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ON A GENERALIZATION OF A RESULT OF WALDSPURGER

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ABSTRACT. We consider a generalization of a trace formula identity of Jacquet, in the context of the symmetric spaces $\mathrm{GL}(2n)/\mathrm{GL}(n) \times \mathrm{GL}(n)$ and G'/H'. Here G' is an inner form of $\mathrm{GL}(2n)$ over F with a subgroup H' isomorphic to $\mathrm{GL}(n,E)$ where E/F is a quadratic extension of number field attached to a quadratic idele class character η of F. A consequence of this identity would be the following conjecture: Let π be an automorphic cuspidal representation of $\mathrm{GL}(2n)$. If there exists an automorphic representation π' of G' which is related to π by the Jacquet-Langlands correspondence, and a vector ϕ in the space of π' whose integral over H' is nonzero, then both $L(1/2,\pi)$ and $L(1/2,\pi\otimes\eta)$ are nonvanishing. Moreover, we have $L(1/2,\pi)L(1/2,\pi\otimes\eta) > 0$. Here the nonvanishing part of the conjecture is a generalization of a result of Waldspurger for GL(2) and the nonnegativity of the product is predicted from the generalized Riemann Hypothesis. In this article, we study the corresponding local orbital integrals for the symmetric spaces. We prove the "fundamental lemma for the unit Hecke functions" which says that unit Hecke functions have "matching" orbital integrals. This serves as the first step toward establishing the trace formula identity and in the same time it provides strong evidence for what we proposed.

0. **Introduction.** Let F be a number field, E a quadratic extension field of F. We use F_A and E_A to denote the adele rings of F and E. Let η be the quadratic character of the idele class group F_A^*/F^* attached to E. For a positive integer m, we denote by G_m the linear group $\mathrm{GL}(m)$. Let G be $G_{2n}(F)$ and let H be $G_n(F) \times G_n(F)$. We embed H into G in the following way:

$$(g_1, g_2) \longmapsto \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}.$$

Let Z be the center of G. For a character χ of $H(F_A)$ trivial on $Z(F_A)H(F)$, we say that an automorphic cuspidal representation π of $G(F_A)$ is (H,χ)-distinguished if π has trivial central character and there exists a vector ϕ in the space of the automorphic realization of π such that the integral

$$\int_{H(F)Z(F_A)\backslash H(F_A)} \phi(h)\chi(h)\,dh \tag{0. 1}$$

is not zero. If π is $(H,1)$-distinguished, we will simply say that it is H-distinguished.

We denote by $X(E:F)$ the set of the isomorphic classes of pairs (G',H'), where G' is an inner form of G over F and $H' \subset G'$ is an algebraic subgroup which is isomorphic to $G_n(E)$ over F. Let Z' be the center of G'. Then we can define the notion of H'-distinguished representation of G' in a similar way. We say that an automorphic cuspidal

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representation π of $G(F_A)$ satisfies Waldspurger's condition if there exists a pair (G', H') in $X(E : F)$ and an automorphic cuspidal representation π' of $G'(F_A)$, which is related to π under the Jacquet-Langlands (Deligne-Kazhdan) correspondence, such that π' is H'-distinguished. When $n = 1$ we have the following remarkable result of Waldspurger.

THEOREM (WALDSPURGER). *Let π be an automorphic cuspidal representation of $G_2(F_A)$. Then π is both H-distinguished and (H, η)-distinguished if and only if π satisfies Waldspurger's condition.*

Here we regard η as a character of $H(F_A)$ by setting

$$\eta(h) = \eta(\det h)$$

for $h \in H(F_A)$.

It is natural to ask for a generalization of this result to G_{2n}. Here we conjecture a partial generalization:

CONJECTURE. *If an automorphic cuspidal representation π of $G_{2n}(F_A)$ satisfies Waldspurger's condition, then π is both H-distinguished and (H, η)-distinguished. In the case that n is odd, the converse is also true.*

Waldspurger first proved his theorem by using the machinery of the Weil representation. But this method is limited to low rank cases. Another approach to the problem of proving the conjecture is the relative trace formula of Jacquet. We proceed to explain this as follows. Let f be a smooth function with compact support on $G(F_A)/Z(F_A)$. We denote by $L_c(G)$ the subspace of cusp forms of the Hilbert space $L^2\big(G(F)Z(F_A)\setminus G(F_A)\big)$. Then f induces an operator $\rho_c(f)$ on $L_c(G)$. Let $K_c(x, y)$ be the kernel of $\rho_c(f)$. We define a distribution $I(f)$ by

$$I(f) = \int_{Z(F_A)H(F)\backslash H(F_A)} \int_{Z(F_A)H(F)\backslash H(F_A)} K_c(x, y)\eta(x)\, dx\, dy.$$

The kernel $K_c(x, y)$ admits a spectral decomposition of the form

$$K_c(x, y) = \sum_{\pi} K_\pi(x, y),$$

where the sum is over all the irreducible cuspidal representations π. Let $\{\phi_i\}$ be an orthonormal basis of π. Then we have

$$K_\pi(x, y) = \sum_i \big(\rho_c(f)\phi_i\big)(x)\bar{\phi}_i(y).$$

From this expression we find

$$\begin{aligned} I_\pi(f) &= \iint K_\pi(x, y)\eta(x)\, dx\, dy \\ &= \sum \Big[\int (\rho_c(f)\phi_i)(x)\eta(x)\, dx\Big]\Big[\int \bar{\phi}_i(y)\, dy\Big]. \end{aligned}$$

So $I_\pi(f)$ is not zero only if π is both H-distinguished and (H, η)-distinguished. In other words only such representations contribute to $I(f)$.

Similarly for $(G',H') \in X(E:F)$ and $f' \in C_c^\infty(G'(F_A)/Z'(F_A))$, we define a distribution $I'(f')$ by

$$I'(f') = \int_{Z'(F_A)H'(F)\backslash H'(F_A)} \int_{Z'(F_A)H'(F)\backslash H'(F_A)} K_c'(x,y)\,dx\,dy$$

where $K_c'(x,y)$ is the kernel representing the operator $\rho_c(f')$ on $L_c(G')$ induced by f'. As before only H'-distinguished representations contribute to $I'(f')$.

Then if n is odd, we should have the following trace formula identity

$$I(f) = \sum_{(G',H')\in X(E:F)} I'(f') \tag{0.2}$$

if the corresponding local components of the function f and the functions $\{f'\}$ have "matching orbital integrals".

If n is even, then for a given pair $(G',H') \in X(E:F)$, the trace formula identity we expected reads:

$$I(f) = I'(f') \tag{0.3}$$

if the local components of f and f' have "matching orbital integrals".

Here the difference in formulating the trace formula identities between the case when n is odd and the case when n is even is suggested by the comparison of the double cosets $H\setminus G/H$ with the double cosets $H'\setminus G'/H'$ for $(G',H') \in X(E:F)$ (see Lemma 1.8 in Section 1.1).

Once these identities are established, the above conjecture can be proved by analyzing the spectral decomposition of both sides. When $n = 1$ this is what Jacquet has done in [J1] where he established (0.2) for GL(2) and reproved the result of Waldspurger. Further exploiting this identity, we proved in [G] that

$$L(1/2,\pi)L(1/2,\pi\otimes\eta) \geq 0$$

for an automorphic cuspidal representation π of $\mathrm{GL}(2,F_A)$ with trivial central character. When combining with some known result concerning the positivity of certain average of $L(1/2,\pi\otimes\eta)$ ([HF]), this implies $L(1/2,\pi) \geq 0$.

Generally for $f = \otimes_v f_v \in C_c^\infty\big(G(F_A)/Z(F_A)\big)$, then we have a finite set S of the places of F such that each $v \notin S$ is unramified in E and f_v is the unit Hecke function of $G(F_v)$. The same is true for the function in the other side. Thus in trying to establish (0.2) or (0.3) for G_{2n}, we should first show that locally the unit Hecke functions have "matching orbital integrals". The purpose of the present paper is to prove that this is the case (see Section 1.2). So we may regard it as the first step toward the generalization of the result of Waldspurger. In the same time it also provides evidence that the identities (0.2) and (0.3) will work for G_{2n}, hence our conjecture.

The distinguishedness of a representation has a strong relation with the properties of L-functions. To explain this we recall a result of Friedberg-Jacquet ([FJ]) as suggested by the work of Bump-Friedberg ([BF]). Let χ be a character of F_A^*/F^*. If an automorphic cuspidal representation π of G is (H,χ)-distinguished, then the exterior square L-function $L(s,\pi,\wedge^2\rho)$ has a pole at $s=1$ and

$$\int_{H(F)Z(F_A)\backslash H(F_A)} \phi\big(\mathrm{diag}(g_1,g_2)\big)\chi\Big(\frac{\det g_1}{\det g_2}\Big)\,d(g_1,g_2)=L(1/2,\pi\otimes\chi)$$

for some cusp form ϕ in the space of π. So we could derive the following result from our conjecture:

If π satisfies Waldspurger's condition, then $L(s,\pi,\wedge^2\rho)$ has a pole at $s=1$ and $L(1/2,\pi)L(1/2,\pi\otimes\eta)\neq 0$.

Furthermore by the trace formula identity (0.2) or (0.3), we could relate the product $L(1/2,\pi)L(1/2,\pi\otimes\eta)$ to, roughly speaking, the square of a integral

$$\int_{H'(F)Z'(F_A)\backslash H'(F_A)} \phi(h)\,dh$$

for a pair $(G',H')\in X(E:F)$ and a cusp form ϕ in the space of a H'-distinguished representation π' of $G'(F_A)$. This in turn should provide information concerning the positivity and some other properties of $L(1/2,\pi)L(1/2,\pi\otimes\eta)$.

To end the introduction, we mention that a necessary tool, which concerns the multiplicity one of the subgroup H in G and H' in G', in carrying on the project we described above becomes available recently ([JR], [G2]). To explain this, we assume that F is a nonarchimedean local field. Suppose π is an irreducible admissible representation of G. Let $\mathrm{Hom}(\pi,\mathbb{C})$ be the dual space of π, and let $\mathrm{Hom}_H(\pi,\mathbb{C})$ be the set of the H-invariant elements in $\mathrm{Hom}(\pi,\mathbb{C})$. Then we have ([JR])

$$\dim \mathrm{Hom}_H(\pi,\mathbb{C})\leq 1.$$

Similarly we have ([G2])

$$\dim \mathrm{Hom}_{H'}(\pi',\mathbb{C})\leq 1$$

where $(G',H')\in X(E:F)$ and π' is an irreducible admissible representation of G'.

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1. Symmetric spaces and orbital integrals.

1.1. In this section, F is a field of characteristic 0. We will study the correspondence between the double classes $H\backslash G/H$ and the double classes $H'\backslash G'/H'$ for $(G',H')\in X(E:F)$. For this purpose, we first recall the description of the geometric structure of the H-conjugacy classes in the symmetric space G/H, as given by Jacquet and Rallis in [JR].

We denote by ϵ_n the matrix

$$\begin{pmatrix} I_n & 0\\ 0 & -I_n\end{pmatrix}.$$

Let θ_n be the involution of G defined by $\theta_n(g) = \epsilon_n g \epsilon_n$. Then we have

$$H = \{g \in G \mid \theta_n(g) = g\}.$$

We consider the variety

$$S_n = \{g\epsilon_n g^{-1}\epsilon_n \mid g \in G\}$$

which is contained in the space

$$P_n = \{g \in G \mid g\epsilon_n g\epsilon_n = I_{2n}\}.$$

In what follows, we often write ϵ, θ, S and P for ϵ_n, θ_n, S_n and P_n respectively if no confusion arises. The group G acts on S by the twisted conjugation

$$(g,s) = gs\theta(g^{-1}), \quad g \in G,\ s \in S.$$

In particular H acts on S by conjugation $(h,s) = hsh^{-1}$ for $h \in H$. The surjective map $\rho\colon G \longrightarrow S$ defined by

$$\rho(g) = g\epsilon g^{-1}\epsilon, \quad g \in G \tag{1.1}$$

satisfies

$$\rho(xgh) = x\rho(g)\theta(x).$$

So it induces an isomorphism between the symmetric space G/H and the space S as G-spaces. The map ρ also induces a one to one correspondence between the H-double cosets in G and H-orbits in S.

Thus we are reduced to study the H-conjugacy classes in S. For $x \in S$, let $x = x_s x_u = x_u x_s$ be its Jordan decomposition in G, where x_s is semisimple and x_u is unipotent. Then both x_s and x_u are in S ([R] and [JR]). Our task is to analyze the semisimple elements in S. We note that an element

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G, \tag{1.2}$$

where A, B, C, D are of size $n \times n$, is in P_n if and only if

$$A^2 = I_n + BC, \quad D^2 = I_n + BC, \quad AB = BD, \quad DC = CA.$$

So when A and D have no eigenvalues ± 1, then B and C are nonsingular. In this case we have:

LEMMA 1.1. *Let $s \in S$ be an element of the form (1.2) where A and D has no eigenvalues ± 1. The s is H conjugate to an element of the form*

$$s(A_1) = \begin{pmatrix} A_1 & A_1 - I_n \\ A_1 + I_n & A_1 \end{pmatrix}$$

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where A_1 has no eigenvalues ± 1. Two such element $s(A_1)$ and $s(A_2)$ are H conjugate if and only if A_1 and A_2 are conjugate. Furthermore $S(A_1)$ is semisimple if and only if A_1 is semisimple.

This is essentially the Lemma 4.3 of [JR]. They actually proved that s is conjugate to an element of the form

$$t(A_1) = \begin{pmatrix} A_1 & I_n \\ A^2 - I_n & A_1 \end{pmatrix} \in S.$$

But it is easily seen that $s(A_1)$ is H conjugate to $t(A_1)$. In fact we have

$$\operatorname{diag}\big((A_1 - I_n)^{-1}, I_n\big) s(A_1) \operatorname{diag}\big((A_1 - I_n), I_n\big) = t(A_1).$$

Let n_1 and n_2 are two integers such that $0 \leq n_1 \leq n_1 + n_2 \leq n$ and let A be an element in $M(n_1, F)$ without eigenvalues ± 1. We denote by $s(A, n_1, n_2)$ the matrix

$$\begin{pmatrix} A & 0 & 0 & A - I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -I_{n-n_1-n_2} & 0 & 0 & 0 \\ A + I_{n_1} & 0 & 0 & A & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -I_{n-n_1-n_2} \end{pmatrix}. \tag{1.3}$$

If $n_1 = n$, $n_2 = 0$, we will simply write $S(A, n_1, n_2)$ as $s(A)$. Then the result of Jacquet-Rallis on the H conjugacy classes of semisimple elements in S can be summarized as follows (*cf.* Proposition 4.1 of [JR]):

PROPOSITION 1.1. *Each semisimple element $s \in S$ is H conjugate to an element of the form $s(A, n_1, n_2)$. The set of the H conjugacy classes of semisimple elements of S is in bijective correspondence with the set of all triples $(n_1, \{A_1\}, n_2)$ where $0 \leq n_1 \leq n$ is an integer, $\{A_1\}$ a semisimple conjugacy class in $M(n, F)$ without the eigenvalues ± 1 and n_2 is an integer with $0 \leq n_2 \leq n - n_1$.*

We say that an element $s \in S$ is θ-regular if it is semisimple and the H orbit of s has the maximal dimension among all the H conjugacy classes in S. This is the same as saying that s is semisimple and the centralizer H^s of s in H has the minimal dimension. We denote by S^r the set of the θ-regular elements of S.

LEMMA 1.2. *A semisimple element $s \in S$ is θ-regular if and only if it is H conjugate to an element of the form*

$$s(A) = \begin{pmatrix} A & A - I_n \\ A + I_n & A \end{pmatrix}$$

where A is a regular element (in the usual sense) in $M(n, F)$ without eigenvalues ± 1. The set of H conjugacy classes of S^r is in one to one correspondence to the conjugacy classes of the regular elements in $M(n, F)$ without eigenvalues ± 1.

PROOF. We first consider an element of the form $s(A)$ where $A \in M(n, F)$ has no eigenvalues ± 1. We have that

$$\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} s(A) \begin{pmatrix} h_1^{-1} & 0 \\ 0 & h_2^{-1} \end{pmatrix} = s(A),$$

if and only if

$$h_1Ah_1^{-1} = A = h_2Ah_2^{-1}, \quad h_1(A - I_n)h_2^{-1} = A - I_n, \quad h_2(A + I_n)h_1^{-1} = A + I_n.$$

It is easily seen that these conditions are equivalent to

$$h_1Ah_1^{-1} = A, h_1 = h_2.$$

So if we identify $G_n(F)$ with the subgroup

$$\{\mathrm{diag}(g,g) \mid g \in G_n(F)\}$$

of H, then $H^{s(A)} = G_n(F)^A$, the centralizer of A in $G_n(F)$. It is well known that $\dim G_n(F)^A \geq n$ and $\dim G_n(F)^A = n$ if and only if A is regular.

Generally, by the above proposition, we can assume that $s = s(A, n_1, n_2)$ where A has no eigenvalues $1, -1$ and $0 \leq n_1 \leq n_1 + n_2$. Then by an easy computation, we find that H^s is the set of the elements of the form

$$\mathrm{diag}(A_1, A_2, A_3, A_1, A_4, A_5)$$

where $A_1 \in G_{n_1}(F)^A, A_2, A_4 \in G_{n_2}(F)$ and $A_3, A_5 \in G_{n_3}(F)\}$ with $n_3 = n - n_1 - n_2$. Thus we have

$$\dim H^s = \dim G_{n_1}(F)^A + 2n_2^2 + 2n_3^2 \geq n_1 + n_2 + n_3 = n.$$

The equality occurs if and only if $\dim G_{n_1}(F)^A = n_1 = n$ and $n_2 = n_3 = 0$. The assertion of the lemma follows.

It is clear that if $s(A) \in S^r$, then $H^{s(A)} = G_n(F)^A$ which is a torus in H. We say that an element $s \in S^r$ is θ-regular elliptic if H^s is a elliptic torus (*i.e.*, H^s/Z is compact). This is equivalent to saying that s is H conjugate to an element of the form $s(A)$ where A is a regular elliptic element in $M(n,F)$. We denote by S^e the set of the θ-regular elliptic elements in S.

Let G^r (resp. G^e) be the set of elements in G whose images under ρ are in S^r (resp. S^e). We say the elements in G^r are θ-regular and the elements in G^e are θ-regular elliptic. For $a \in G_n(F)$ such that $\det(I_n - a) \neq 0$, we denote by $g(a)$ the matrix

$$\begin{pmatrix} I_n & a \\ I_n & I_n \end{pmatrix}$$

which is in G. Then we have:

LEMMA 1.3. *Each θ-regular element of G is in the same H double coset as an element of the form $g(a)$ where a is a regular element in $G_n(F)$ such that* $\det(I_n - a) \neq 0$. *It is θ-regular elliptic if and only if a is regular elliptic. The set $H\backslash G^r/H$ is in bijective correspondence with the conjugacy classes of regular elements in $G_n(F)$ without eigenvalue* 1.

PROOF. We consider the inverse Cayley map λ from the set

$$W = \{a \in G_n(F) \mid \det(I_n - a) \neq 0\}$$

to the set

$$U = \{A \in M(n,F) \mid \det(I_n + A)\det(I_n - A) \neq 0\}$$

defined by

$$\lambda(a) = (I_n + a)(I_n - a)^{-1}.$$

Clearly λ is bijective and induces a one to one correspondence between the $G_n(F)$ conjugacy classes of W and the $G_n(F)$ conjugacy classes of U, which maps regular classes (elliptic regular classes) in W to regular classes (elliptic regular classes) in U. On the other hand it is easy to check that

$$\rho\big(g(a)\big) = s\big(\lambda(a)\big)$$

for $a \in W$. Here $s\big(\lambda(a)\big)$ is θ-regular (θ-regular elliptic) if and only if $\lambda(a)$ is regular (regular elliptic), which is equivalent to saying that a is regular (regular elliptic). Recall that ρ induces a bijective correspondence between the H-double classes in G and the H conjugacy classes in S. The assertion of the lemma follows immediately from Lemma 1.2.

Next we consider the double cosets of the pairs in $X(E:F)$. Given $(G', H') \in X(E:F)$, we can find a central simple algebra L' of dimension $4n^2$ over F and a subalgebra $M' \subset L'$ isomorphic to $M(n,E)$ over F such that G' (resp. H') is the multiplicative group of L' (resp. M'). For $\gamma \in F^*$, we use $L_{\gamma,n}$ to denote the algebra

$$\left\{ \begin{pmatrix} \alpha & \gamma\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \,\middle|\, \alpha, \beta \in M(n,E) \right\}.$$

Then $L_{\gamma,n}$ is a central simple algebra of dimension $4n^2$ over F with the subalgebra

$$M_{E,n} = \{\operatorname{diag}(\alpha, \bar{\alpha}) \mid \alpha \in M(n,E)\}$$

isomorphic to $M(n,E)$. In particular if $\gamma \in NE^*$ then $(L_{\gamma,n}, M_{E,n})$ is isomorphic to $\big(M(2n,F), M(n,E)\big)$ over F. Let $N(E:F)$ be a set of representatives of F^*/NE^*. It is easily seen that there exists one and only one $\gamma \in N(E:F)$ such that (L', M') is isomorphic to $(L_{\gamma,n}, M_{E,n})$ over F. Let $G_{\gamma,n}$ be the multiplicative group of $L_{\gamma,n}$ and let $H_{E,n}$ be that of $M_{E,n}$. Thus we find that

$$\{(G_{\gamma,n}, H_{E,n}) \mid \gamma \in N(E:F)\}$$

gives us a set of representatives of $X(E:F)$. Therefore we just need to study the $H_{E,n}$-double cosets of $G_{\gamma,n}$.

Let $E = F(\sqrt{\tau})$ where $\tau \in F^*$ is not a square. We denote by $\epsilon_{E,n}$ the matrix

$$\begin{pmatrix} \sqrt{\tau} I_n & 0 \\ 0 & -\sqrt{\tau} I_n \end{pmatrix}.$$

Let $\theta_{\gamma,n}$ be the involution on $G_{\gamma,n}$ defined by

$$\theta_{\gamma,n}(g) = \epsilon_{E,n} g \epsilon_{E,n}^{-1}, \quad g \in G_{\gamma,n}.$$

Then $H_{E,n} = \{g \in G_{\gamma,n} \mid \theta_{\gamma,n}(g) = g\}$. As before we consider the variety

$$S_{\gamma,n} = \{g\theta_{\gamma,n}(g^{-1}) \mid g \in G_{\gamma,n}\} \subset G_{\gamma,n}.$$

In what follows, we will drop the second index n from the notations if no confusions arises. Then it can be proved (see [G2]) that

$$S_\gamma = \{g \in G_\gamma \mid (g\epsilon_E)^2 = I_{2n}\}.$$

So an element

$$g = \begin{pmatrix} A & \gamma B \\ \bar{B} & \bar{A} \end{pmatrix} \in G_\gamma$$

is in S_γ if and only it satisfies the following conditions:

$$(1.4) \qquad A^2 = I_n + \gamma B\bar{B}, \quad AB = B\bar{A}.$$

As before G_γ operates on S_γ by the twisted conjugation

$$(g, x) = gx\theta_\gamma(g^{-1}), \quad g \in G_\gamma,\ x \in S_\gamma$$

and in particular the subgroup H_E of G_γ operates on S_γ by conjugation. The surjective map $\rho_\gamma \colon G_\gamma \longrightarrow S_\gamma$ defined by

$$\rho_\gamma(g) = g\epsilon_E\theta_\gamma(g^{-1})$$

induces an isomorphism between the symmetric space G_γ/H and S_γ as G_γ-spaces. It also induces a bijective correspondence between the H_E double cosets in G_γ and the H_E conjugacy classes in S_γ.

Before describing the H_E conjugacy classes in S_γ, we first recall some facts about the twisted conjugation in $G_n(E)$. Two elements $g_1, g_2 \in G_n(E)$ are called *twisted conjugate* if there is a $g \in G_n(E)$ such that $g_1 = gg_2\bar{g}^{-1}$. If $g \in G_n(E)$, we will write $N(g)$ for $g\bar{g}$ and call it the norm of g.

LEMMA 1.4 ([AC], LEMMA 1.1 IN CHAPTER 1). *(1) If $g \in G_n(E)$, then Ng is conjugate to an element h of $G_n(F)$; h is uniquely determined modulo conjugation in $G_n(F)$.*

(2) If Ng_1 and Ng_2 are conjugate in $G_n(E)$, then g_1 and g_2 are twisted conjugate.

In other words, the norm map is an injection from the set of twisted conjugacy classes in $G_n(E)$ to the set of conjugacy classes in $G_n(F)$. We will write $\mathcal{N}(g)$ for the conjugacy class in $G_n(F)$ so obtained. We denote by $NG_n(E)$ the subset of the elements $h \in G_n(F)$ which are conjugate in $G_n(E)$ to Ng for some $g \in G_n(E)$. In fact, if $h \in NG_n(E)$, we can find a $g \in G_n(E)$ such that $h = Ng$. Let $g \in G_n(E)$ such that Ng is conjugate to $h \in G_n(F)$. Then we say g is twisted regular (twisted regular elliptic) if h is regular (regular elliptic) in $G_n(F)$.

We are back to the symmetric space. For $x \in S_\gamma$, let $x = x_ux_s = x_sx_u$ be its Jordan decomposition where x_s is semisimple and x_u is unipotent. Then we also have that $x_s \in S_\gamma$ and $x_u \in S_\gamma$ ([R]).

LEMMA 1.5. *Let*

$$s = \begin{pmatrix} A & \gamma B \\ \bar{B} & \bar{A} \end{pmatrix}$$

be an element in S_γ such that A has no eigenvalues ± 1. Then s is H_E-conjugate to an element of the form

$$s_\gamma(A_1, B_1) = \begin{pmatrix} A_1 & \gamma B_1 \\ \bar{B}_1 & A_1 \end{pmatrix}$$

where $A_1 \in M(n,F)$ has no eigenvalues ± 1 such that $A_1^2 - I_n = \gamma B_1 \bar{B}_1$ and $A_1 B_1 = B_1 A_1$. Furthermore s is semisimple if and only if A_1 is semisimple. Two such elements $s_\gamma(A_1, B_1)$ and $s_\gamma(A_2, B_2)$ are H_E conjugate to each other if and only if A_1 and A_2 are conjugate to each other.

PROOF. The first assertion is proved in [G2]. Now we assume that

$$s = \begin{pmatrix} A & \gamma B \\ \bar{B} & A \end{pmatrix}$$

where $A \in M(n,F)$ has no eigenvalues ± 1 such that $A^2 - I_n = \gamma B \bar{B}$ and $AB = BA$. We have that

$$\operatorname{diag}(\gamma B, I_n)^{-1} \begin{pmatrix} A & \gamma B \\ \bar{B} & A \end{pmatrix} \operatorname{diag}(\gamma B, I_n) = \begin{pmatrix} A & I_n \\ A^2 - I_n & A \end{pmatrix} = t(A) \in S.$$

In other words s is conjugate to $t(A)$ in $\mathrm{GL}(2n, E)$. Applying the Lemma 4.3 of [JR], we find that s is semisimple if and only if A is semisimple. To prove the last assertion, we note that for two such elements $S_\gamma(A_1, B_1)$ and $S_\gamma(A_2, B_2)$, if we set $\alpha_i = \gamma^{-1} B_i (A_i + I_n)^{-1} \in G_n(E)$ $(i = 1, 2)$, then

$$A_i = (I_n + \alpha_i \bar{\alpha}_i)(I_n - \alpha_i \bar{\alpha}_i)^{-1}, \quad B_i = 2\gamma \alpha_i (I_n - \alpha_i \bar{\alpha}_i)^{-1}.$$

It is easy to check that if $h \in G_n(E)$, then

$$\operatorname{diag}(h, \bar{h}) s_\gamma(A_1, B_1) \operatorname{diag}(h^{-1}, \bar{h}^{-1}) = s_\gamma(A_2, B_2)$$

if and only if $h \alpha_1 \bar{h}^{-1} = \alpha_2$. In other words, $s_\gamma(A_1, B_1)$ is H_E conjugate to $S_\gamma(A_2, B_2)$ if and only if α_1 is twisted conjugate to α_2. By Lemma 1.3, this last condition is equivalent to the condition that $\alpha_1 \bar{\alpha}_1$ is conjugate to $\alpha_2 \bar{\alpha}_2$, which is the same as saying that A_1 is conjugate to A_2. We are done.

Let n_1, n_2 be two integers such that $0 \leq n_1 \leq n_1 + n_2 \leq n$ and let A be an element in $M(n_1, F)$ such that $A^2 - I_{n_1} \in \gamma N G_{n_1}(E)$. Then we denote by $s_\gamma(A, n_1, n_2)$ the matrix

$$(1.5) \qquad \begin{pmatrix} A & 0 & 0 & \gamma B_A & 0 & 0 \\ 0 & I_{n_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -I_{n-n_1-n_2} & 0 & 0 & 0 \\ \bar{B}_A & 0 & 0 & A & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -I_{n-n_1-n_2} \end{pmatrix}$$

where B_A is a fixed matrix such that $A^2 - I_n = \gamma B_A \bar{B}_A, B_A A = A B_A$. We will write $s_\gamma(A, n, 0)$ simply as $s_\gamma(A)$. We have:

PROPOSITION 1.2. *Each semisimple element $s \in G_\gamma$ is conjugate to an element of the form $s_\gamma(A, n_1, n_2)$ where A is semisimple. The set of H conjugacy classes of semisimple elements of S_γ is in one to one correspondence with the set of all triples $(n_1, \{A\}, n_2)$, where $0 \le n_1 \le n$ is an integer, $\{A\}$ a semisimple conjugacy classes in $M(n,F)$ with $A^2 - 1 \in \gamma NG_n(E)$, and n_2 is an integer with $0 \le n_2 \le n - n_1$.*

PROOF. In [G2], we have proved each element $s \in S_\gamma$ is H_E conjugate to an element of the form

$$(1.6) \qquad \begin{pmatrix} A_1 & 0 & 0 & \gamma B_1 & 0 & 0 \\ 0 & A_2 & 0 & 0 & \gamma B_2 & 0 \\ 0 & 0 & A_3 & 0 & 0 & \gamma B_3 \\ \bar{B}_1 & 0 & 0 & A_1 & 0 & 0 \\ 0 & \bar{B}_2 & 0 & 0 & \bar{A}_2 & 0 \\ 0 & 0 & \bar{B}_3 & 0 & 0 & \bar{A}_3 \end{pmatrix},$$

where if we set

$$s_i = \begin{pmatrix} A_i & \gamma B_i \\ \bar{B}_i & \bar{A}_i \end{pmatrix} \in G_{\gamma, n_i},$$

then $s_i \in S_{\gamma,n_i}$ such that $A_1 \in M(n,F)$ with $A_1^2 - I_n = \gamma B_1 \bar{B}_1$, $A_1 B_1 = B_1 A_1$, and both s_2 and $-s_3$ are unipotent elements. Observing that in this case the element s is in fact conjugate to $\mathrm{diag}(s_1, s_2, s_3)$ in $\mathrm{GL}(2n, E)$, we have that s is semisimple if and only if s_1, s_2, s_3 are semisimple. Equivalently A_1 is semisimple by the above lemma, $s_2 = I_{2n_2}$ and $s_3 = -I_{2n_3}$. Hence s is semisimple if and only if it is H_E conjugate to an element of the form $s_\gamma(A, n_1, n_2)$ where A is semisimple. The last assertion of the proposition follows immediately from Lemma 1.5.

As before we say that a semisimple element $s \in S_\gamma$ is θ_γ-regular if the H_E conjugacy class of s has the maximal dimension among all the H_E conjugacy classes in S_γ. We denote by S_γ^r the set of θ_γ-regular elements in S_γ.

LEMMA 1.6. *Each element in S_γ^r is H_E conjugate to an element of the form $s_\gamma(A)$ where A is regular. The H_E conjugacy classes is in a bijective correspondence with the conjugacy classes in $M(n,F)$ of those elements A such that $A^2 - I_n \in \gamma NG_n(E)$.*

PROOF. We first consider an element of the form $s_\gamma(A)$ in S_γ. We set

$$\alpha_A = \gamma^{-1} B_A (I_n + A)^{-1} \in G_n(E).$$

Then as in the proof of Lemma 1.5, we have that

$$\mathrm{diag}(h, \bar{h}) s_\gamma(A) \,\mathrm{diag}(h^{-1}, \bar{h}^{-1}) = s_\gamma(A), \quad h \in G_n(E)$$

if and only if $h\alpha_A \bar{h}^{-1} = \alpha_A$. Thus if we identify H_E with $G_n(E)$ in the natural way, then $H_E^{s_\gamma(A)}$ is the twisted centralizer $G_n(E)^{\alpha_A}_{-}$ of α_A in $G_n(E)$. Hence we have $\dim H_E^{s_\gamma(A)} \ge n$ and $\dim H_E^{s_\gamma(A)} = n$ if and only if α_A is twisted regular which means that $\alpha_A \bar{\alpha}_A$, hence $A = (I_n + \alpha_A \bar{\alpha}_A)(I_n - \alpha_A \bar{\alpha}_A)$, is regular. If this is the case, we have $H_E^{s_\gamma(A)} = G_n(F)^A$ if we regard $G_n(F)$ as a subgroup of H_E.

Generally, for a semisimple element $s \in S_\gamma$, we can assume that $s = s_\gamma(A, n_1, n_2)$ where $A \in M(n_1, F)$ such that $A^2 - I_{n_1} = \gamma B_A \bar{B}_A$ and $0 \leq n_1 \leq n_1 + n_2 \leq n$. Then we have

$$\dim H_E^s = \dim H_{E,n_1}^{s_\gamma(A)} + 2n_2^2 + 2n_3^2 \geq n_1 + n_2 + n_3 = n$$

and the equality occurs if and only if $n_1 = n, n_2 = n_3 = 0$ and A is regular. Hence the assertion of the lemma.

We say that an element $s \in S_\gamma^r$ is θ_γ- regular elliptic if H_E^s is an elliptic torus. We denote by S_γ^e the set of such elements. Then for $s \in S_\gamma$, it is easily seen that $s \in S_\gamma^e$ if and only if s is conjugate to an element $s_\gamma(A) \in S_\gamma$ such that A is regular elliptic in $M(n, F)$.

We say that an element $g \in G_\gamma$ is θ_γ-regular (resp. θ_γ-regular elliptic) if $\rho_\gamma(g) \in S_\gamma^r$ (resp. S_γ^e). We use G_γ^r and G_γ^e to denote the set of the θ_γ-regular elements and the set of the θ_γ-regular elliptic elements respectively. For $\alpha \in G_n(E)$ with $\det(I_n - \gamma\alpha\bar{\alpha}) \neq 0$, we denote by $g_\gamma(\alpha)$ the matrix

$$\begin{pmatrix} I_n & \gamma\alpha \\ \bar{\alpha} & I_n \end{pmatrix}.$$

LEMMA 1.7. *Each element $g \in G_\gamma^r$ is in the same double coset as an element $g_\gamma(\alpha)$ where $\alpha\bar{\alpha}$ is a regular element in $G_n(F)$ such that* $\det(I_n - \gamma\alpha\bar{\alpha}) \neq 0$. *Furthermore g is θ_γ-regular elliptic if and only if $\alpha\bar{\alpha}$ is elliptic. The double classes $H_E \setminus G_\gamma^r / H_E$ is in one to one correspondence with the twisted conjugacy classes of twisted regular elements in $G_n(E)$ whose norms have no eigenvalue γ^{-1}.*

PROOF. Let $W_\gamma = W \cap \gamma NG_n(E)$ and $U_\gamma = U \cap \{A \in M(n, F) \mid A^2 - I_n \in \gamma NG_n(E)\}$. (See the proof of Lemma 1.4 for the notations.) For $a \in W$, we have

$$\lambda(a)^2 - I_n = 4a(I_n - a)^{-2}.$$

Here $4(I_n - a)^{-2} \in NG_n(E)$. So $\lambda(a)^2 - I_n \in \gamma NG_n(E)$ if and only if $a \in \gamma NG_n(E)$. Hence the inverse Cayley map λ induces a bijective correspondence between W_γ and U_γ. For $\alpha \in G_n(E)$ such that $\gamma\alpha\bar{\alpha} \in W_\gamma$, we have

$$\rho_\gamma\big(g_\gamma(\alpha)\big) = s_\gamma\big(\lambda(\gamma\alpha\bar{\alpha})\big).$$

Here $s_\gamma\big(\lambda(\gamma\alpha\bar{\alpha})\big)$ is θ_γ-regular (θ_γ-regular elliptic) if and only if $\lambda(\gamma\alpha\bar{\alpha})$ is regular (regular elliptic), which is the same as saying that α is twisted regular (twisted regular elliptic). Then the first and the second assertions follow from Lemma 1.6.

It is easily seen that $g_\gamma(\alpha_1)$ and $g_\gamma(\alpha_2)$ are in the same double coset if and only if α_1 and α_2 are twisted conjugate. Recall that we have a bijective map $\mathcal{N}$ between the twisted conjugacy classes in $G_n(E)$ and the conjugacy classes in $NG_n(E)$. The the map $\mathcal{N}_\gamma$ defined by $\mathcal{N}_\gamma(\alpha) = \gamma\mathcal{N}(\alpha)$ is a bijective map between the twisted conjugacy classes in $\{\alpha \in G_n(E) \mid \det(I_n - \gamma\alpha\bar{\alpha}) \neq 0\}$ and the conjugacy classes in W_γ. Hence the last assertion of the lemma.

We now consider the correspondence between the H_E conjugacy classes in S_γ and the H conjugacy classes in S. We note that both S_γ and S are subsets of $G_{2n}(E)$. The following simple fact will be used repeatedly. Let

$$g_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$$

are two elements in $G_{2n}(E)$ where A_i are elements in $M(n,E)$. If g_1 is $H(E)$ conjugate to g_2, then A_1 is conjugate to A_2.

We first consider the H_E conjugacy class of $s_\gamma(A)$ where, as usual, $A \in M(n,F)$ and $A^2 - I_n \in \gamma NG_n(E)$. We have

$$\begin{pmatrix} (A-I_n)B_A^{-1} & 0 \\ 0 & I_n \end{pmatrix} s_\gamma(A) \begin{pmatrix} B_A(A-I_n)^{-1} & 0 \\ 0 & I_n \end{pmatrix} = s(A).$$

In other words $s_\gamma(A)$ is $H(E)$ conjugate to $s(A)$. Now suppose $s_\gamma(A)$ is $H(E)$ conjugate to another element

$$s_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in S.$$

Then A_1 is conjugate to A. Thus by Lemma 1.1, we have that s_1 is H conjugate to $s(A)$. On the other hand if

$$s_2 = \begin{pmatrix} A_2 & \gamma B_2 \\ \bar{B}_2 & \bar{A}_2 \end{pmatrix} \in S_\gamma$$

is also $H(E)$ conjugate to $s(A)$. Then A_2 is conjugate to the element A. So by Lemma 1.5, s_2 is H_E conjugate to $s_\gamma(A)$. Combining this considerations with Proposition 1.1 and Proposition 1.2, we get:

PROPOSITION 1.3. *Suppose s is a semisimple element of S_γ. Then there is $g \in H(E)$ such that $gsg^{-1} \in S$. This establishes an injection of the H_E semisimple conjugacy classes in S_γ into the H semisimple conjugacy classes in S which carries the conjugacy class of $s_\gamma(A,n_1,n_2)$ in S_γ to the conjugacy class of $s(A,n_1,n_2)$ in S where $A \in M(n,F)$ such that $A^2 - I_n \in \gamma NG_n(E)$ and $0 \le n_1 \le n_1 + n_2 \le n$.*

We can in fact define an injection of the general H_E conjugacy classes in S_γ into the H conjugacy classes in S in exactly the same way. Recall that each element in S_γ is H_E conjugate to an element of the form (1.6). The same thing is true for an element in S. So we are reduced to study the element $s \in S_\gamma$ such that s or $-s$ is unipotent. The point here is that in this case we can find $h \in H_E$ such that hsh^{-1} is in $S_\gamma \cap S$ (see [G2]), and we can also prove that this induces an injection of the H_γ conjugacy classes of elements in S_γ of this type into the H conjugacy classes of the same kind of elements in S. Since this will not be used in this paper, we omit the details.

Finally we study more especially the θ-regular elliptic elements. We denote by Y_γ^e the subset

$$\big\{\{s(A)\} \mid A \text{ is regular elliptic such that } A^2 - I_n \in \gamma NG_n(E)\big\}$$

of the set of the H conjugacy classes $\{S^e\}$ in S, which is the image of the set of the H_E conjugacy classes $\{S_\gamma^e\}$ in S_γ. Then we have

LEMMA 1.8. *When n is odd, the subsets Y^e_γ and $Y^e_{\gamma_1}$ are disjoint if $\gamma\gamma_1^{-1}$ is not in NE^*. Furthermore we have*

$$\bigcup_{\gamma \in F^*/NE^*} Y^e_\gamma = \{S^e\}.$$

So we have a bijective correspondence between the disjoint union $\bigsqcup_{\gamma \in F^*/NE^*}\{S^e_\gamma\}$ and $\{S^e\}$.

When n is even, then for each γ we have

$$Y^e_\gamma = \big\{\{s(A)\} \mid A \text{ is regular elliptic in } G_n(F) \text{ with } \det(A^2 - I_n) \in NG_n(E)\big\}$$

which does not depend on γ.

PROOF. We consider the element $S(A) \in S$ where $A \in U$ (see the proof of Lemma 1.3). Then there exists $a \in W$ such that

$$A = \lambda(a) = (I_n + a)(I_n - a)^{-1}.$$

It is clear that A is regular elliptic if and only if a is regular elliptic. Then as in the proof of Lemma 1.7, we have

$$A^2 - I_n = 4a(I_n - a)^{-2}.$$

Since $4(I_n - a)^{-2} \in NGL_n(E)$, so $A^2 - I_n \in \gamma NG_n(E)$ if and only if $a \in \gamma NG_n(E)$. It is well known that if a is regular elliptic, then $\gamma^{-1}a \in NG_n(E)$ if and only if $\det \gamma^{-1}a \in NE^*$ (Lemma 1.3, Chapter 1 in [AC]). This implies that if A is regular elliptic, then $A^2 - I_n \in \gamma NG_n(E)$ if and only if

$$\gamma^n \det(A^2 - I_n) \in NE^*. \tag{1.7}$$

Now if n is odd, then the condition (1.7) is equivalent to

$$\gamma \det(A^2 - I_n) \in NE^*.$$

Note that we also have

$$\bigcup_{\gamma \in F^*/NE^*} NE^* = F^*.$$

Then we obtain the first part of the lemma.

If n is even, then the condition (1.7) is equivalent to

$$\det(A^2 - I_n) \in NE^*.$$

The second part of the lemma follows.

By this lemma, we can also find easily that, if n is odd, then we have a injection from the disjoint union $\bigsqcup_{\gamma \in F^*/NE^*}\{S^r_\gamma\}$ to $\{S^r\}$. But it is not surjective except for $n = 1$.

1.2. We now assume that F is a non-archimedean local field, E is a quadratic extension field of F. Let η be the quadratic character of F^* attached to E. We use $|\ |_F$ to denote the absolute value on F for which we have $|\varpi|_F = q^{-1}$ where ϖ is a uniformizer of F and q is the number of the elements of the residue field of F.

We call an element $s \in S$ relevant if $\eta|_{H^s} = 1$. It is easily seen that a semisimple element $s(A, n_1, n_2)$ is relevant if and only if $n_1 = n, n_2 = 0$ (see the proof of Lemma 1.2). Recall that the H conjugacy class of a semisimple element in S is closed ([R]). Then for $f \in C_c^\infty(S)$ and a relevant semisimple element $s \in S$, the following orbital integral is well defined

$$(1.8) \qquad H(s;f;\eta) = \int_{H/H^s} f(hsh^{-1})\eta(\det h)\,dh.$$

If $s = s(A)$, we will simply denote $H\big(s(A) : f; \eta\big)$ by $H(A;f;\eta)$.

REMARK. For a relevant semisimple element $s \in S$ and $h \in H$, we have

$$H(hsh^{-1};f;\eta) = \eta(\det h)H(s;f;\eta).$$

So our orbital integral is not constant on a double coset. But for a semisimple element $A \in M(n, F)$ without eigenvalues ± 1 and an element $h \in G_n$, we have that

$$H(hAh^{-1};f;\eta) = \eta(\det h^2)H(A;f;\eta) = H(A;f;\eta).$$

Hence $H(A;f;\eta)$ depends only on the conjugacy classes of A under G_n.

For $g \in G$, we let

$$(1.9) \qquad H_g = H \cap g^{-1}Hg = \{h \in H \mid hg = gh' \text{ for some } h' \in H\}.$$

We call g relevant if $\eta|_{H_g} = 1$. It is easy to check that $g \in G$ is relevant if and only if $\rho(g)$ is relevant in S. For a function $f \in C_c^\infty(G)$ and a relevant element $g \in G$ such that $\rho(g)$ is semisimple (this is the same as saying that HgH is closed [R]), the following orbital integral is convergence

$$(1.10) \qquad H(g;f;\eta) = \int_{H/H_g}\int_H f(h_1gh_2)\eta(\det h_1)\,dh_1\,dh_2.$$

If $g = g(a)$ where a is an element in G_n without eigenvalue 1, then we simply denote $H(g;f;\eta)$ by $H(a;f;\eta)$. If for $f \in C_c^\infty(G)$, we define a function $\rho(f) \in C_c^\infty(S)$ by the formula

$$\rho(f)\big(\rho(g)\big) = \int_H f(gh)\,dh,$$

then we have

$$H(g;f;\eta) = H\big(\rho(g); \rho(f); \eta\big)$$

for $g \in G$, and

$$H(a;f;\eta) = H\big(\lambda(a); \rho(f); \eta\big),$$

for $a \in W$. So $H(g;f;\eta)$ is not constant on the double cosets, but $H(a;f;\eta)$ depends only on the conjugacy classes of a in G_n.

Let f be a smooth function with compact support on S_γ. We define an orbital integral for f at a semisimple element $s \in S_\gamma$ by

$$H_\gamma(s;f) = \int_{H_E/H_E^s} f(hsh^{-1})\,dh.$$

If $s = s_\gamma(A)$, we use $H_\gamma(A;f)$ to denote $H_\gamma(s;f)$. In this case, $H_\gamma(s;f)$ depends only on the H_E conjugacy classes of s. Hence $H_\gamma(A;f)$ depends only on the conjugacy classes of A.

For $g \in G_\gamma$ such that $\rho_\gamma(g)$ is semisimple, then H_EgH_E is closed. So we can also define an orbital integral of $f \in C_c^\infty(G)$ at g as follows

$$H_\gamma(g;f) = \int_{H_E/(H_E)_g} \int_{H_E} f(h_1gh_2)\,dh_1\,dh_2 \tag{1.11}$$

where $(H_E)_g = H_E \cap g^{-1}H_Eg$. If $g = g_\gamma(\alpha)$ where $\alpha \in G_n(E)$ such that $\det(I_n - \gamma\alpha\bar{\alpha}) \neq 0$, we will denote $H_\gamma(g;f)$ by $H_\gamma(\alpha;f)$. As before we have

$$H_\gamma(g;f) = H_\gamma\big(\rho_\gamma(g);\rho_\gamma(f)\big), \quad \text{and} \quad H_\gamma(\alpha;f) = H_\gamma\big(\lambda(\gamma\alpha\bar{\alpha});\rho_\gamma(f)\big)$$

where

$$\rho_\gamma(f)\big(\rho_\gamma(g)\big) = \int_{H_E} f(gh)\,dh.$$

If $\gamma = 1$, then G_1 is isomorphic to G over F. For convenience, we will denote (G_1, H_E) by (G', H'), and denote the orbital integral $H_1(g;f)$ by $H(g;f)$ in all cases. Let R_F and R_E be the rings of the integers of F and E. We set $K = G(R_F)$ and $K' = G_{2n}(R_E) \cap G'$. They are maximal compact subgroups of G and G' respectively. Suppose F has odd residual characteristic and E is unramified over F. Then as usual, one of the ingredients in the comparison of trace formulas is the "fundamental lemma" which involves matching the orbital integrals of Hecke functions. In our case, the (conjectural) fundamental lemma has the following form.

FUNDAMENTAL LEMMA. *Let f be a Hecke function on G and let f' be the corresponding Hecke function on G'. Then for a regular element $a \in G_n(F)$ without eigenvalue 1, we have*

$$H(a;f;\eta) = \begin{cases} H(\alpha;f'), & \text{if } a = \alpha\bar{\alpha} \\ 0, & \text{if } a \notin NG_n(E). \end{cases} \tag{1.12}$$

The main result of this paper is that the fundamental lemma is true for unit Hecke functions. More precisely we have:

THEOREM. *Let f_0 and f_0' be the characteristic functions of K and K' respectively. Then the identity (1.12) is true when f and f' are f_0 and f_0' respectively.*

This theorem is the initial step in our project described in the introduction. We also expect that we can deduce the fundamental lemma for general Hecke functions from this result for the unit Hecke functions following the ideas of Clozel [C] or Labesse [L] for stable base change.

Another local result we need is the existence of the transfer of smooth function f with compact support on G to the smooth function f_γ with compact support on G_γ where f and $\{f_\gamma\}$ have "matching" orbital integrals at regular elements. In the case $G = G_4$, this was done by the author in an unpublished note.

The remaining part of the paper is arranged as follows. In Section 2 we prove a reduction formula that can be used to reduce the comparison of orbital integrals of Hecke functions to the case of elliptic orbital integrals. Then for a regular elliptic element $r = \alpha\bar{\alpha}$ in $G_n(F)$, we construct a Hecke function Ψ_r of $G_n(F)$ and a Hecke function Φ_α of $G_n(E)$ such that $H(r;f_0;\eta)$ and $H(\alpha;f_0')$ are represented by the usual conjugate orbital integral of Ψ_r at r and the usual twisted conjugate orbital integral of Φ_α at α. We then prove that Ψ_r and Φ_α are related by the base change map. These are done in Section 3. In conclusion we prove our theorem in Section 4.

2. **A reduction formula.** In this section we will prove a reduction formula which relates the orbital integrals on $G(F)$ to the orbital integrals on the Levi subgroups of $G(F)$. This formula can be used to reduce the comparison of the orbital integrals of Hecke functions to the case of elliptic orbital integrals.

2.1. We first introduce some notations. Let $(m_1, m_2, \ldots, m_l)$ be a partition of a positive integer m. Let T_i be a subgroup of $G_{m_i}(F)$. For $t_i \in T_i$ $(i = 1, 2, \ldots, l)$, we use $(t_1, \ldots, t_l)$ to denote the element $\mathrm{diag}(t_1, \ldots, t_l)$ of $G_m(F)$. Then $T_1 \times \cdots \times T_l$ is the subgroup of $G_m(F)$ consisting of such elements.

Given a regular semisimple element r of $G_n(F)$, we may assume

$$r = (r_1, \ldots, r_l) \tag{2.1}$$

up to a conjugation. Here r_i is a regular elliptic element of $G_{n_i}(F)$. Let P_1 be the standard parabolic subgroup of $G_n(F)$ associated to the partition $n = n_1 + \cdots + n_l$. We have a decomposition

$$P_1 = M_1 N_1 = N_1 M_1$$

where M_1 is the standard Levi subgroup and N_1 is the unipotent radical of P_1.

We use P to denote the standard parabolic subgroup of $G(F)$ of type $(2n_1, \ldots, 2n_l)$. Let M and N be the standard Levi subgroup and the unipotent radical respectively of P. Then we have

$$M = G_{2n_1}(F) \times \cdots \times G_{2n_l}(F). \tag{2.2}$$

We now define an automorphism θ of G by

$$\begin{pmatrix} (A_{i,j})_{1\le i,j\le l} & (B_{i,j})_{1\le i,j\le l} \\ (C_{i,j})_{1\le i,j\le l} & (D_{i,j})_{1\le i,j\le l} \end{pmatrix} \mapsto (Q_{i,j})_{1\le i,j\le l} \tag{2.3}$$

where $A_{i,j}$, $B_{i,j}$, $C_{i,j}$, $D_{i,j}$ are $n_i \times n_j$ matrices and

$$Q_{i,j} = \begin{pmatrix} A_{i,j} & B_{i,j} \\ C_{i,j} & D_{i,j} \end{pmatrix} \tag{2.4}$$

is $2n_i \times 2n_j$ matrix. Let

$$x_0 = \begin{pmatrix} 1_{n_1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1_{n_1} & 0 & \cdots & 0 \\ 0 & 1_{n_2} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1_{n_2} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1_{n_l} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1_{n_l} \end{pmatrix}.$$

The map θ is actually the inner automorphism defined by $\theta(g) = x_0 g x_0^{-1}$. Since $x_0 \in K$, We have $\theta(K) = K$. We remind that $P_1 \times P_1$ is regarded as a subgroup of G_{2n} in the way that it is a subgroup of $G_n \times G_n \subset G_{2n}$. Let H_M be the image of $M_1 \times M_1$ under the map θ. Then H_M is the subgroup

$$G_{n_1}(F) \times G_{n_1}(F) \times \cdots \times G_{n_l}(F) \times G_{n_l}(F) \tag{2.5}$$

of M. We can extend the definition of our orbital integrals to (M, H_M) in a natural way. Let N_2 be the subgroup $\theta(N_1 \times N_1)$. It is easy to see that each element of N_2 has the form

$$\begin{pmatrix} 1_{2n_1} & (U_{12}, U'_{12}) & \cdots & (U_{1l}, U'_{1l}) \\ 0 & 1_{2n_2} & \cdots & (U_{2l}, U'_{2l}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1_{2n_l} \end{pmatrix}$$

where U_{ij} and U'_{ij} are $n_i \times n_j$ matrices and (U_{ij}, U'_{ij}) is used to denote

$$\begin{pmatrix} U_{ij} & 0 \\ 0 & U'_{ij} \end{pmatrix}.$$

It is clear that N_2 is contained in N.

In order to state our proposition we need to introduce another notation. For a Hecke function f on G, we define a function $f^{(P)}$ on M as follows:

$$f^{(P)}(m) = \delta_P(m)^{\frac{1}{2}} \int_N f(mn)\, dn. \tag{2.6}$$

Here δ_P is the usual module on P. We have

$$\delta_P(m) = \frac{dmnm^{-1}}{dn} = \prod_{i<j} \frac{|\det m_i|^{m_j}}{|\det m_j|^{m_i}} \tag{2.7}$$

for $n \in N$ and $m = (m_1, \ldots, m_l) \in M$. The function $f^{(P)}$ is actually a Hecke function of M and the map

$$f \longmapsto f^{(P)}$$

is an isomorphism from $H(G,K)$ to $H(M, M\cap K)$. In particular $f^{(P)}$ is the unit Hecke function of M if f is the unit Hecke function of $G(F)$.

Back to the element r given in (2.1). Let F_1 be a finite Galois extension field of F such that r splits in $G_n(F_1)$. Let r_{im_i} $(1 \leq m_i \leq n_i)$ be the eigenvalues of r_i in $G_{n_i}(F_1)$ $(1 \leq i \leq l)$. Then the element

$$\prod_{\substack{1\leq m_i\leq n_i\\ 1\leq m_j\leq n_j}} (r_{im_i} - r_{jm_j})$$

is in F. We use $\lambda(r)$ to denote the value

$$\prod_{1\leq i<j\leq l} \frac{|\det(1_{n_i} - r_i)|_F^{n_j} |\det(1_{n_j} - r_j)|_F^{n_i}}{\left|\prod_{\substack{1\leq m_i\leq n_i\\ 1\leq m_j\leq n_j}} (r_{im_i} - r_{jm_j})\right|_F}. \tag{2.8}$$

2.2. We have:

PROPOSITION 2.1. *Let $f \in H(G,K)$ and let $r \in G_n(F)$ be as in (2.1), then:*

$$H(r;f;\eta) = \lambda(r) H\left(\left(\begin{pmatrix} 1_{n_1} & r_1 \\ 1_{n_1} & 1_{n_1}\end{pmatrix}, \dots, \begin{pmatrix} 1_{n_l} & r_l \\ 1_{n_l} & 1_{n_l}\end{pmatrix}\right); f^{(P)}; \eta\right). \tag{2.9}$$

PROOF. For any positive integer m, we normalize the Haar measure on $G_m(F)$ so that the mass of $K_m = G_m(R_F)$ is one. By the definition of our orbital integral we have:

$$H(r;f;\eta) = \int_{H(F)/H_r} \int_{H(F)} f\left(h_1 \begin{pmatrix} 1_n & r \\ 1_n & 1_n\end{pmatrix} h_2\right) \eta(\det h_1)\, dh_1\, dh_2.$$

Since r is a regular semisimple element in M_1, it is easily seen that H_r is equal to

$$(M_1 \times M_1) \cap r^{-1}(M_1 \times M_1) r$$

which is contained in $M_1 \times M_1$. Applying the Iwasawa Decomposition

$$H(F) = \big(H(F) \cap K\big)(N_1 \times N_1)(M_1 \times M_1)$$

to the integral we get:

$$H(r;f;\eta) = \int_{M_1\times M_1/H_r} \int_{M_1\times M_1} \int_{N_1\times N_1} \int_{N_1\times N_1} f\left(n_1 m_1 \begin{pmatrix} 1_n & r \\ 1_n & 1_n\end{pmatrix} m_2 n_2\right) \eta(\det m_1)\, dn_1\, dn_2\, dm_1\, dm_2.$$

We now consider the action of the map θ. Since f is a Hecke function, we have $f\circ\theta = f$. Let $\theta(r)$ be the image of

$$\begin{pmatrix} 1_n & r \\ 1_n & 1_n\end{pmatrix}$$

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under θ. Then we find

$$\theta(r) = \left(\begin{pmatrix} 1_{n_1} & r_1 \\ 1_{n_1} & 1_{n_1} \end{pmatrix}, \ldots, \begin{pmatrix} 1_{n_l} & r_l \\ 1_{n_l} & 1_{n_l} \end{pmatrix} \right) \tag{2.10}$$

which is in M. Therefore with the action of θ, our integral becomes

$$\int_{H_M/(H_M)_{\theta(r)}} \int_{H_M} \int_{N_2} \int_{N_2} f\big(n_1 m_1 \theta(r) m_2 n_2\big) \eta(\det m_1)\, dn_1\, dn_2\, dm_1\, dm_2. \tag{2.11}$$

where

$$(H_M)_{\theta(r)} = H_M \cap \theta(r)^{-1} H_M \theta(r)$$

which is equal to

$$\theta\big((M_1 \times M_1) \cap r^{-1}(M_1 \times M_1) r\big) = \theta(H_r).$$

Let

$$m = m_1 \theta(r) m_2 \in M.$$

Thus we can write the integrand of our integral in the form

$$f(n_1 m n_2) = f(m m^{-1} n_1 m n_2).$$

Consider the map

$$(n_1, n_2) \longmapsto m^{-1} n_1 m n_2 \tag{2.12}$$

from $N_2 \oplus N_2$ to N. We will find that it is bijective. Let $J(m)$ be the Jacobian of this map. We obtain:

$$H(r; f; \eta) = \int_{H_M/(H_M)_{\theta(r)}} \int_{H_M} \int_N f(mn) \big(1/J(m)\big) \eta(\det m_1)\, dn\, dm_1\, dm_2. \tag{2.13}$$

To continue, we must compute $J(m)$. First of all we write (2.12) in the form

$$(n_1, n_2) \longmapsto m_2^{-1} [\theta(r)^{-1} (m_1^{-1} n_1 m_1) \theta(r) (m_2 n_2 m_2^{-1})] m_2. \tag{2.14}$$

By formula (2.7), we find

$$\delta_P\Big(\theta\big((x_1, x_2)\big)\Big) = \delta_{P_1}(x_1 x_2)^2$$

for $x_1, x_2 \in M_1$. Recall that $H_M = \theta(M_1 \times M_1)$ and $N_2 = \theta(N_1 \times N_1)$. It follows that the Jacobian of the map

$$n_1 \longmapsto m_1^{-1} n_1 m_1 \quad \text{and} \quad n_2 \longmapsto m_2 n_2 m_2^{-1}$$

from $N_2 \longrightarrow N_2$ are

$$\delta_P(m_1)^{-1/2} \quad \text{and} \quad \delta_P(m_2)^{1/2}$$

respectively.

Next we consider the Jacobian J of the map

$$(n_1, n_2) \longmapsto \theta(r)^{-1} n_1 \theta(r) n_2$$

from $N_2 \oplus N_2$ to N. Let

$$n_1 = \begin{pmatrix} 1_{2n_1} & (U_{12}, U'_{12}) & \cdots & (U_{1l}, U'_{1l}) \\ 0 & 1_{2n_2} & \cdots & (U_{2l}, U'_{2l}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1_{2n_l} \end{pmatrix}$$

and

$$n_2 = \begin{pmatrix} 1_{2n_1} & (V_{12}, V'_{12}) & \cdots & (V_{1l}, V'_{1l}) \\ 0 & 1_{2n_2} & \cdots & (V_{2l}, V'_{2l}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1_{2n_l} \end{pmatrix}$$

where U_{ij}, U'_{ij}, V_{ij}, V'_{ij} are $n_i \times n_j$ matrices. We denote the matrix

$$\begin{pmatrix} 1_{n_i} & r_i \\ 1_{n_i} & 1_{n_i} \end{pmatrix}$$

by X_i for $i = 1, 2, \ldots, l$. Let W_{ij} $(1 \leq i \leq l, 1 \leq j \leq n_i)$ be the matrix

$$X_i^{-1}(U_{ij}, U'_{ij})X_j + (V_{ij}, V'_{ij}). \tag{2.15}$$

Then we have

$$\theta(r)^{-1} n_1 \theta(r) n_2 = \begin{pmatrix} 1_{2n_1} & W_{12} & \cdots & W_{1l} \\ 0 & 1_{2n_2} & \cdots & W_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1_{2n_l} \end{pmatrix}.$$

The formula (2.15) determines a F-linear map from $F^{4n_in_j}$ to $F^{4n_in_j}$. Let J_{ij} be the corresponding Jacobian. Then we have $J = \prod_{1 \leq i < j \leq l} J_{ij}$. So we are reduced to computing J_{ij}. From (2.15), we find W_{ij} is

$$\begin{pmatrix} * + V_{ij} & (1_{n_i} - r_1)^{-1}(U_{ij}r_j - r_iU'_{ij}) \\ (1_{n_i} - r_i)^{-1}(U'_{ij} - U_{ij}) & * + V'_{ij} \end{pmatrix}.$$

Here $*$ indicates matrices not related to V_{ij} and V'_{ij}. Thus the only nontrivial part we need to compute is the Jacobian of the map

$$(U_{ij}, U'_{ij}) \longmapsto (U_{ij}r_j - r_iU'_{ij}, U'_{ij} - U_{ij})$$

from $F^{2n_in_j}$ to $F^{2n_in_j}$. We now go for a moment to the field F_1. Let

$$r_i = s_i(r_{i1}, r_{i2}, \ldots, r_{in_i})s_i^{-1}$$

and

$$r_j = s_j(r_{j1}, r_{j2}, \ldots, r_{jn_j})s_j^{-1}$$

for $s_i \in G_{n_i}(F_1)$ and $s_j \in G_{n_j}(F_1)$. Then by a suitable change of basis of $F_1^{2n_in_j}$, we just need to consider the Jacobian of the map

$$(U_{ij}, U'_{ij}) \longmapsto \big(U_{ij}(r_{j1}, \ldots, r_{jn_j}) - (r_{i1}, \ldots, r_{in_i})U'_{ij}, U'_{ij} - U_{ij}\big).$$

In this case an elementary computation shows that the result is

$$\Big|\prod_{\substack{1\le m_i\le n_i\\ 1\le m_j\le n_j}} (r_{im_i} - r_{jm_j})\Big|_F.$$

Thus we find

$$J_{ij} = \frac{\big|\prod_{\substack{1\le m_i\le n_i\\ 1\le m_j\le n_j}} (r_{im_i} - r_{jm_j})\big|_F}{|\det(1_{n_i} - r_i)|_F^{2n_j}}.$$

Taking the product of all J_{ij}, we obtain

$$\begin{aligned} J &= \prod_{1\le i<j\le l} \frac{\big|\prod_{\substack{1\le m_i\le n_i\\ 1\le m_i\le n_i}} (r_{im_i} - r_{jm_j})\big|_F}{|\det(1_{n_i} - r_i)|_F^{2n_j}} \\ &= \prod_{i<j} \frac{\big|\prod_{\substack{1\le m_i\le n_i\\ 1\le m_j\le n_j}} (r_{im_i} - r_{jm_j})\big|_F}{|\det(1_{n_i} - r_i)|^{n_j}|\det(1_{n_j} - r_j)|^{n_i}} \prod_{i<j} \frac{|\det(1_{n_j} - r_j)|^{n_i}}{|\det(1_{n_i} - r_i)|^{n_j}} \\ &= \lambda(r)^{-1}\delta_P\big(\theta(r)\big)^{-1/2}. \end{aligned}$$

Finally the Jacobian of the map $n \longmapsto m_2^{-1}nm_2$ from N to N is $\delta_P(m_2)^{-1}$. Therefore

$$J(m) = \delta_P(m_1)^{-1/2}\delta_p(m_2)^{1/2}\lambda(r)^{-1}\delta_P\big(\theta(r)\big)^{-1/2}\delta_P(m_2)^{-1}$$

which can be simplified to

$$\delta_P(m)^{-1/2}\lambda(r)^{-1}.$$

Hence our integral becomes

$$\begin{aligned} &\lambda(r)\int_{H_M/(H_M)_{\theta(r)}}\int_{H_M}\Big[\delta_P(m)^{1/2}\int_N f(mn)dn\Big]\eta(\det m_1)\,dm_1\,dm_2 \\ &\qquad = \lambda(r)\int_{H_M/(H_M)_{\theta(r)}}\int_{H_M} f^{(P)}\big(m_1\theta(r)m_2\big)\eta(\det m_1)\,dm_1\,dm_2. \end{aligned}$$

This ends the proof of our proposition.

2.3. We turn to the H'-double orbital integrals on $G'(F)$. Let α belong to $G(E)$ and let $r = \alpha\bar{\alpha}$. Suppose r is a regular element in $G_n(F)$ which has the form as in (2.1). We may assume

$$\alpha = (\alpha_1, \ldots, \alpha_l)$$

where $r_i = \alpha_i\bar{\alpha}_i$ is a regular elliptic element in $G_{n_i}(F)$. To continue we need to introduce another variant of $G(F)$. Let $G'' \subset G_{2n}(E)$ be the subgroup which consists of the elements of the form

$$g'' = \begin{pmatrix} \begin{pmatrix} \alpha_{11} & \beta_{11} \\ \bar{\beta}_{11} & \bar{\alpha}_{11} \end{pmatrix} & \cdots & \begin{pmatrix} \alpha_{1l} & \beta_{1l} \\ \bar{\beta}_{1l} & \bar{\alpha}_{1l} \end{pmatrix} \\ \vdots & \vdots & \vdots \\ \begin{pmatrix} \alpha_{l1} & \beta_{l1} \\ \bar{\beta}_{l1} & \bar{\alpha}_{l1} \end{pmatrix} & \cdots & \begin{pmatrix} \alpha_{ll} & \beta_{ll} \\ \bar{\beta}_{ll} & \bar{\alpha}_{ll} \end{pmatrix} \end{pmatrix}$$

where α_{ij} and β_{ij} are $n_i \times n_j$ matrices over E. Regarded as an algebraic group over F, G'' is isomorphic to both G and G'. Let $P'' \subset G''(F)$ be the parabolic group which corresponds to $P \subset G(F)$. Then $g'' \in P''$ if and only if $\alpha_{ij} = \beta_{ij} = 0$ when $j < i$. Let

$$M'' = G'_{2n_1}(F) \times \cdots \times G'_{2n_l}(F)$$

be a Levi subgroup and N'' be the unipotent radical of P''.

The automorphism θ of G defined in (2.3) can be regarded as an automorphism of $G(E)$. Then it is easy to see that the restriction of θ to G' induces an isomorphism between G' and G''. In what follows we will not make a distinction between $G_m(E)$ and its embedding in $G'_{2m}(F)$ for any $m > 0$. Thus the subgroup

$$H_{M''} = G_{n_1}(E) \times G_{n_2}(E) \times \cdots \times G_{n_l}(E)$$

of M'' is just $\theta\big(M_1(E)\big)$. We can extend the definition of our orbital integral to $(M'', H_{M''})$ in a natural way.

PROPOSITION 2.2. *Let f' be a Hecke function of $G'(F)$ and let $f'' = f' \circ \theta^{-1}$ be the corresponding one of $G''(F)$. Then*

$$H(\alpha; f') = \lambda(r) H\left(\left(\begin{pmatrix} 1_{n_1} & \alpha_1 \\ \bar{\alpha}_1 & 1_{n_1} \end{pmatrix}, \ldots, \begin{pmatrix} 1_{n_l} & \alpha_l \\ \bar{\alpha}_l & 1_{n_l} \end{pmatrix} \right); f''^{(P'')} \right). \tag{2.16}$$

PROOF. We normalize the Haar measure on $G_n(E)$ by $\int_{K(E)} dg = 1$. For $g \in G_n(E)$, we set $g = knm$ by Iwasawa decomposition, where $k \in K_n(E)$, $n \in N_1(E)$ and $m \in M_1(E)$. Then $dg = dk\, dn\, dm$. So we get

$$H(\alpha; f') = \int_{M_1(E)} \int_{M_1(E)/G_n(E)^-_{\bar{\alpha}}} \int_{N_1(E)} \int_{N_1(E)}$$
$$f'\left((n_1, \bar{n}_1)(h, \bar{h}) \begin{pmatrix} 1_n & \alpha \\ \bar{\alpha} & 1_n \end{pmatrix} (g, \bar{g})(n_2, \bar{n}_2) \right) dn_1\, dn_2\, dh\, dg.$$

Let N''_2 be the image of $N_1(E)$ under the map θ. Then N''_2 is a subgroup of N'', which consists of the elements of the form

$$\begin{pmatrix} 1_{2n_1} & (U_{11}, \bar{U}_{11}) & \cdots & (U_{1l}, \bar{U}_{1l}) \\ 0 & 1_{2n_2} & \cdots & (U_{2l}, \bar{U}_{2l}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1_{2n_l} \end{pmatrix}.$$

Here we use $(U_{ij}, \bar{U}_{ij})$ to denote the matrix

$$\begin{pmatrix} U_{ij} & 0 \\ 0 & \bar{U}_{ij} \end{pmatrix}.$$

Let $\theta(\alpha)$ be the element

$$\theta\left(\begin{pmatrix} 1_n & \alpha \\ \bar{\alpha} & 1_n \end{pmatrix}\right) = \left(\begin{pmatrix} 1_{n_1} & \alpha_1 \\ \bar{\alpha}_1 & 1_{n_1} \end{pmatrix}, \ldots, \begin{pmatrix} 1_{n_l} & \alpha_l \\ \bar{\alpha}_l & 1_{n_l} \end{pmatrix}\right). \tag{2.17}$$

Then we can change our orbital integral into

$$\begin{aligned} &\int_{H_{M''}/(H_{M''})_{\theta(\alpha)}} \int_{H_{M''}} \int_{N_2''} \int_{N_2''} f''\big(n_1 h_1 \theta(\alpha) h_2 n_2\big)\, dn_1\, dn_2\, dh_1\, dh_2 \\ &\qquad = \iint \int_{N''} f''(m''n'')\big(1/J(m'')\big)\, dn''\, dh_1\, dh_2 \end{aligned}.$$

Here $m'' = h_1\theta(\alpha)h_2 \in M''$ and $J(m'')$ is the Jacobian of the map

$$(n_1, n_2) \longmapsto m''^{-1} n_1 m'' n_2 \tag{2.18}$$

from $N_2'' \oplus N_2''$ to N''. We need to compute $J(m'')$. To do this we express (2.18) in the form

$$(n_1, n_2) \longmapsto h_2^{-1}[\theta(\alpha)^{-1}(h_1^{-1} n_1 h_1)\theta(\alpha)(h_2 n_2 h_2^{-1})]h_2.$$

First of all the Jacobians of the maps $n_1 \longmapsto h_1^{-1} n_1 h_1$ and $n_2 \longmapsto h_2 n_2 h_2^{-1}$ from N_2'' to N_2'' are

$$\delta_{P_1(E)}\big(\theta^{-1}(h_1)\big)^{-1} = \delta_{P''}(h_1)^{-1/2} \quad \text{and} \quad \delta_{P_1(E)}\big(\theta^{-1}(h_2)\big) = \delta_{P''}(h_2)^{1/2}$$

respectively. Next we consider the Jacobian J' of the map

$$(n_1, n_2) \longmapsto \theta(\alpha)^{-1} n_1 \theta(\alpha) n_2.$$

from $N_2'' \oplus N_2''$ to N''. We proceed as in the last proposition. Let

$$n_1 = \begin{pmatrix} 1_{2n_1} & (U_{12}, \bar{U}_{12}) & \cdots & (U_{1l}, \bar{U}_{1l}) \\ 0 & 1_{2n_2} & \cdots & (U_{2l}, \bar{U}_{2l}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1_{2n_l} \end{pmatrix}$$

and

$$n_2 = \begin{pmatrix} 1_{2n_1} & (V_{12}, \bar{V}_{12}) & \cdots & (V_{1l}, \bar{V}_{1l}) \\ 0 & 1_{2n_l} & \cdots & (V_{2l}, \bar{V}_{2l}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1_{2n_l} \end{pmatrix}.$$

We use X_i' to denote the element

$$\begin{pmatrix} 1_{n_i} & \alpha_i \\ \bar{\alpha}_i & 1_{n_i} \end{pmatrix}.$$

Let W'_{ij} be the matrix

$$X'^{-1}_i(U_{ij}, \bar{U}_{ij})X'_j + (V_{ij}, \bar{V}_{ij}) \tag{2.19}$$

for $1 \le i < j \le l$. Then we have

$$\theta(\alpha)^{-1} n_1 \theta(\alpha) n_2 = \begin{pmatrix} 1_{2n_1} & W'_{12} & \cdots & W'_{1l} \\ 0 & 1_{2n_2} & \cdots & W'_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1_{2n_l} \end{pmatrix}.$$

As before if we denote by J'_{ij} the Jacobian of the F-linear map from $F^{4n_in_j}$ to $F^{4n_in_j}$ determined by (2.19), then $J' = \prod_{1\le i<j\le l} J'_{ij}$. We now compute J'_{ij}. We will do this over $E_1 = F_1 \otimes E$. Here E_1 is a quadratic extension field of F_1 if E is not isomorphic to a subfield of F_1. Otherwise we have $E_1 \cong F_1 \oplus F_1$. In either case we will still use $x \longmapsto \bar{x}$ to denote the only nontrivial F_1 -automorphism of E_1. Since $r = \alpha\bar{\alpha}$ splits over F_1, there exists $s_k \in G_{n_k}(E_1)$ for each $k \in \{1, \ldots, l\}$ such that

$$\alpha_k = s_k(\alpha_{k1}, \ldots, \alpha_{kn_k})\bar{s}_k^{-1}$$

where $\alpha_{kt} \in E_1$ and $\alpha_{kt}\bar{\alpha}_{kt} = r_{kt}$. Thus we can write W'_{ij} in the following form:

$$\begin{pmatrix} * + V_{ij} & Y_{ij} \\ \bar{Y}_{ij} & \bar{*} + \bar{V}_{ij} \end{pmatrix}$$

where $*$ is not related to V_{ij} and

$$\begin{aligned}(1_{n_i} - \alpha_i\bar{\alpha}_i)Y_{ij} &= U_{ij}\alpha_j - \alpha_i\bar{U}_{ij} \\ &= U_{ij}s_j(\alpha_{j1}, \ldots, \alpha_{jn_j})\bar{s}_j^{-1} - s_i(\alpha_{i1}, \ldots, \alpha_{in_i})\bar{s}_i^{-1}\bar{U}_{ij} \\ &= s_i[(s_i^{-1}U_{ij}s_j)(\alpha_{j1}, \ldots, \alpha_{jn_j}) - (\alpha_{i1}, \ldots, \alpha_{in_i})(\bar{s}_i^{-1}\bar{U}_{ij}\bar{s}_j)]\bar{s}_j^{-1}.\end{aligned}$$

Then through an elementary computation we obtain

$$J'_{ij} = \frac{\left|\prod_{\substack{1\le m_i\le n_i \\ 1\le m_j\le n_j}} (r_{im_i} - r_{jm_j})\right|_F}{|\det(1_{n_i} - r_i)|_F^{2n_j}}.$$

Thus by taking the product of all J'_{ij}, we find

$$J' = \lambda(r)^{-1}\delta_{P''}\big(\theta(\alpha)\big)^{-1/2}.$$

Finally the Jacobian of the map $n \longmapsto h_2^{-1}nh_2$ from N'' to N'' is

$$\delta_{P''}(h_2)^{-1/2}.$$

Hence

$$J(m'') = \delta_{P''}(h_1)^{-1/2}\delta_{P''}(h_2)^{1/2}\lambda(r)^{-1}\delta_{P''}\big(\theta(\alpha)\big)^{-1/2}\delta_{P''}(h_2)^{-1}$$

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which is

$$\delta_{P''}(m'')^{-1/2}\lambda(r)^{-1}.$$

So our orbital integral becomes

$$\lambda(r)\int_{H_{M''}/(H_{M''})_{\theta(\alpha)}}\int_{H''_M}\left[\delta_{P''}(m'')^{1/2}\int_{N''}f''(m''n'')\,dn''\right]dh_1\,dh_2$$

which equals to

$$\lambda(r)H\big(\theta(\alpha);f''^{(P'')}\big).$$

From these two propositions the comparison of the orbital integrals of Hecke functions is reduced to the case of the orbital integrals at regular elliptic elements.

3. **Orbital integrals at regular elliptic elements.** In this section we restrict ourself to the unit Hecke functions f_0 and f_0'. We will compare the orbital integrals $H(a;f_0;\eta)$ and $H(\alpha;f_0')$ where a is regular elliptic in $G_n(F)$ and α is twisted regular elliptic in $G_n(E)$.

3.1. We need some preliminaries. Let $V = F^{2n}$ be the $2n$-dimensional vector space of column vectors over F. Let $(e_1, e_2, \ldots, e_{2n})$ be the natural basis of V. Then the exterior n-space $\wedge^n V$ is a vector space with basis $\{e_{i_1}\wedge\cdots\wedge e_{i_n} \mid 1\le i_1<\cdots<i_n\le 2n\}$. For a vector

$$v=\sum_{1\le i_1<\cdots<i_n\le 2n} a_{i_1\cdots i_n}e_{i_1}\wedge\cdots\wedge e_{i_n}$$

in $\wedge^n V$, we define

$$|v|_F=\max(|a_{i_1\cdots i_n}|_F;\,1\le i_1<\cdots<i_n\le 2n).$$

The group $G(F)$ acts on $\wedge^n V$ by

$$g(e_{i_1}\wedge\cdots\wedge e_{i_n})=ge_{i_1}\wedge\cdots\wedge ge_{i_n}.$$

LEMMA 3.1. *Given $g\in G(F)$. Then $g\in KH$ if and only if*

$$|g(e_1\wedge\cdots\wedge e_n)|_F\bullet|g(e_{n+1}\wedge\cdots\wedge e_{2n})|_F=|\det g|_F. \tag{3.1}$$

We remark that as a function on $G(F)$

$$\frac{|g(e_1\wedge\cdots\wedge e_n)|_F|g(e_{n+1}\wedge\cdots\wedge e_{2n})|_F}{|\det g|_F}$$

is invariant under K on the left and invariant under H on the right.

PROOF. Clearly if $g\in KH$, then the relation is satisfied.

As for the "if" part, We apply the following Iwasawa decomposition for $g\in G(F)$

$$g=k\begin{pmatrix}1_n & X\\ 0 & 1_n\end{pmatrix}h$$

where $k \in K$ and $h \in H$. Because the element

$$\begin{pmatrix} 1_n & \varpi^m 1_n \\ 0 & 1_n \end{pmatrix} \quad m \geq 0$$

is in K and $\varpi^m 1_n + X$ is nonsingular when m is large enough, we may assume $\det X \neq 0$ in the above decomposition. By the Cartan decomposition for $G_n(F)$, we can express X as

$$k_1(\varpi^{i_1}, \dots, \varpi^{i_n})k_2$$

with $k_1, k_2 \in K_n$ and $i_1 \leq \dots \leq i_n$. Thus we can write

$$g = k \begin{pmatrix} 1_n & (\varpi^{i_1}, \dots, \varpi^{i_n}) \\ 0 & 1_n \end{pmatrix} h$$

if we change k, h suitably. We may further assume $i_1 \leq \dots \leq i_n \leq 0$ by adjusting k. In this case we have

$$|g(e_1 \wedge \dots \wedge e_n)|\,|g(e_{n+1} \wedge \dots \wedge e_{2n})| = q^{i_1+\dots+i_n}|\det g|.$$

Therefore if

$$|g(e_1 \wedge \dots \wedge e_n)|\,|g(e_{n+1} \wedge \dots \wedge e_{2n})| = |\det g|$$

then $i_1 = \dots = i_n = 0$ and $g \in KH$.

The subgroup H' is not a Levi subgroup of G'. But we still have a similar description for $K'H'$. To that end we need a preparatory lemma.

LEMMA 3.2. *Every double coset $K'g'H'$ has a representative g'_0 such that g'_0 has integral entries and*

$$|g'_0(e_1 \wedge \dots \wedge e_n)|_E = 1.$$

PROOF. Let

$$g' = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}.$$

Multiplying g' on the left by

$$\begin{pmatrix} 1_n & \varpi^m 1_n \\ \varpi^m 1_n & 1_n \end{pmatrix} \in K'$$

for large enough m if necessary, we may assume $\det \alpha \neq 0$ and $\det \beta \neq 0$. So we can take g' of the form

$$\begin{pmatrix} 1_n & \beta \\ \bar{\beta} & 1_n \end{pmatrix}.$$

Let

$$\beta = k_1(\varpi^{i_1}, \dots, \varpi^{i_t}, \varpi^{i_{t+1}}, \dots, \varpi^{i_n})k_2.$$

Here $k_1, k_2 \in K_n(E)$ and $i_1 \geq \cdots \geq i_t > 0 \geq i_{t+1} \geq \cdots \geq i_n$. Then the element

$$\begin{aligned} g_0' &= g'\big(\bar{k}_2^{-1}(1,\ldots,1,\varpi^{-i_{t+1}},\ldots,\varpi^{-i_n}), k_2^{-1}(1,\ldots,1,\varpi^{-i_{t+1}},\ldots,\varpi^{-i_n})\big) \\ &= \begin{pmatrix} \bar{k}_2^{-1}(1,\ldots,1,\varpi^{-i_{t+1}},\ldots,\varpi^{-i_n}) & k_1(\varpi^{i_1},\ldots,\varpi^{i_t},1,\ldots,1) \\ \bar{k}_1(\varpi^{i_1},\ldots,\varpi^{i_t},1,\ldots,1) & k_2^{-1}(1,\ldots,1,\varpi^{-i_{t+1}},\ldots,\varpi_{-i_n}) \end{pmatrix} \\ &= (\bar{k}_2^{-1},\bar{k}_1) \times \begin{pmatrix} (1,\ldots,1,\varpi^{-i_{t+1}},\ldots,\varpi^{-i_n}) & \bar{k}_2 k_1(\varpi^{i_1},\ldots,\varpi^{i_t},1,\ldots,1) \\ (\varpi^{i_1},\ldots,\varpi^{i_t},1,\ldots,1) & \bar{k}_1^{-1}k_2^{-1}(1,\ldots,1,\varpi^{-i_{t+1}},\ldots,\varphi^{-i_n}) \end{pmatrix} \end{aligned}$$

has integral entries. We have

$$\begin{aligned} &|g_0'(e_1 \wedge \cdots \wedge e_n)| \\ &= |(e_1+\varpi^{i_1}e_n) \wedge \cdots \wedge (e_t+\varpi^{i_t}e_{t+n}) \wedge (\varpi^{-i_{t+1}}e_{t+1}+e_{n+t+1}) \wedge \cdots \wedge (\varpi^{-i_n}e_n+e_{2n})| \\ &= \Big|e_1 \wedge \cdots \wedge e_t \wedge e_{n+t+1} \wedge \cdots \wedge e_{2n} + \sum a_{l_1\cdots l_n} e_{l_1} \wedge \cdots \wedge e_{l_n}\Big| \end{aligned}$$

where $a_{l_1\cdots l_n}$ are integers. Hence $|g_0'(e_1 \wedge \cdots \wedge e_n)| = 1$.

LEMMA 3.3. *An element $g' \in G'$ is in $K'H'$ if and only if*

$$|g'(e_1 \wedge \cdots \wedge e_n)|_E \bullet |g'(e_{n+1} \wedge \cdots \wedge e_{2n})|_E = |\det g'|_E. \tag{3.2}$$

PROOF. Let $g' = k'g_0'h'$ where g_0' has integral entries and satisfies

$$|g_0'(e_1 \wedge \cdots \wedge e_n)|_E = 1.$$

We also have

$$|g_0'(e_{n+1} \wedge \cdots \wedge e_{2n})| = |g_0'(e_1 \wedge \cdots \wedge e_n)| = 1.$$

Then we find

$$|g'(e_1 \wedge \cdots \wedge e_n)|_E |g'(e_{n+1} \wedge \cdots \wedge e_{2n})|_E = |\det h'|_E.$$

Thus

$$|g'(e_1 \wedge \cdots \wedge e_n)|_E |g'(e_{n+1} \wedge \cdots \wedge e_{2n})|_E = |\det g'|_E$$

if and only if $|\det g_0'| = 1$ which is equivalent to $g_0' \in K'$, hence $g' \in K'H'$.

The following simple lemma is crucial in later computations.

LEMMA 3.4. *Let r be a regular elliptic element in $G_n(F)$ and let*

$$x_r = |\det(1_n - r)|_F, \quad y_r = |\det r|_F,$$

then $x_r \leq y_r$ or $y_r < x_r = 1$.

PROOF. Let F_1 be the splitting field of the characteristic polynomial of r. Then F_1 is a finite Galois extension of F. Let $\alpha_1, \ldots, \alpha_n$ be the eigenvalues of r in F_1. Then r is conjugate to $(\alpha_1, \ldots, \alpha_n)$ in $G_n(F_1)$. Thus

$$x_r = |(1-\alpha_1)\cdots(1-\alpha_n)|_F, \quad y_r = |\alpha_1 \cdots \alpha_n|_F.$$

Since r is regular elliptic, the characteristic polynomial of r is irreducible. Therefore $\alpha_1, \ldots, \alpha_n$ are conjugate to each other. Hence we have

$$|\alpha_1|_{F_1} = \cdots = |\alpha_n|_{F_1}, \quad |1 - \alpha_1|_{F_1} = \cdots = |1 - \alpha_n|_{F_1}.$$

Now $|1-\alpha_i|_{F_1} \leq |\alpha_i|_{F_1}$ or $|\alpha_i|_{F_1} < |1-\alpha_i|_{F_1} = 1$. The conclusion of the lemma follows.

We now fix some notations. Let A be an $n \times n$ matrix over F. We will denote by $\|(A)_k\|$ the maximum of the absolute values of determinants of all $k \times k$ submatrices of A. In particular $\|(A)_1\|$ is the maximum of the absolute values of all the entries and $\|(A)_n\| = |\det A|$. We will simple denote $\|(A)_1\|$ by $\|A\|$. For convenience, we define $\|(A)_0\| = 1$.

3.2. Let r be an element of $G_n(F)$ such that $\det(1_n - r) \neq 0$. Let x_r and y_r be defined as in Lemma 3.4. We use Φ_r to denote the characteristic function of the set of $(X, Y) \in G_n(F) \times G_n(F)$ such that

$$\begin{aligned} &\sup\big(\|(X)_k\|; k = 0, \ldots, n\big) \sup\big(\|(Y)_k\|; k = 0, \ldots, n\big) = x_r \\ &|\det XY| = y_r. \end{aligned} \tag{3.3}$$

Then Φ_r is compactly supported and bi-K_n invariant for both X and Y. Let Ψ_r be the function on $G_n(F)$ defined by

$$\Psi_r(g) = \int_{G_n(F)} \Phi(gh^{-1}, h)\eta(\det h)\, dh. \tag{3.4}$$

It is clear that Ψ_r is a Hecke function of $G_n(F)$.

LEMMA 3.5. *With the Hecke function Ψ_r defined above, we have*

$$H(r; f_0; \eta) = \int_{G_n(F)/G_n(F)_r} \Psi_r(grg^{-1})\, dg. \tag{3.5}$$

Here $G_n(F)_r$ is the centralizer of r in $G_n(F)$.

The right-hand side of (3.5) is the usual orbital integral with respect to conjugation in $G_n(F)$. We will call it the conjugate orbital integral.

PROOF. Recalling the definition of the integral $H(r; f; \eta)$, we have:

$$\begin{aligned} &H(r; f_0; \eta) \\ &= \int_{H/H_r} \int_H f_0\left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \begin{pmatrix} 1_n & r \\ 1_n & 1_n \end{pmatrix} \begin{pmatrix} h_3 & 0 \\ 0 & h_4 \end{pmatrix} \right) \eta(\det h_1 h_2)\, d(h_1, h_2)\, d(h_3, h_4) \end{aligned}$$

which is

$$\int_{H/H_r} \int_H f_0\left(\begin{pmatrix} h_1 h_3 & h_1 r h_4 \\ h_2 h_3 & h_2 h_4 \end{pmatrix} \right) \eta(\det h_1 h_2)\, d(h_1, h_2)\, d(h_3, h_4).$$

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We now change the variables $h_3 \longmapsto h_1h_3$, $h_4 \longmapsto h_2h_4$, $h_2 \longmapsto h_2h_1^{-1}$. Then our integral becomes:

$$\int\int f_0\left(\begin{pmatrix} 1_n & h_1rh_1^{-1}h_2^{-1} \\ h_2 & 1_n \end{pmatrix} h\right)\eta(\det h_2)\,dh\,d(h_1,h_2).$$

Let Φ be the function on the symmetric space G/H defined by

$$\Phi(g) = \int_H f_0(gh)\,dh$$

and let $\Phi(X, Y)$ be

$$\Phi\left(\begin{pmatrix} 1_n & X \\ Y & 1_n \end{pmatrix}\right)$$

for $X, Y \in G_n(F)$. Thus we obtain

$$H(r;f_0) = \int_{G_n(F)}\int_{G_n(F)/G_n(F)_r} \Phi(h_1rh_1^{-1}h_2^{-1}, h_2)\eta(\det h_2)\,dh_1\,dh_2.$$

By Lemma 3.1, we have that Φ is the characteristic function of the set of

$$(X, Y) \in G_n(F) \times G_n(F)$$

such that

$$\left|\begin{pmatrix} 1_n & X \\ Y & 1_n \end{pmatrix}(e_1 \wedge \cdots \wedge e_n)\right| \left|\begin{pmatrix} 1_n & X \\ Y & 1_n \end{pmatrix}(e_{n+1} \wedge \cdots \wedge e_{2n})\right| = |\det(1_n - XY)|.$$

By a simple computation, we find

$$\left|\begin{pmatrix} 1_n & X \\ Y & 1_n \end{pmatrix}(e_1 \wedge \cdots \wedge e_n)\right| = \sup\big(\|(Y)_k\|; k = 0, 1, \ldots, n\big)$$

and

$$\left|\begin{pmatrix} 1_n & X \\ Y & 1_n \end{pmatrix}(e_{n+1} \wedge \cdots \wedge e_{2n})\right| = \sup\big(\|(X)_k\|; k = 0, 1, \ldots, n\big).$$

On the other hand when $X = h_1rh_1^{-1}h_2^{-1}$ and $Y = h_2$ we have

$$|\det(1_n - XY)| = |\det(1_n - r)|, \quad |\det XY| = |\det r|.$$

Thus we find

$$\Phi(h_1rh_1^{-1}h_2^{-1}, h_2) = \Phi_r(h_1rh_1^{-1}h_2^{-1}, h_2)$$

for all $h_1, h_2 \in G_n(F)$. The assertion of the lemma follows immediately.

Similarly for $\alpha \in G_n(E)$ such that $\alpha\bar{\alpha} = r \in G_n(F)$ and $\det(1_n - r) \neq 0$, we use Φ_α to denote the characteristic function of the set of $X \in G_n(E)$ where

$$\sup(\|(X)_k\|_E; k = 0, 1, \ldots, n) = x_r$$

$$|\det X|_E = y_r. \tag{3.6}$$

Then Φ_α is a Hecke function on $G_n(E)$.

LEMMA 3.6. *With the above notations, we have:*

$$H(\alpha;f_0') = \int_{G_n(E)/G_n(E)_\alpha^-} \Phi_\alpha(h\alpha\bar{h}^{-1})\,dh. \tag{3.7}$$

Here

$$G_n(E)_\alpha^- = \{h \in G_n(E) \mid h\alpha\bar{h}^{-1} = \alpha\}$$

is the twisted centralizer of α in $G_n(E)$.

The right-hand side is the usual orbital integral with respect to the twisted conjugation in $G_n(E)$. We will call it the twisted conjugate orbital integral.

PROOF. By definition the orbital integral $H(\alpha;f_0')$ is equal to

$$\int_{G_n(E)/G_n(E)_\alpha^-}\int_{G_n(E)} f_0'\left(\begin{pmatrix} h & 0 \\ 0 & \bar{h}\end{pmatrix}\begin{pmatrix} 1_n & \alpha \\ \bar{\alpha} & 1_n\end{pmatrix}\begin{pmatrix} h_1 & 0 \\ 0 & \bar{h}_1\end{pmatrix}\right) dh_1\,dh$$
$$= \int\int f_0'\left(\begin{pmatrix} 1_n & h\alpha\bar{h}^{-1} \\ \bar{h}\bar{\alpha}h^{-1} & 1_n\end{pmatrix}\begin{pmatrix} h_1 & 0 \\ 0 & \bar{h}_1\end{pmatrix}\right) dh_1\,dh.$$

Let Φ' be the function on G'/H' defined by

$$\Phi'(g) = \int_{H'} f'(gh)\,dh$$

and let $\Phi'(X)$ be

$$\Phi'\left(\begin{pmatrix} 1_n & X \\ \bar{X} & 1_n\end{pmatrix}\right).$$

Then our integral becomes

$$\int_{G_n(E)/G_n(E)_\alpha^-} \Phi'(h\alpha\bar{h}^{-1})\,dh.$$

By Lemma 3.3, Φ' is the characteristic function of the set of $X \in G_n(E)$ such that

$$\left|\begin{pmatrix} 1_n & X \\ \bar{X} & 1_n\end{pmatrix}(e_1\wedge\cdots\wedge e_n)\right|_E \left|\begin{pmatrix} 1_n & X \\ \bar{X} & 1_n\end{pmatrix}(e_{n+1}\wedge\cdots\wedge e_{2n})\right|_E = |\det(1_n - X\bar{X})|_E.$$

This condition is equivalent to

$$\sup\big(\|(X)_k\|_E; k = 0,\ldots,n\big) = |\det(1_n - X\bar{X})|_F.$$

If $X = h\alpha\bar{h}^{-1}$, we have

$$|\det X|_E = |\det\alpha|_E = |\det r|_F = y_r, \quad |\det(1_n - X\bar{X})|_F = x_r.$$

Thus we find

$$\Phi'(h\alpha\bar{h}^{-1}) = \Phi_\alpha(h\alpha\bar{h}^{-1})$$

for any $h \in G_n(E)$. Hence our lemma.

From these two lemmas we find that to compare the orbital integral of f_0 at r with the orbital integral of f_0' at α, it suffices to compare the conjugate orbital integrals of Ψ_r with the twisted orbital integrals of Φ_α. In this case we have:

PROPOSITION 3.7. *Let r be a regular elliptic element in $G_n(F)$ such that* $\det(1_n - r) \neq 0$. *Then*

(1) if $r = \alpha\bar{\alpha}$ for some $\alpha \in G_n(E)$, then Ψ_r and Φ_α have matching (conjugate and twisted conjugate) orbital integrals at regular split elements of $G_n(F)$, i.e., for a regular split element t of $G_n(F)$

$$\int_{G_n(F)/G_n(F)_t} \Psi_r(gtg^{-1})\,dg = \begin{cases} \int_{G_n(E)/G_n(E)_\alpha^-} \Phi_\alpha(g\beta\bar{g}^{-1})\,dg, & \text{if } t = \beta\bar{\beta} \\ 0, & \text{if } t \notin NG_n(E). \end{cases} \tag{3.8}$$

(2) if $r \notin NG_n(E)$, then the conjugate orbital integral of Ψ_r vanishes at all split regular elements of $G_n(F)$.

PROOF. Since the orbital integrals depend only on conjugacy classes, we just need to consider the diagonal matrices. We use A_n to denote the subgroup of diagonal matrices of $G_n(F)$.

We first compute the conjugate orbital integral of Ψ_r at the matrix $a = (a_1, \ldots, a_n)$ where $a_i \neq a_j$ if $i \neq j$. For convenience we use $A_r(a)$ to denote this integral. Then we have:

$$A_r(a) = \int_{G_n(F)/A_n(F)} \Psi_r(gag^{-1})\,dg$$

which is equal to

$$\int_{N_n} \Psi_r(nan^{-1})\,dn$$

by the Iwasawa decomposition $G_n(F) = K_n N_n A_n$. Here N_n is the group of the upper triangular unipotent elements of $G_n(F)$.

Recalling the definition of the function Ψ_r we have

$$A_r(a) = \int_{N_n} \int_{G_n(F)} \Phi_r(nan^{-1}g^{-1}, g)\eta(\det g)]\,dn\,dg$$

Then using the Iwasawa decomposition $G_n(F) = K_n N_n A_n$ again and the fact that Φ_r is bi-K_n invariant for both variables, we find

$$\begin{aligned} A_r(a) &= \int_{A_n} \int_{N_n} \int_{N_n} \Phi_r(nan^{-1}m^{-1}n_1^{-1}, n_1 m)\eta(\det m)\,dn\,dn_1\,dm \\ &= \int_{A_n} \int_{N_n} \int_{N_n} \Phi_r\big(a(a^{-1}nan^{-1})m^{-1}n_1^{-1}, n_1 m\big)\eta(\det m)\,dn\,dn_1\,dm. \end{aligned}$$

We now consider the map

$$n \longmapsto a^{-1}nan^{-1}$$

from N_n to N_n. Its Jacobian is

$$\Delta(a) = \left|\det\big(\mathrm{Ad}(a) - 1\big)\big|_{\mathrm{Lie}(N_n)}\right|_F$$

where $\mathrm{Lie}(N_n)$ is the Lie algebra of N_n. So our integral becomes:

$$\begin{aligned} &\Delta(a)^{-1} \int_{A_n} \int_{N_n} \int_{N_n} \Phi(anm^{-1}n_1^{-1}, n_1 m)\eta(\det m)\,dn\,dn_1\,dm \\ &= \Delta(a)^{-1} \int_{A_n} \int_{N_n} \int_{N_n} \Phi\big(am^{-1}(mnm^{-1})n_1^{-1}, m(m^{-1}n_1 m)\big)\eta(\det m)\,dn\,dn_1\,dm \end{aligned}$$

Next we change the variables

$$n \longmapsto mnm^{-1} \quad n_1 \longmapsto m^{-1} n_1 m$$

and then $n \longmapsto nn_1^{-1}$ to obtain

$$A_r(a) = \Delta(a)^{-1} \int_{A_n} \int_{N_n} \int_{N_n} \Phi_r(am^{-1}n, mn_1)\eta(\det m)\, dn\, dn_1\, dm.$$

Recall that the function Φ_r is the characteristic function of the subset of $(X, Y) \in G_n(F) \times G_n(F)$ which satisfy the conditions

$$\begin{gathered} \sup\bigl(\|(X)_k\|; k = 0, \ldots, n\bigr) \sup\bigl(\|(Y)_k\|; k = 0, \ldots, n\bigr) = x_r \\ |\det XY| = y_r. \end{gathered} \tag{3.9}$$

This subset is empty unless $y_r \leq x_r$ and $1 \leq x_r$. Assuming this is the case, we must have

$$x_r = y_r > 1 \quad \text{or} \quad y_r \leq x_r = 1$$

by Lemma 3.4.

To continue we distinguish these two cases.

We first consider the case $y_r \leq x_r = 1$. Then $(X, Y) \in G_n(F) \times G_n(F)$ satisfies the conditions (3.9) if and only if

$$\begin{gathered} \sup\bigl(\|(X)_k\|; k = 0, 1, \ldots, n\bigr) = 1 \\ \sup\bigl(\|(Y)_k\|; k = 0, 1, \ldots, n\bigr) = 1 \end{gathered}$$

and

$$|\det XY| = y_r.$$

These conditions are equivalent to

$$\|X\| \leq 1, \quad \|Y\| \leq 1, \quad |\det XY| = y_r.$$

Now let

$$m = (x_1, \ldots, x_n)$$

and let u_{ij} and v_{ij} be the (i,j)-entry of n and n_1 respectively. So when $|\det a| = y_r$ our integral becomes

$$\begin{aligned} &\Delta(a)^{-1} \prod_{1 \leq k \leq n} \int_{|a_k| \leq |x_k| \leq 1} \Bigl[\prod_{i<j} \Bigl(\int_{|u_{ij}| \leq |x_i/a_i|} du_{ij}\Bigr) \Bigl(\int_{|v_{ij}| \leq 1/|x_i|} dv_{ij}\Bigr)\Bigr] \eta(\det x_i)\, d^* x_i \\ &\qquad = \frac{1}{\Delta(a)|a_1^{n-1} a_2^{n-2} \cdots a_{n-1}|_F} \prod_{1 \leq i \leq n} \int_{|a_i| \leq |x_i| \leq 1} \eta(x_i)\, d^* x_i. \end{aligned}$$

We conclude that

$$A_r(a) = \begin{cases} \frac{1}{\Delta(a)|a_1^{n-1} \cdots a_{n-1}|_F} & \text{if } |a_i| \leq 1, a_i \in NE^*, |a_1 \cdots a_n| = y_r \\ 0 & \text{otherwise.} \end{cases}$$

In particular if $y_r \notin |NE^*|$, then $A_r(a)$ is always zero.

Next we consider the case $x_r = y_r > 1$. Then the conditions in (3.9) become

$$\begin{aligned} \sup\big(\|(X)_k\|; k = 0, \dots, n\big) &= |\det X| \\ \sup\big(\|(Y)_k\|; k = 0, \dots, n\big) &= |\det Y| \\ |\det XY| &= x_r = y_r. \end{aligned} \tag{3.10}$$

Let

$$X = k_1(\varpi^{i_1}, \dots, \varpi^{i_n})k_2$$

where $k_1, k_2 \in K_n$ and $i_1 \leq \cdots \leq i_n$. Then we have

$$\|(X)_k\| = q^{-(i_1+\dots+i_k)}.$$

So

$$\sup\big(\|(X)_k\|; k = 0, \dots, n\big) = |\det X|$$

if and only if $i_1 \leq \cdots \leq i_n \leq 0$. Since

$$X^{-1} = k_1^{-1}(\varpi^{-i_1}, \dots, \varpi^{-i_n})k_2^{-1}$$

we find that $\|X^{-1}\| = q^{i_n}$. Thus the above condition is equivalent to $\|X^{-1}\| \leq 1$. Similarly

$$\sup\big(\|(Y)_k\|; k = 0, \dots, n\big) = |\det Y|$$

if and only if $\|Y^{-1}\| \leq 1$.

Hence the conditions in (3.10) are simply

$$\|X^{-1}\| \leq 1, \quad \|Y^{-1}\| \leq 1, \quad |\det XY| = x_r = y_r.$$

Therefore $A_r(a)$ equals to

$$\Delta(a)^{-1} \iiint \eta(\det m)\, dn\, dn_1\, dm$$

where the integrals are over the domain

$$\|n^{-1}ma^{-1}\| \leq 1, \quad \|n_1^{-1}m^{-1}\| \leq 1, \quad |\det a| = x_r = y_r.$$

Now we change the variables from (n, n_1) to (n^{-1}, n_1^{-1}). The domain for our integral becomes

$$\|nma^{-1}\| \leq 1, \|n_1m^{-1}\| \leq 1, |\det a| = x_r = y_r.$$

Thus in the case $|\det a| = x_r = y_r$ we find

$$\begin{aligned} A_r(a) &= \Delta(a)^{-1} \prod_{1 \leq i \leq n} \int_{1 \leq |x_i| \leq |a_i|} \Big[\prod_{i<j} \Big(\int_{|u_{ij}| \leq |a_i/x_j|} du_{ij}\Big) \Big(\int_{|v_{ij}| \leq |x_j|} dv_{ij}\Big)\Big] \eta(x_i)\, d^*x_i \\ &= \frac{|a_2a_3^2 \cdots a_n^{n-1}|}{\Delta(a)} \prod_{1 \leq i \leq |a_i|} \int_{1 \leq |x_i| \leq |a_i|} \eta(x_i)\, d^*x_i. \end{aligned}$$

In conclusion we get

$$A_r(a) = \begin{cases} \frac{|a_2 a_3^2 \cdots a_n^{n-1}|}{\Delta(a)}, & \text{if } |a_i| \geq 1, a_i \in NE^*, |a_1 \cdots a_n| = y_r \\ 0, & \text{otherwise.} \end{cases}$$

In particular A_r is always zero if $y_r \notin |NE^*|$.

We now pass to the computation of the twisted conjugate orbital integrals of Φ_α at diagonal matrices. We assume $\alpha\bar{\alpha} = r$ is a regular elliptic element of $G_n(F)$. Let $\beta = (\beta_1, \ldots, \beta_n) \in A_n(E)$ such that $\beta_i\bar{\beta}_i \neq \beta_j\bar{\beta}_j$ if $i \neq j$. We use $B_\alpha(\beta)$ to denote the orbital integral

$$\int_{G_n(E)/A_n(F)} \Phi_\alpha(g\beta\bar{g}^{-1})\, dg.$$

Then it is equal to

$$\int_{A_n(E)/A_n(F)} \int_{N_n(E)} \Phi_\alpha(nm\beta\bar{m}^{-1}\bar{n}^{-1})\, dn\, dm = \int_{N_n(E)} \Phi_\alpha(n\beta\bar{n}^{-1})\, dn.$$

We now change n to $\beta^{-1}n\beta\bar{n}^{-1}$. The Jacobian is

$$\left|\det\left(1 - \mathrm{Ad}(\beta) \circ \sigma\right)\Big|_{\mathrm{Lie}(N_n(E))}\right|$$

where σ is the nontrivial element of $\mathrm{Gal}(E/F)$. This Jacobian is actually equal to $\Delta(N\beta)$ ([K], p. 376). Hence our integral becomes

$$\Delta(N\beta)^{-1} \int_{N_n(E)} \Phi_\alpha(\beta n)\, dn.$$

Recall Φ_α is the characteristic function of the set of $X \in G_n(E)$ such that

$$|\det X| = y_r, \sup\left(\|(X)_k\|; k = 0, \ldots, n\right) = x_r. \tag{3.11}$$

So $\Phi_\alpha \equiv 0$ if $x_r < 1$ or $x_r < y_r$. As before the only remaining cases are

$$x_r = y_r > 1 \quad \text{and} \quad y_r \leq x_r = 1.$$

We first consider the case $y_r \leq x_r = 1$. Then the conditions for Φ_α becomes

$$\|X\|_E \leq 1, \quad |\det X| = x_r.$$

So when $|\det\beta|_E = x_r$ we have

$$\begin{aligned} B_\alpha(\beta) = \Delta(N\beta)^{-1} \int_{\|\beta n\|_E \leq 1} dn &= \begin{cases} \Delta(N\beta)^{-1} \prod_{i<j} \left(\int_{|u_{ij}|_E \leq |\beta_i|^{-1}} du_{ij}\right), & \text{if } |\beta_i| \leq 1 \\ 0, & \text{otherwise.} \end{cases} \\ &= \begin{cases} \frac{1}{\Delta(N\beta)|\beta_1^{n-1} \cdots \beta_{n-1}|_E}, & \text{if } |\beta_i|_E \leq 1 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

If $|\det\beta|_E \neq x_r$, the integral is zero.

Next we consider the case $y_r = x_r > 1$. Then the condition for Φ_α is

$$\sup\left(\|(X)_k\|; k = 0, \ldots, n\right) \leq |\det X|_E = x_r = y_r.$$

As before this is equivalent to

$$\|X^{-1}\|_E \le 1, \quad |\det X|_E = x_r = y_r.$$

Hence if $|\det\beta|_E = x_r = y_r$ we have

$$\begin{aligned}
B_\alpha(\beta) &= \Delta(N\beta)^{-1}\int_{|n^{-1}\beta^{-1}|_E\le 1} dn = \Delta(N\beta)^{-1}\int_{|n\beta^{-1}|_E\le 1} dn \\
&= \begin{cases} \Delta(N\beta)^{-1}\prod_{i<j}(\int_{|u_{ij}|_E\le|\beta_j|_E} du_{ij}), & \text{if } |\beta_i| \ge 1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{|\beta_2\cdots\beta_n^{n-1}|_E}{\Delta_{(N\beta)}} & \text{if } |\beta_i|_E \ge 1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

If $|\det\beta|_E \ne y_r$, then $B_\alpha(\beta) = 0$.

So we have finished the computations. Comparing the results, we find the assertions of the proposition are true.

COROLLARY 3.7. *With the notations as in the proposition, we have*

$$\Psi_r = 0 \quad \textit{if } r \notin NG_n(E)$$

and

$$\Psi_r = b\Phi_\alpha \quad \textit{if } r = \alpha\bar{\alpha}.$$

Here $b: H\big(G_n(E), K_n(E)\big) \longrightarrow H\big(G_n(F), K_n(F)\big)$ is the base change map.

PROOF. We assume $\Phi_\alpha = 0$ if $r \notin NG_n(E)$. Let $f = b\Phi_\alpha$. Then f and Φ_α have matching (conjugate and twisted conjugate) orbital integrals ([AC], Chapter 1). Thus f and Ψ_r have the same conjugate orbital integrals at regular diagonal matrices.

Let S be the Satake transform from $H\big(G_n(F), K_n\big)$ to $H(A_n, A_n \cap K_n)$. Then for regular $a \in A_n$ we have

$$Sf(a) = \Delta(a)\delta_{P_n}(a)^{-1/2}\int_{G_n(F)/A_n(F)} f(gag^{-1})\,dg$$

where $P_n = A_nN_n$ is the minimal standard parabolic subgroup of G_n ([C], p. 147). The same is true for Ψ_r. Therefore $sf = s\Psi_r$. But S is one to one. So $f = \Psi_r$.

4. **Conclusion.**

Now we are ready to prove our theorem.

PROOF OF THE THEOREM. Let $r \in G_n(F)$ be a regular semisimple element such that $\det(1_n - r) \ne 0$.

If r is regular elliptic, we have

$$\int_{G_n(F)/G_n(F)_r} \Psi_r(grg^{-1})\,dg = \begin{cases} \int_{G_n(E)/G_n(E)_{\bar{\alpha}}} \Phi(g\alpha\bar{g}^{-1})\,dg, & \text{if } r = \alpha\bar{\alpha} \\ 0, & \text{if } r \notin NG_n(E) \end{cases}$$

by the above corollary. By Lemma 3.5 and 3.6, we obtain:

$$H(r;f_0;\eta) = \begin{cases} H(\alpha;f'_0), & \text{if } r = \alpha\bar{\alpha} \\ 0, & \text{if } r \notin NG_n(E). \end{cases} \tag{4.1}$$

If r is not regular elliptic, we may assume

$$r = (r_1, \ldots, r_l)$$

where r_i is a regular elliptic element in $G_{n_i}(F)$. Proposition 2.1 tells us

$$H(r;f_0;\eta) = \lambda(r) \prod_{1 \leq i \leq l} H(r_i;f_{i0};\eta)$$

where f_{i0} is the unit Hecke function of $G_i(F)$. On the other hand if

$$\alpha = (\alpha_1, \ldots \alpha_l)$$

where $\alpha_i\bar{\alpha}_i = r_i$ is regular elliptic in $G_{n_i}(F)$, we have

$$H(\alpha;f'_0) = \lambda(r) \prod_{1 \leq i \leq l} H(\alpha_i,f'_{i0}).$$

Here f'_{i0} is the unit Hecke function of G'_{2n_i}. Therefore the identity (4.1) is also satisfied in this case. This ends the proof of our theorem.

We remark that we could also consider the orbital integrals over nonregular elements. For GL(4), this was done by the author in an unpublished note where he established the Shalike germ expansion theorem for our orbital integrals and applied it to obtain the relations between orbital integrals over elements which are not relatively regular.

REFERENCES

[AC] J. Arthur and L. Clozel, *Simple algebras, base change, and advanced theory of the trace formula*, Ann. Math. Stud. **120**, Princeton Univ. Press, 1989.

[BF] D. Bump and S. Friedberg, *The exterior square L-functions on* GL(n). In: Festschrift in honor of I. I. Piatetski-Shapiro, Part II, Israel Math. Conf. Proc. **3**, 1990, 47–65.

[C] P. Cartier, *Representations of p-adic groups*. In: Proc. Sympos. Pure Math. **33**, Amer. Math. Soc., Providence, Rhode Island, 1979, 111–155.

[Cl] L. Clozel, *The fundamental lemma for stable base change,* Duke Math. J. **61**(1990), 255–302.

[DKV] P. Deligne, D. Kazhdan and M.-F. Vigneras, *Representations des algèbres centrales simples p-adiques*. In: Representations des Groupes Reductifs sur un Corps Local, Herman, Paris, 1984, 33–117.

[FJ] S. Friedberg and H. Jacquet, *Linear periods,* J. Reine Angew. Math. **443**(1993), 91–139.

[G1] J. Guo, *On the positivity of the central critical values of automorphic L-functions for* GL(2), Duke Math. J., to appear.

[G2] ———, *Uniqueness of generalized Waldspurger model for* GL($2n$), (1995), preprint .

[J1] H. Jacquet, *Sur une resultat de Waldspurger,* Ann. Sci. École Norm. Sup. (4) **19** (1986), 185–229.

[J2] ———, *On the nonvanishing of some L-functions,* Proc. Indian Acad. Sci. Math. Sci. **97**(1987), 117–155.

[JR] H. Jacquet and S. Rallis, *Uniqueness of linear periods,* Compositio Math., to appear.

[JS] H. Jacquet and J. Shalika, *On the exterior square L-function*. In: Automorphic Forms, Shimura Varieties, and L-functions, (eds. L. Clozel and J. S. Milne), Perspectives in Math. **11**, Academic Press, 143–225.

[K] R. Kottwitz, *Orbital integrals on* GL(3), Amer. J. Math. **102**(1980), 327–384.

[L] J. P. Labesse, *Fonctions elementaires et lemme fondamental pour le changement de base stable,* Duke Math. J. **61**(1990), 519–530.

[R] R. Richardson, *Orbits, invariants, and representations associated to involutions of reductive groups,* Invent. Math. **66**(1982), 287–312.

[W1] J.-L. Waldspurger, *Sur les valeurs de certains fonctions L automorphes en leur centre de symetrie,* Compositio Math. **54**(1985), 173–242.
[W2] ———, *Quelques proprietes arithmetiques des forms modulaires de poids demi-entier,* Compositio Math. **54**(1985), 121–171.

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