\alpha_i^{a_i} = \widehat{H'}^{a_i}(1 - g_i)^{a_i}$. Since $(1 - g_i)^p = 0$, it follows that if $r \geqslant np$, then the above product of r terms is 0, hence clearly $(\alpha KL)^{np} = \{0\}$. Thus αKG is a non-zero nil ideal, $JKG \neq \{0\}$, and the result follows.

In a later paper I will show that if G is a finitely generated metabelian group, then $JKG = (JKH)(KG)$ for some finite normal subgroup H of G. In particular, JKG is nilpotent.

Added in proof. A systematic study of the Jacobson radical and the nilpotent radical of twisted group algebras can be found in "Radicals of twisted group rings", to appear in the Proceedings of the London Mathematical Society. The paper also contains the result on finitely generated metabelian groups mentioned in the last paragraph above. This result is obtained as a corollary of Theorem 1 and of certain general considerations.

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