

BOUNDED AND RELATIVELY BOUNDED SETS

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All subsets of a field that are intersections of almost-orders of the field are characterised, and all ring topologies on a field which are not finer than any nontrivial locally bounded ring are characterised.

Dürbaum [1] and Kowalsky [2] give conditions which guarantee the intersection of almost-orders is again an almost-order. Dürbaum [1] gives an example of an intersection of almost-orders that is neither an almost-order nor the set $\{0, 1, -1\}$. We characterise the subsets of a field that are intersections of almost-orders.

Kowalsky [2] characterises those ring topologies that are a supremum of locally bounded ring topologies. A slight modification of his proof establishes Theorem 2 below in which the ring topologies finer than some proper locally bounded ring topology are characterised.

In this note K denotes a (commutative) field, and, for $D \subseteq K$, we let $D^* = \{x \in D : x \neq 0\}$. By a proper topology we mean a topology that is neither trivial nor discrete. Value groups of valuations are written multiplicatively. The supremum (Durchschnitt in [2]) of a collection T of topologies is denoted by $\vee T$. We refer the reader to [2] for the definition of a bounded set and the definition of a locally

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bounded topology (called a bounded topology in [2]).

By an almost-order (Fastordnung) we mean a subset A of a field K which satisfies

(01) $0, 1, -1 \in A; A \neq K$;

(02) $x, y \in A$ implies $xy \in A$;

(03) each element in K is a quotient of elements in A ;

(04) there exists $z \in A, z \neq 0$, such that $z(x + y) \in A$ for all $x, y \in A$. (Such an element z will be called a 2-addiator.)

The notion of almost-order is well-known. It is defined here because the definition we have given differs from the one appearing in [2]: Kowalsky omits the requirements $-1 \in A$ and $A \neq K$. With obvious modifications, the results we use from [2] apply to almost-orders as defined here. (We note that if A satisfies (01)-(03) and there exists $w \in K$ such that $w(x + y) \in A$ for all $x, y \in A$, then A satisfies (04)).

The utility of almost-orders is established in [2], where it is shown that, for each almost-order A of a field K , the collection $\{xA : x \in K^*\}$ is a neighbourhood base at zero for a ring topology τ_A on K for which A is a τ_A -bounded neighbourhood of zero, and, conversely, every proper locally bounded topology is induced by some almost-order in this fashion.

THEOREM 1. *A subset E of a field K is an intersection of some collection of almost-orders of K if and only if $0, 1, -1 \in E$, E is a multiplicative semigroup, and E is a bounded subset in some proper locally bounded ring topology on K .*

Proof. The conditions are obviously necessary. Suppose, on the other hand, they are satisfied. Let A be an almost-order of a topology τ_A in which E is bounded. Since $1 \in E, A \subseteq EA$; so EA , which is a product of τ_A -bounded sets is a τ_A -bounded neighbourhood of zero. We show EA is an almost-order inducing the same topology as A . EA satisfies (01) because EA is a bounded and, hence, proper subset of K . Obviously EA satisfies (02) and (03). To verify that EA satisfies (04), choose $a/b, a, b \in A^*$, such that $(a/b)E \subseteq A$. If z is a 2-addiator of A , then za is a 2-addiator of EA .

Next, we observe that, for each $c \in (EA)^*$, $E_c = E \cup (cEA)$ is an almost-order. Verification of (01)-(03) is straightforward. We see that cz , where z is a 2-addiator of EA , is a 2-addiator of E_c :

$$cz(E_c + E_c) \subseteq cz(EA + EA) \subseteq cEA \subseteq E_c.$$

Then $E = \bigcap_{c \in (EA)^*} E_c$, and the proof is complete.

The proof above shows that if a set is an intersection of a collection of almost-orders, it is an intersection of a collection of almost-orders all of which induce the same topology.

COROLLARY. *A subset E of a field K is an intersection of some collection of almost-orders of K if and only if $0, 1, -1 \in E$, E is a multiplicative semigroup, and there is a proper valuation v on K and an element $\gamma \in v(K^*)$ such that $v(x) \leq \gamma$ for all $x \in E$.*

Proof. Each locally bounded topology contains a topology induced by a valuation (see [2]).

We recall that a subset M in a commutative topological ring (S, τ) is *relatively bounded* if there is a neighbourhood V of zero such that $V(M \pm M) \subseteq M$. Clearly the empty set and $\{0\}$ are relatively bounded subsets of any topological ring. Any other relatively bounded subset M of a field with a ring topology is a neighbourhood of zero by virtue of the containments

$$0 \in V(M \pm M) \subseteq M;$$

$$\forall m \in V(M \pm M) \subseteq M, \quad m \in M^*.$$

If M and N are relatively bounded, so are $-M$ and $M \cap N$. Therefore each relatively bounded neighbourhood of zero contains a symmetric relatively bounded neighbourhood. Certainly the entire ring S is relatively bounded. A relatively bounded set other than the empty set, $\{0\}$ and S will be called *proper*.

THEOREM 2. *A proper ring topology τ on a field is finer than some proper locally bounded ring topology if and only if there is at least one proper τ -relatively bounded subset.*

Proof. Suppose there is a relatively bounded neighbourhood of zero $U \neq K$. By the discussion following the definition of relative boundedness, we may assume that U is symmetric. We define the set

$$A = \{x \in K : xU \subseteq U\},$$

which was shown in [2, p. 179] to be an almost-order and a neighbourhood of zero. Since A is a τ -neighbourhood of zero, $\tau_A \leq \tau$: the τ_A -neighbourhoods xA , $x \in K^*$, are τ -neighbourhoods of zero.

Conversely, suppose that there is a proper locally bounded topology ρ weaker than τ . Any ρ -bounded neighbourhood of zero U is a proper τ -relatively bounded set: Since $(U + U) \cup (U - U)$ is ρ -bounded, there is an ρ -neighbourhood V of zero (which is therefore a τ -neighbourhood of zero) such that $V(U \pm U) \subseteq U$.

We state formally some definitions for the property "local boundedness" which have obvious analogues for any property.

DEFINITION. *A nontrivial ring topology is said to be s -locally bounded (n -locally bounded, z -locally bounded) if it is the supremum of some family of locally bounded ring topologies (respectively, not the supremum of any family of locally bounded ring topologies, not finer than any nontrivial locally bounded ring topology).*

Note that in the analogues of the above definition only weaker ring topologies are considered.

An n -locally bounded topology is sometimes referred to as being of type N . The following four properties are mutually exclusive and each property might be thought of as being "less related" to local boundedness than the one before it:

- (1) locally bounded,
- (2) s -locally bounded but not locally bounded,
- (3) n -locally bounded but not z -locally bounded,
- (4) z -locally bounded.

EXAMPLE 1: (a) The p -adic topology τ_p on the rationals is locally bounded.

(b) The supremum over all primes p of the topologies τ_p is s -locally bounded but not locally bounded.

(c) If τ_m is a topology defined by the scheme in [3, Section 1], then the topology $\tau_m \vee \tau_p$ is n -locally bounded but not z -locally bounded.

(d) The topology τ_m described above is z -locally bounded.

Podewski [5] shows each countable field has 2^{2^c} (where c is the cardinality of the continuum) n -locally bounded topologies.

A ring topology on a ring S is σ -bounded if S is the union of a countable collection of bounded subsets. In [6, proof of Theorem 3] (in [7, Question 2 of Section 2]) it is shown that each proper neighbourhood of zero in a nontrivial σ -bounded ring topology (ring topology with a nondiscrete countable subset) on a field is also a neighbourhood of zero in some coarser nontrivial first countable ring topology. Thus each nontrivial ring topology which is σ -bounded or has a nondiscrete countable subset is s -first countable.

Thus, all ring topologies on a countable field K are s -first countable. Since the assignment $\tau \rightarrow \{V_n\}$, where $\{V_n\}$ is any fixed neighbourhood base at zero for τ , defines an injective map into the set of sequences of subsets of K , there are at most 2^c first countable ring topologies on K . Hence, there are 2^{2^c} s -first countable, but not first countable, ring topologies on each countable field.

A supremum of uncountably many independent (as defined in [8] or [9]) first countable ring topologies is s -first countable but not first countable. If K is an uncountable field, then K is an extension of a subfield $F(x)$, where F is an uncountable subfield and x is transcendental over F . Then $\{\tau_{x-a}\}_{a \in F}$, where τ_{x-a} is any topology induced by an extension of the $(x - a)$ -adic valuation on $F(x)$ to K , is an uncountable collection of independent first countable ring topologies.

Any valuation-induced topology which is not first countable is z -first countable (because topologies induced by valuations are minimal).

EXAMPLE 2. Let v be a valuation on K such that the topology τ_v induced by v is not first countable, and let w be a nontrivial valuation inducing a first countable topology τ_w . Then $\tau_v \vee \tau_w$ is not first countable: The supremum of independent topologies is not discrete, and each sequence of nonzero elements is bounded away from zero in the topology τ_v and, hence, also in the supremum. By construction the supremum is not z -first countable. We do not know whether the supremum is s -first countable or n -first countable.

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