IDEMPOTENT IDEALS IN PERFECT RINGS

GERHARD MICHLER

1. Introduction. All rings considered in this note have an identity element, and all R-modules are unitary.

Bass (**2**) defined a left perfect ring as a ring R satisfying the minimum condition on principal *right* ideals. A commutative ring R is perfect if and only if R is a direct sum of finitely many local rings whose radicals are T-nilpotent. Therefore, the commutative perfect rings with finite global projective dimension are just the direct sums of finitely many commutative fields, and hence they trivially satisfy the minimum condition for *all* ideals. However, in the non-commutative case, even hereditary perfect rings are not necessarily right or left artinian (cf. Example 3.4).

Each left perfect ring R has only finitely many idempotent (two-sided) ideals (Corollary 2.3), where the ideal X of R is called idempotent, if $X = X^2$. Hence, it makes sense to consider *minimal idempotent ideals* of the left perfect ring R, i.e., ideals of R which are minimal in the set of all idemponent ideals of R. The right R-module structure of the minimal idempotent ideals of the right hereditary, left perfect ring R decides whether or not R is right artinian, since the following theorem holds.

The left perfect, right hereditary ring R is right artinian if and only if the minimal idempotent ideals of R are finitely generated right R-modules (Theorem 3.7).

The assertion of this theorem holds more generally for all left perfect rings R with the property that each epimorphic image of R has finite global projective dimension (Lemma 4.1 and Corollary 4.3). Hence, the requirement of Theorem 3.7 that R be right hereditary can be replaced by the apparently weaker assumption that the principal right ideals of R be projective right R-modules (Corollary 4.5).

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For notation and terminology, the reader is referred to (**2**).

2. Generators of idempotent ideals. Throughout this section, J denotes the Jacobson radical of the left perfect ring R. The element $a \in R$ is *central modulo J*, if $a + J$ lies in the centre of R/J. The idempotent $e \in R$ is called

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centrally primitive if e lies in the centre of R, and if e is not the sum of two orthogonal idempotents e_i contained in the centre of R. In this section it is shown that there is a one-to-one correspondence between the minimal idempotent ideals of R and the centrally primitive idempotents of R/J.

PROPOSITION 2.1. *If X is an idempotent ideal of the left perfect ring R, then $X = ReR$, where e is an idempotent of R which is central modulo J.*

Proof. We may assume that $X \neq 0$. If $X \leqq J$, then $X = JX$, since X is idempotent. Hence, $X = 0$ (see **2**, p. 473), as R is left perfect. Thus, $\bar{X} = (X + J)/J$ is a non-zero ideal of R/J. Since R/J is a semi-simple artinian ring, $\bar{X} = \bar{e}\bar{R} = \bar{R}\bar{e}$ for some central idempotent $\bar{e} \in \bar{R} = R/J$. Therefore, $\bar{e} = x + J$ for some $x \in X$. Since J is locally nilpotent, there exists an idempotent $e \in R$ (see **5**, Lemma 1.12), which is a finite sum of powers of x and satisfies $e \equiv x \bmod J$. Hence, $e \in X$ and $\bar{e} = e + J$. Furthermore, $ReR \leqq X$ and $ReR + J = X + J$. Thus,

$$X = X^2 \leqq (ReR + J)X \leqq ReR + JX \leqq X.$$

Therefore, $J(X/ReR) = X/ReR$, whence $X = ReR$; see (**2**, p. 473).

PROPOSITION 2.2. *There is a one-to-one correspondence between the idempotent ideals $K \neq 0$ of the left perfect ring R and the two-sided ideals $\bar{K} \neq 0$ of the semi-simple artinian ring R/J.*

Proof. Let $\bar{K} \neq 0$ be a two-sided ideal of $\bar{R} = R/J$. Then $\bar{K} = \bar{R}\bar{e}\bar{R}$, where $0 \neq \bar{e}$ is a central idempotent of $\bar{R}$. Since R is left perfect, there is an idempotent $e \neq 0$ in R such that $\bar{e} = e + J$. Clearly, $K_1 = ReR$ is an idempotent ideal of R satisfying $\bar{K}_1 = (K_1 + J)/J = \bar{K}$. Assume that K_2 is another idempotent ideal of R with $\bar{K}_2 = \bar{K}$. Then, by Proposition 2.1, there is an idempotent $f \neq 0$ of R which is central modulo J, and for which $K_2 = RfR$ holds. Thus, $\bar{e} = \bar{f} = f + J$, since $\bar{f}$ is a generator of the ideal $\bar{K}$ of the semi-simple artinian ring $\bar{R}$. Therefore, e and $1 - e$ and f and $1 - f$ are two pairs of orthogonal idempotents of R. Hence (see **1**, p. 122), there exists an element $h \in J$ with $e = (1 - h)^{-1}f(1 - h)$. Thus,

$$K_1 = ReR = R(1 - h)^{-1}f(1 - h)R = RfR = K_2,$$

which completes the proof of Proposition 2.2.

COROLLARY 2.3. *The left perfect ring R has exactly 2^s idempotent ideals V, where s is the number of simple components of R/J.*

Proof. This follows at once from Propositions 2.1 and 2.2 and the well-known fact that the semi-simple artinian ring $\bar{R} = R/J$ has exactly 2^s ideals.

Since each left perfect ring R has only finitely many idempotent ideals by Corollary 2.3, R has minimal idempotent ideals X, where the idempotent (two-sided) ideal X of R is a *minimal idempotent ideal* of R, if $X \neq 0$, and if X does not contain an idempotent ideal $0 \neq Y \neq X$ of R. By Propositions

2.1 and 2.2, each minimal idempotent ideal X of R has the form $X = ReR$, where $e = e^2$ is centrally primitive modulo J.

3. The structure of certain left perfect rings with finite global projective dimension. In this section we show that the left perfect ring R with finite left global dimension is a split extension of a left T-nilpotent ring J by a semi-simple artinian ring S, if $K \otimes_R K$ is a projective left R-module for each minimal idempotent ideal K of R. By (**2**, p. 467), the ring J is left T-nilpotent if for each sequence $a_0, a_1, a_2, \ldots$ of elements of J there exists a positive integer N depending on the sequence such that $a_0a_1 \ldots a_N = 0$.

Assertion (a) and statement (b) of the following lemma are a special case of (**8**, Theorem 2.5 and Lemma 2.8, respectively). The proof of assertion (c) is due to the referee, and it is shorter than the original one.

LEMMA 3.1. *Let $e \neq 0$ be an idempotent of the ring R and let $T = eRe$. Then the following statements hold:*

(a) *$Re \otimes_T eR \cong ReR \otimes_R ReR$ as right and left R-modules;*

(b) *eR is a projective left T-module if and only if $ReR \otimes_R ReR$ is a projective left R-module;*

(c) *The left T-module eR is finitely generated if and only if $ReR \otimes_R ReR$ is a finitely generated left R-module.*

Proof. It remains to prove assertion (c). If eR is a finitely generated left T-module, then $Re \otimes_T eR$ is obviously a finitely generated left R-module. Hence, $ReR \otimes_R ReR$ is finitely generated as a left R-module by (a).

If, conversely, $Re \otimes_T eR$ is finitely generated as a left R-module, then the obvious R-module epimorphism $Re \otimes_T eR \to ReR$ guarantees that ReR is a finitely generated left ideal of R. Hence, $ReR = \sum_{i=1}^r Rz_i$ with $z_i = x_iey_i$; $x_i, y_i \in R$. If $x \in R$, then $ex \in ReR$. Thus, $ex = \sum_{i=1}^r f_iz_i$ with $f_i \in R$; whence,

$$ex = e^2x = \sum_{i=1}^{r} ef_iz_i = \sum_{i=1}^{r} (ef_ix_ie)ey_i = \sum_{i=1}^{r} t_i(ey_i),$$

where $t_i = ef_ix_ie \in T$. Thus, $ey_1, ey_2, \ldots, ey_r$ generate eR as a left T-module.

If X is a subset of the ring R, then $X_l(R) = \{x \in R \mid xX = 0\}$ is the left annihilator and $X_r(R) = \{y \in R \mid Xy = 0\}$ is the right annihilator of X. R is *local* if R is a division ring modulo its Jacobson radical J.

LEMMA 3.2. *A left perfect local ring R with finite weak global dimension is a division ring.*

Proof. As R is left perfect, l. gl. dim R = w. gl. dim R; see (**2**, Theorem P). Furthermore, the Jacobson radical J of R is left T-nilpotent, and therefore J is locally nilpotent. Since R is local, J consists of all the zero-divisors of R. Assume now that $a \neq 0$ is a zero-divisor, and let

$$0 \to P_m \xrightarrow{\mu_m} P_{m-1} \xrightarrow{\mu_{m-1}} P_{m-2} \to \ldots \to P_1 \xrightarrow{\mu_1} P_0 \xrightarrow{\mu_0} Ra$$

be a projective resolution of Ra of shortest length $m \leqq \text{l. gl. dim}\, R < \infty$. Since R is left perfect, $\ker \mu_i \leqq JP_{i+1}$ $(i = 0, 1, \ldots, m-1)$, which implies that $P_m = \ker \mu_{m-1} \leqq JP_{m-1}$. However, P_m is a free left R-module; see (**7**, Theorem 2). Hence, there exists $0 \neq p \in P_m$ such that $p_l(R) = 0$. Since

$$p = j_1p_1 + j_2p_2 + \ldots + j_kp_k \in JP_{m-1},$$

where $j_i \in J$ and $p_i \in P_{m-1}$ $(i = 1, 2, \ldots, k)$, we obtain

$$p_l(R) \geqq \bigcap_{i=1}^{k} (j_i)_l(R).$$

Let S be the subring of R generated by $j_1, j_2, \ldots, j_k$. Then S is nilpotent, since J is locally nilpotent. Hence, $S^v = 0$, but $S^{v-1} \neq 0$ for some positive integer $v \neq 1$. Thus,

$$0 \neq S^{v-1} \leqq \bigcap_{i=1}^{k} (j_i)_l(R) \leqq p_l(R) = 0.$$

This contradiction implies that $m = 0$. Thus, Ra is free, and a is left regular. Since R is local, a is a unit.

THEOREM 3.3. *The following properties of the left perfect ring R with* l. gl. dim $R < \infty$ *are equivalent:*

(1) *$K \otimes_R K$ is a projective left R-module for each minimal idempotent ideal K of R;*

(2) *R is a split extension of a left T-nilpotent ring J by a semi-simple artinian subring S of R such that $eJe = 0$ for each centrally primitive idempotent e of S;*

(3) *$K \otimes_R K$ is a projective right R-module for each minimal idempotent ideal K of R.*

Furthermore, if R satisfies the equivalent conditions (1), (2), *and* (3), *and if S_1 and S_2 are (semi-simple artinian) subrings of R such that $R = S_i + J$, $S_i \cap J = 0$ $(i = 1, 2)$, then there exists an element $h \in J$ such that $S_1 = (1-h)S_2(1-h)^{-1}$.*

Proof. (1) implies (2): Since R is left perfect, there exist orthogonal idempotents $e_i \neq 0$ $(i = 1, 2, \ldots, n)$ whose sum is the identity of the ring R such that $\bar{e}_i = e_i + J$ is centrally primitive in $\bar{R} = R/J$ for all i. Hence,

$$(*) \quad R = e_1Re_1 + e_2Re_2 + \ldots + e_nRe_n + J, \quad \textit{and } R = S + J$$
$$\text{if } S = e_1Re_1 + e_2Re_2 + \ldots + e_nRe_n.$$

By Lemma 3.1, each right R-module $P_i = e_iR$ is left T_i-projective, where $T_i = e_iRe_i$. From this, we deduce that l. gl. dim $T_i \leqq$ l. gl. dim $R < \infty$, by applying (**4**, p. 27, Theorem 8). Since $J_i = e_iJe_i$ is the Jacobson radical of T_i, and $T_i/J_i \cong \bar{e}_i(R/J)\bar{e}_i$ for each i, we obtain that each T_i is left perfect. Since $\bar{e}_i$ is centrally primitive in $\bar{R}$, T_i/J_i is simple artinian. Thus, for each i, there exists a local ring D_i such that $T_i \cong (D_i)_{n_i}$; see (**6**, p. 56), where $(D_i)_{n_i}$

denotes the ring of all $n_i \times n_i$ matrices with entries in D_i, and where n_i is a positive integer. By (**4**, p. 20, Theorem 1) l. gl. dim D_i = l. gl. dim $T_i < \infty$. Clearly, the Jacobson radical $J(D_i)$ of D_i is also left T-nilpotent. Therefore, each ring D_i is a local, left perfect ring with finite left global projective dimension. Thus, D_i is a division ring by Lemma 3.2. Hence, T_i is a simple artinian ring, and $e_iJe_i = 0$ for $i = 1, 2, \ldots, n$. Therefore, S is a semi-simple artinian subring of R with $S \cap J = 0$. From $(*)$ we obtain $R = S + J$, and therefore $R/J \cong S$. Hence, (2) holds.

(2) implies (1): Let K be a minimal idempotent ideal of R. Let e_i $(i = 1, 2, \ldots, n)$ be the centrally primitive idempotents of S. Then (2) and Propositions 2.1 and 2.2 imply that $K = Re_jR$ for some $j \in \{1, 2, \ldots, n\}$. From $e_jJe_j = 0$ follows $e_jSe_j = e_jRe_j = T_j$. Thus, T_j is a simple artinian ring. Hence, e_jR is a projective left T_j-module. Therefore, Lemma 3.1 implies that $Re_jR \otimes_R Re_jR = K \otimes_R K$ is left R-projective. Hence, (2) and (1) are equivalent.

The implication (2) $\Rightarrow$ (3) is proved similarly as the implication (2) $\Rightarrow$ (1).

(3) implies (2): As the proof of the implication (1) $\Rightarrow$ (2) shows, it suffices to prove that if $K = ReR$ is a minimal idempotent ideal of R, then $T = eRe$ has finite weak global dimension. Applying Lemma 3.1 we obtain from (3) that Re is right T-projective. Hence, w. gl. dim $T \leqq$ w. gl. dim $R < \infty$; see (**4**, Theorem **7**). Therefore, the conditions (1), (2), and (3) are equivalent.

Uniqueness of S. Suppose that the semi-simple artinian subring S of R satisfies $R = S + J$, $S \cap J = 0$ and $eJe = 0$ for each centrally primitive idempotent of S. Assume that S has n centrally primitive idempotents e_i. If A is another subring of R such that $R = A + J$ and $A \cap J = 0$, then A has also n centrally primitive idempotents f_i which can be ordered so that $\bar{e}_i = e_i + J = \bar{f}_i = f_i + J$ in R/J, since $S \cong R/J \cong A$. By (**1**, Theorem 3), there exists an element $h \in J$ such that

$$(1 - h)e_i(1 - h)^{-1} = f_i \quad \text{for } i = 1, 2, \ldots, n.$$

Thus, $f_iAf_i \leqq (1 - h)e_iRe_i(1 - h)^{-1}$, and $A \leqq (1 - h)S(1 - h)^{-1}$. Now, $(1 - h)S(1 - h)^{-1} \cap J = 0$ and $R = A + J = (1 - h)S(1 - h)^{-1} + J$ imply that

$$\begin{aligned}(1 - h)S(1 - h)^{-1} &= (1 - h)S(1 - h)^{-1} \cap (A + J) \\ &= A + [(1 - h)S(1 - h)^{-1} \cap J] = A.\end{aligned}$$

This completes the proof of Theorem 3.3.

In general, a left perfect ring R with finite global projective dimension satisfying the equivalent conditions (1), (2), and (3) of Theorem 3.3 is not necessarily left or right artinian as the following example shows.

Example 3.4. Let V be an infinite-dimensional vector space over the rationals Q. Let R be the ring of all 2×2 matrices of the form

$$\begin{pmatrix} q_1 & v \\ 0 & q_2 \end{pmatrix}, \qquad q_1, q_2 \in Q, v \in V,$$

where addition is component-wise and multiplication is matrix multiplication. Then R is a semi-primary ring with l. gl. dim $R = 1$; see (**3**, Lemma 4.1). Hence, R satisfies the hypothesis and condition (1) of Theorem 3.3; however, R is obviously not left artinian. In this example, the minimal idempotent ideal ReR, where

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

is not finitely generated as a left ideal of R; hence, $ReR \otimes_R ReR$ is not a finitely generated left R-module. On the other hand, we shall now show that a left perfect ring with l. gl. dim $R < \infty$ satisfying the equivalent conditions (1), (2), and (3) of Theorem 3.3 is left artinian if and only if each minimal idempotent ideal K of R has the property that $K \otimes_R K$ is a finitely generated left R-module.

COROLLARY 3.5. *The following properties of the left perfect ring R with* l. gl. dim $R < \infty$ *are equivalent:*

(1) *$K \otimes_R K$ is a finitely generated projective left R-module for each minimal idempotent ideal K of R;*

(2) *R is a left artinian split extension of a nilpotent ring J by a semi-simple artinian subring S of R, and $eJe = 0$ for every centrally primitive idempotent e of S.*

Proof. If (1) holds, then because of Theorem 3.3 it remains to prove that R is left artinian. Let $e_1, e_2, \ldots, e_n$ be the centrally primitive idempotents of the semi-simple artinian subring S occurring in Theorem 3.3. Since $(Re_iR) \otimes_R (Re_iR)$ is a finitely generated left R-module by (1), e_iR is a finitely generated left T_i-module by Lemma 3.1, where $T_i = e_iRe_i$ for $i = 1, 2, \ldots, n$. From $S = T_1 \oplus T_2 \oplus \ldots \oplus T_n$ and $R = e_1R \dotplus e_2R \dotplus \ldots \dotplus e_nR$, one deduces that R is a finitely generated left S-module. Since S is a semi-simple artinian ring, R is a noetherian left S-module. Hence, R is left noetherian. Therefore, R is left artinian (see **2**, p. 475), since R is left perfect. Thus (1) implies (2).

If R satisfies (2), then $K \otimes_R K$ is left R-projective for every minimal idempotent ideal K of R by Theorem 3.3. Since R is left artinian, K is finitely generated as a left R-module, completing the proof of Corollary 3.5.

As a special case of Corollary 3.5 we have the following result.

COROLLARY 3.6. *The left perfect, left hereditary ring R is left artinian if and only if the minimal idempotent ideals of R are finitely generated left ideals of R.*

Another consequence of Theorem 3.3 is the following.

THEOREM 3.7. *The left perfect, right hereditary ring R is right artinian if and only if the minimal idempotent ideals of R are finitely generated right ideals of R.*

Proof. Since each right ideal in a right artinian ring with identity is finitely generated, it suffices to prove the "if" part. As R is left perfect and right hereditary, R is left hereditary (**2**, Theorem P). Hence, by Theorem 3.3, R is a split extension of a left T-nilpotent ring J by a semi-simple artinian subring S of R such that $e_iJe_i = 0$ for each centrally primitive idempotent $e_i \in S$ ($i = 1, 2, \ldots, n$). Thus, by the proof of Corollary 3.5, the ring R is right noetherian. Therefore, R is semi-primary, since J is locally nilpotent. Furthermore, the right R-module R_R has a composition series, since J is finitely generated and nilpotent, and R/J is artinian, Therefore, R is right artinian.

4. Left perfect triangular rings. In this section we show that the assertion of Theorem 3.7 remains true under the weaker hypothesis that the principal right ideals of the left perfect ring R be projective. Actually, we only need that R is triangular in the following sense.

Definition (cf. Chase (**3**, p. 21)). Let R be a semi-perfect ring with Jacobson radical J. Then R is *triangular* if any complete set $e_1, e_2, \ldots, e_r$ of mutually orthogonal primitive idempotents of R can be indexed so that $e_iJe_i = 0$ whenever $i \geqq j$.

The following lemma was proved by Chase (**3**, Theorem 4.1) for semi-primary rings, i.e., for semi-perfect rings whose Jacobson radical is nilpotent.

LEMMA 4.1. *The following properties of the left perfect ring R with Jacobson radical J are equivalent:*

(1) *R is triangular;*

(2) gl. dim$(R/K) < \infty$ *for any two-sided ideal K in R;*

(3) gl. dim$(R/J^2) < \infty$*;*

(4) l. gl. dim$(R/J^2) < \infty$.

If any (and hence all) of these conditions hold, then R is semi-primary and gl. dim $R < s$, *where s is the number of simple components of R/J.*

Proof. If R is triangular, then there exists a complete set of mutually orthogonal primitive idempotents e_i ($i = 1, 2, \ldots, r$) such that $e_iJe_j = 0$ for $i \geqq j$. Thus,

$$J = e_1Je_2 + e_1Je_3 + \ldots + e_1Je_r + e_2Je_3 + \ldots + e_2Je_r + \ldots + e_{r-1}Je_r.$$

This implies that J is nilpotent. Thus, R is semi-primary. Hence, (2) follows from (1) by (**3**, Theorem 4.1). The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are immediate.

Thus, we may assume that (4) holds. If $J^2 = 0$, then R is semi-primary and triangular by (**3**, pp. 21, 22). If $J^2 \neq 0$, then $\bar{R} = R/J^2$ is a semi-primary ring with l. gl. dim $\bar{R}$ = r. gl. dim $\bar{R} < \infty$ whose Jacobson radical $\bar{J} = J/J^2$ satisfies $\bar{J}^2 = 0$. Hence, any complete set of mutually orthogonal primitive idempotents $e_1, e_2, \ldots, e_r$ of R can be indexed so that $e_iJe_j \leqq J^2$ if $i \geqq j$. Repeating Chase's argument (**3**, p. 22), this implies that

$$e_iJe_j \leqq J^k \quad \text{for all } k = 1, 2, \ldots \quad \text{whenever } i \geqq j.$$

Let $T = \bigcap_{k=1}^{\infty} J^k$. Then $e_i J e_j \leqq T$ for all $i \geqq j$. If $S = R/T$, then the Jacobson radical of S is $J' = J/T$. Therefore, S is left perfect and triangular. Hence, J' is nilpotent, which implies that $J^t \leqq T$ for some positive integer t. Thus, $J^t \leqq J^{t+1} \leqq J^t$, and $J^t = J(J^t)$. Since R is left perfect, we obtain $J^t = 0$; see (**2**, p. 473). Thus, $e_i J e_j \leqq J^t = 0$ whenever $i \geqq j$. Therefore, R is triangular.

The proof of the following result is due to the referee, and it is simpler than the original one.

PROPOSITION 4.2. *If K is a minimal idempotent ideal of the triangular, left perfect ring R, then $K \otimes_R K$ is a projective left and right R-module.*

Proof. By Propositions 2.1 and 2.2, the minimal idempotent ideal K of R is generated as a two-sided ideal by an idempotent $e \neq 0$ of R which is centrally primitive modulo J. Hence, e is a sum of finitely many orthogonal primitive idempotents e_i. Since R is triangular, this implies that $eJe = 0$. Hence, $T = eRe$ is a simple artinian ring. Thus, eR is a projective left T-module, and Re is a projective right T-module; whence, $K \otimes_R K$ is a projective left and right R-module by Lemma 3.1.

COROLLARY 4.3. *Let R be a triangular, left perfect ring. Then*

(1) *R is left artinian if and only if each minimal idempotent ideal K of R is a finitely generated left ideal;*

(2) *R is right artinian if and only if each minimal idempotent ideal K of R is a finitely generated right ideal.*

The *proof* follows at once from Lemma 4.1, Proposition 4.2, and Corollary 3.5.

The following lemma is a generalization of a theorem by Chase (**3**, p. 23, Theorem 4.2), who proved it for semi-primary rings.

LEMMA 4.4. *If every principal right ideal of the left perfect ring R is a projective right R-module, then R is triangular.*

Proof. The proof is by induction on the number s of simple components of the semi-simple artinian ring $\bar{R} = R/J$. If $s = 1$, then R is isomorphic to a ring of $n \times n$ matrices over a left perfect, local ring D. By (**3**, Lemma 4.4), the principal right ideals of D are projective. Hence, $x_r = \{y \in D \mid xy = 0\}$ is a direct summand of D for each $0 \neq x \in D$. Thus, x_r is generated by an idempotent. Since D is local, $x_r = 0$. Therefore, D is a division ring since D is left perfect. Hence, R is triangular.

Assume that Lemma 4.4 is true for all $s' < s$ and that R is not semi-simple. Since R is left perfect, R satisfies the minimum condition on principal right ideals; see (**2**, Theorem P). Let $0 \neq xR$ be a minimal right ideal of R. If $xJ \neq 0$, then $xJ = xR$, which implies that $(xJ)J = (xR)J = xRJ = xJ = xR$. Since xR, and thus xJ, is a projective right R-module, $0 \neq xJ = 0$; see (**2**, Proposition 2.7). This contradiction shows that $xJ = 0$. Since xR is projective, $x_r(R) = fR$ for some idempotent $f \neq 0$ of R. As xR is minimal, $e_0 = (1 - f)$ generates a minimal right ideal of R. Now, $xJ = 0$ implies that $e_0 J = 0$.

Since R is left perfect, there exists a complete set of mutually orthogonal primitive idempotents $e_i \neq 0$ $(i = 1, 2, \ldots, n)$ such that $e_n = e_0$. By rearrangement, we may assume that $e_{k+1}R \cong e_nR$, but $e_iR \not\sim e_nR$ for $i \leqq k$. If $e = e_{k+1} + e_{k+2} + \ldots + e_n$, then $eJ = 0$, and therefore $eJe = 0$. Hence, eRe is a semi-simple artinian ring. From (**6**, p. 53, Proposition 1) we obtain $eR(1 - e) = 0$ and $(1 - e)R(1 - e)$ has $s - 1$ simple components. Since the principal right ideals of the ring $(1 - e)R(1 - e)$ are projective (**3**, Lemma 4.4), $(1 - e)R(1 - e)$ is triangular, by induction. Now, $R = eRe \dotplus (1 - e)Re \dotplus (1 - e)R(1 - e)$ implies that R is triangular.

An immediate consequence of Corollary 4.3 and Lemma 4.4 is the following.

COROLLARY 4.5. *If every principal right ideal of the left perfect ring R is a projective right R-module, then R is right artinian if and only if each minimal idempotent ideal K of R is a finitely generated right ideal of R.*

COROLLARY 4.6. *If the principal right ideals of the left perfect ring R are projective, then R is left artinian if and only if the minimal idempotent ideals of R are finitely generated left ideals of R.*

Proof. By Lemma 4.4, a left perfect ring whose principal right ideals are projective is triangular. Thus, Corollary 4.6 follows at once from Corollary 4.3.

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Universität Tübingen,
Tübingen, West Germany