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SOME SPACES ARE NOT THE DOMAIN OF A CLOSED LINEAR OPERATOR IN A BANACH SPACE

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Let $P(\partial)=\sum_{|\alpha|\leqq m} a_\alpha \partial^\alpha$ be a linear partial differential operator with C^∞-coefficients. The study of $P(\partial)$ as an operator in $L^2(\mathbb{R}^n)$ usually starts with the investigation of the minimal operator P_0 which is the closure of $P(\partial)$ acting on $C_c^\infty(\mathbf{R}^n)=\mathcal{D}(\mathbf{R}^n)$. In the case of constant coefficients it is known that the domain $D(P_0)$ of P_0 at least contains the space $\mathcal{S}(\mathbf{R}^n)$ (cf. Schechter [4, p. 58, Lemma 1.2]). However, the question arises whether spaces such as $\mathcal{D}(\mathbf{R}^n)$ or $\mathcal{S}(\mathbf{R}^n)$ may occur as the domain of any closed linear operator in $L^p(\mathbf{R}^n)$. The purpose of this note is to show that these and other spaces familiar from the theory of distributions cannot be the domain of a closed linear operator in $L^p(\mathbf{R}^n)$. The proofs depend on the following easy consequences of known closed graph theorems.

PROPOSITION. *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be Banach spaces. Moreover let $D\subset E$ be a linear subspace and let $A:D\to F$ be a linear map whose graph $G:=\{(x, Ax)\in E\times F;\ x\in D\}$ is closed in the product $(E, \|\cdot\|_E)\times(F, \|\cdot\|_F)$. Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ denote the topologies on D induced by the norms $\|\cdot\|_E \mid D$ and $\|\cdot\|_A : D\to \mathbf{R}$, $\|x\|_A := \|x\|_E+\|Ax\|_F$, respectively.*

Clearly $\mathfrak{T}_2\supset\mathfrak{T}_1$, and $(D, \mathfrak{T}_2)$ is a Banach space. Thus dim(D) *is either finite or uncountable.*

(a) *D is closed in $(E, \|\cdot\|_E)$ if and only if $(D, \mathfrak{T}_1)$ is barrelled.*

(b) *Let $\mathfrak{T}$ be a locally convex topology on D which is stronger than $\mathfrak{T}_1$. If $(D, \mathfrak{T})$ is barrelled, $\mathfrak{T}$ is stronger than $\mathfrak{T}_2$. If $(D, \mathfrak{T})$ is an LF-space (i.e. a countable inductive limit of Fréchet spaces, cf. Grothendieck [2, p. 13]), then $\mathfrak{T}=\mathfrak{T}_2$.*

Proof. (a) Let $(D, \mathfrak{T}_1)$ be barrelled. Since every Banach space is a Pták space and because of $\mathfrak{T}_1\subset\mathfrak{T}_2$ the closed graph theorem from Horváth [3, p. 301, Thm. 4] yields $\mathfrak{T}_1=\mathfrak{T}_2$. Thus D is a complete hence closed subspace of $(E, \|\cdot\|_E)$. The converse statement is trivial.

(b) Because of $\mathfrak{T}_2\supset\mathfrak{T}_1$ and $\mathfrak{T}\supset\mathfrak{T}_1$ the diagonal $\Delta:=\{(x, x);\ x\in D\}$ is a closed subset of $(D, \mathfrak{T})\times(D, \mathfrak{T}_2)$. If $(D, \mathfrak{T})$ is barrelled, [3, p. 301, Thm. 4] yields that the identity map $(D, \mathfrak{T})\to(D, \mathfrak{T}_2)$ is continuous, whence $\mathfrak{T}\supset\mathfrak{T}_2$.

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Now let $(D, \mathfrak{T})$ be an LF-space. Then $(D, \mathfrak{T})$ is in particular barrelled whence $\mathfrak{T} \supset \mathfrak{T}_2$. From [3, p. 304, Prop. 10] we obtain that the identity map $(D, \mathfrak{T}_2) \to (D, \mathfrak{T})$ is continuous whence $\mathfrak{T}_2 \supset \mathfrak{T}$.

EXAMPLES. (1) Let $p \in [1, \infty)$, and let F be a Banach space.

(a) None of the spaces $\mathcal{K}(\mathbf{R}^n)$, $\mathcal{D}^m(\mathbf{R}^n)$ $(m \in \mathbf{N})$, $\mathcal{D}(\mathbf{R}^n)$ (cf. Horváth [3, p. 164, ex. 5, p. 171, ex. 8, p. 165, ex. 6]), and $L^p_c(\mathbf{R}^n)$ (cf. Treves [6, p. 32/33]) can occur as the domain of a closed linear operator from $L^p(\mathbf{R}^n)$ into F. In fact, each of these spaces caries a natural LF-space-topology, which is stronger than the norm topology induced by $L^p(\mathbf{R}^n)$, and which is not normable (Horváth [3, p. 175, ex. 6a]).

(b) None of the spaces $\mathcal{S}(\mathbf{R}^n)$ (Horváth [3, p. 91, ex. 14]) and $\mathcal{D}_{L^p}(\mathbf{R}^n)$ (Schwartz [5, p. 199]) can occur as the domain of a closed linear operator from $L^p(\mathbf{R}^n)$ into F. In fact, these spaces are non-normable Fréchet spaces in their natural topology (thus in particular LF-spaces), and embed continuously into $L^p(\mathbf{R}^n)$.

(2) Let E, F be Banach spaces and let L be a proper dense subspace of E which is barrelled in its relative topology. (On account of Valdivia [7, Thm. 3] L is barrelled if $\dim(E/L)$ is at most countable.) Then there does not exist a linear map $A : L \to F$ whose graph is closed in $E \times F$.

Now let (Ω, Σ, μ) be a measure space such that $\dim(L^\infty(\Omega, \Sigma, \mu))$ is infinite. It follows from the Nikodým boundedness theorem (cf. Diestel, Uhl [1, p. 14, Thm. 1]) that the proper dense subspace $M_0(\Sigma) := \text{span } \{\chi_A;\ A \in \Sigma\}$ of $L^\infty(\Omega, \Sigma, \mu)$ is barrelled. Thus $M_0(\Sigma)$ cannot be the domain of a closed linear operator in $L^\infty(\Omega, \Sigma, \mu)$.

(3) Let I be an infinite set and let $p \in [1, \infty)$. $\mathbf{C}^{(I)} := \{x \in \mathbf{C}^I;\ \{i \in I;\ x(i) \neq 0\}$ is finite$\}$ is not the domain of a closed linear operator in $l^p(I)$, since there does not exist a Banach-space-topology on $\mathbf{C}^{(I)}$ stronger than the $\|\cdot\|_p$-topology. In fact, I contains a countable infinite subset J, $L := \{x \in \mathbf{C}^{(I)};\ x(i) = 0$ for all $i \notin J\}$ is a $\|\cdot\|_p$-closed linear subspace of $\mathbf{C}^{(I)}$ whose dimension is countably infinite, hence L does not admit a Banach-space-topology.

REFERENCES

1. J. Diestel and J. J. Uhl, Jr., *Vector measures*, American Mathematical Society, Providence, Rhode Island 1977.

2. A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Memoirs of the AMS, No. 16, American Mathematical Society, Providence, Rhode Island 1966.

3. J. Horváth, *Topological vector spaces and distributions, vol. I*, Addison-Wesley Publishing Comp., Reading, Massachusetts 1966.

4. M. Schechter, *Spectra of partial differential operators*, North-Holland Publishing Comp., Amsterdam 1971.

5. L. Schwartz, *Théorie des distributions*, Hermann, Paris 1966.

6. F. Treves, *Linear partial differential equations with constant coefficients*, Gordon and Breach, New York 1966.

7. M. Valdivia, *Absolutely convex sets in barrelled spaces*, Ann. Inst. Fourier Grenoble 21, 1971, pp. 3–13.

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