# NORMAL AND CANONICAL REPRESENTATIONS IN FREE PRODUCTS OF LATTICES†

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**1. Introduction.** In solving the word problem for free lattices, Whitman [**4**] showed that free lattices admit *canonical representations*, that is, of all polynomials over the generating set representing an element of the lattice, the polynomial of shortest length is unique up to commutativity and associativity. These well-defined shortest polynomials have proved very important in analyzing the internal structure of free lattices in detail; see, e.g., [**5**].

Sorkin [**3**] proved that the free product of chains also admits canonical representations; these were exploited by Rolf [**2**]. In the above-mentioned paper, Sorkin also suggested that the free product of two copies of $2^2$ does not admit canonical representations.

In this paper we show that, except for one pathological case, Sorkin's result on canonical representations in free products of chains is optimal for free products of lattices (Theorem 4). However, we succeed in characterizing minimal polynomials in free products of lattices in general (Theorems 1, 2) and, among all minimal polynomials representing an element of the free product, we choose a well-defined one, the *normal* representation, unique up to commutativity and associativity (Theorem 3). In the study of free products of lattices, this normal representation should serve the same purpose as canonical representations do in the study of free lattices.

The principle of duality applies to lattice polynomials as well as to lattices; in dualizing, we exchange $\bigvee$ and $\bigwedge$, $A_{(\lambda)}$ and $A^{(\lambda)}$, and replace $\subseteq$ by $\supseteq$. (The notation is explained in § 2.) Consequently, in the following only one of the two possible dual forms of each theorem and definition is stated; whenever necessary we use the dual of a stated theorem without further comment.

Theorem 4 first appeared in my thesis, *Free lattices generated by partially ordered sets*, the University of Manitoba, 1968. The proof presented in this paper is much shorter than that presented in the thesis, as I had not yet developed Theorems 1 and 2 at the time I wrote the thesis.

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**2. Preliminaries.** In this paper we use the notation of [**1**]. Let $(L_\lambda \mid \lambda \in \Lambda)$ be a family of mutually disjoint lattices, let $Q = \bigcup (L_\lambda \mid \lambda \in \Lambda)$, and let $P(Q)$ denote the set of lattice polynomials over $Q$. The operational symbols in $P(Q)$ are $\bigvee$ and $\bigwedge$, while $\vee$ and $\wedge$ denote the operations in a lattice. If $A \in P(Q)$, then the *length* of $A$, written $l(A)$, is $1 +$ the number of occurrences of $\bigvee$ and $\bigwedge$ in $A$; thus $l(A) = 1$ if and only if $A \in L_\lambda$ for some $\lambda \in \Lambda$. The symbol "=" denotes formal equality of polynomials; e.g., $A \bigvee (B \bigvee C) \neq (A \bigvee B) \bigvee C$. A polynomial $A \in P(Q)$ is said to be a $\bigvee$-*polynomial* if $A = B \bigvee C$, $B, C \in P(Q)$. The dual concept is a $\bigwedge$-polynomial. Thus for each $A \in P(Q)$ one and only one of the following three holds:

(i) $A \in Q$;

(ii) $A$ is a $\bigvee$-polynomial;

(iii) $A$ is a $\bigwedge$-polynomial.

We define an equivalence relation $\equiv$ on $P(Q)$; the relation $\equiv$ is the smallest equivalence relation satisfying the following two conditions and their duals:

(i) $A_0 \bigvee (A_1 \bigvee A_2) \equiv (A_0 \bigvee A_1) \bigvee A_2$ for all $A_0, A_1, A_2 \in P(Q)$;

(ii) if $A_0, A_1, B_0, B_1 \in P(Q)$ and $A_0 \equiv B_0, A_1 \equiv B_1$, then $A_0 \bigvee A_1 \equiv B_1 \bigvee B_0$.

If $A \equiv B$, then $A$ is said to be *equivalent to* $B$ *up to commutativity and associativity.*

The following lemma is proved by a straightforward, but rather lengthy, inductive argument.

LEMMA 1. (i) *The relation* $\equiv$ *is preserved under* $\bigvee$ *and* $\bigwedge$.

(ii) *For all* $A_0, A_1, B_0, B_1 \in P(Q)$, $A_0 \bigvee A_1 \not\equiv B_0 \bigwedge B_1$.

(iii) *If* $A \in P(Q)$ *is a* $\bigvee$-*polynomial, then* $A$ *can be written as*

$$A \equiv A_0 \bigvee \ldots \bigvee A_{r-1}, \qquad r > 1,$$

*where, for each* $i < r$, $A_i$ *is not a* $\bigvee$-*polynomial. This representation is unique up to a permutation of the integers* $0, \ldots, r-1$.

Let $L$ be the free product of the family $(L_\lambda \mid \lambda \in \Lambda)$. Each $A \in P(Q)$ represents an element $\langle A \rangle$ of $L$. Clearly, if $A, B \in P(Q)$ and $A \equiv B$ then $\langle A \rangle = \langle B \rangle$. A polynomial $A \in P(Q)$ is said to be *minimal* if no shorter polynomial in $P(Q)$ represents $\langle A \rangle$; we also say that $A$ is a *minimal representation* of $\langle A \rangle$.

We recall several results of [**1**].

*Definition* A ([**1**]). For each $A \in P(Q)$ and each $\lambda \in \Lambda$, existence and the value of the *lower* $\lambda$-*cover of* $A$, $A_{(\lambda)} \in L_\lambda$, are defined as follows:

(i) If $A \in L_\lambda$, then $A_{(\lambda)}$ exists and $A_{(\lambda)} = A$; $A_{(\mu)}$ does not exist if $\mu \neq \lambda$;

(ii) If $A = B \bigvee C$, then $A_{(\lambda)}$ exists if and only if at least one of $B_{(\lambda)}$, $C_{(\lambda)}$ exists; $A_{(\lambda)} = B_{(\lambda)}$ (respectively $C_{(\lambda)}$) if only $B_{(\lambda)}$ (respectively $C_{(\lambda)}$) exists, and $A_{(\lambda)} = B_{(\lambda)} \vee C_{(\lambda)}$ (join in $L_\lambda$) if both $B_{(\lambda)}$ and $C_{(\lambda)}$ exist;

(iii) If $A = B \bigwedge C$, then $A_{(\lambda)}$ exists if and only if $B_{(\lambda)}$ and $C_{(\lambda)}$ both exist and in this event $A_{(\lambda)} = B_{(\lambda)} \wedge C_{(\lambda)}$ (meet in $L_\lambda$).

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The *upper* $\lambda$-*cover*, $A^{(\lambda)}$, is defined dually.

There is a quasi-order $\subseteq$ on $P(Q)$; $A \subseteq B$ if and only if $\langle A\rangle \leqq \langle B\rangle$ in $L$. The equivalence relation $\cong$ on $P(Q)$ is defined similarly; $A \cong B$ if and only if $\langle A\rangle = \langle B\rangle$, that is, if and only if $A \subseteq B$ and $B \subseteq A$. The major result of [**1**] is the solution of the word problem for free products of lattices, that is, the characterization of the quasi-order $\subseteq$ on $P(Q)$.

THEOREM A ([**1**]). *If $A, B \in P(Q)$, then $A \subseteq B$ if and only if at least one of the following six conditions holds:*

(1) $A = B$;

(2) *there is a $\lambda \in \Lambda$ such that $A^{(\lambda)}$ and $B_{(\lambda)}$ exist and $A^{(\lambda)} \leqq B_{(\lambda)}$;*

(3) $A = A_0 \vee A_1$, *where $A_0 \subseteq B$ and $A_1 \subseteq B$;*

(4) $A = A_0 \wedge A_1$, *where $A_0 \subseteq B$ or $A_1 \subseteq B$;*

(5) $B = B_0 \vee B_1$, *where $A \subseteq B_0$ or $A \subseteq B_1$;*

(6) $B = B_0 \wedge B_1$, *where $A \subseteq B_0$ and $A \subseteq B_1$.*

The following properties of the covers and of the quasi-order $\subseteq$ are useful.

THEOREM B ([**1**]). (a) *If $A \in P(Q)$, $\lambda \in \Lambda$, and $A_{(\lambda)}$, $A^{(\lambda)}$ exist, then $A_{(\lambda)} \leqq A^{(\lambda)}$.*

(b) *If $A \in P(Q)$, $\lambda, \mu \in \Lambda$, and $A_{(\lambda)}$, $A^{(\mu)}$ both exist, then $\lambda = \mu$.*

(c) *If $\Lambda$ consists of only two elements, $\Lambda = \{\lambda, \mu\}$, then, for each $A \in P(Q)$, if $A_{(\mu)}$ does not exist, then $A^{(\lambda)}$ exists, and dually.*

(d) *Let $A, B \in P(Q)$ and $\lambda \in \Lambda$. If $A \subseteq B$ and $A_{(\lambda)}$ exists, then $B_{(\lambda)}$ exists and $A_{(\lambda)} \leqq B_{(\lambda)}$, and dually.*

**3. Normal representations.** In this section we characterize the minimal polynomials in $P(Q)$. In general, a minimal representation of an element $x \in L$ is not unique, but among all the minimal representations of $x$ there is a well-defined unique *normal* representation.

LEMMA 2. *Let $A \in P(Q)$, $\lambda \in \Lambda$, and suppose that $A_{(\lambda)}$ and $A^{(\lambda)}$ exist. If $A^{(\lambda)} \leqq A_{(\lambda)}$, then $A \cong A_{(\lambda)}$.*

*Proof.* By (i) of Definition A, $(A_{(\lambda)})_{(\lambda)} = (A_{(\lambda)})^{(\lambda)} = A_{(\lambda)}$. Thus $A^{(\lambda)} \leqq (A_{(\lambda)})_{(\lambda)}$, and so $A \subseteq A_{(\lambda)}$ by condition (2) of Theorem A. Similarly, $A_{(\lambda)} \subseteq A$ since $(A_{(\lambda)})^{(\lambda)} \leqq A_{(\lambda)}$; thus $A \cong A_{(\lambda)}$.

THEOREM 1. *Let $(L_\lambda | \lambda \in \Lambda)$ be a family of lattices, $Q = \bigcup(L_\lambda | \lambda \in \Lambda)$, and $A \in P(Q)$.*

(a) *If $l(A) = 1$, then $A$ is minimal.*

(b) *If $A$ is a $\vee$-polynomial, and if $A \equiv A_0 \vee \ldots \vee A_{r-1}$, $r > 1$, with no $A_i$ a $\vee$-polynomial, then $A$ is minimal if and only if the following five conditions hold:*

(I) *each $A_i$, $i < r$, is minimal;*

(II) *for each $i < r$, $A_i \not\subseteq A_0 \vee \ldots \vee A_{i-1} \vee A_{i+1} \vee \ldots \vee A_{r-1}$;*

(III) *if $i < r$, $l(A_i) > 1$, $\lambda \in \Lambda$, and $A_i^{(\lambda)}$, $A_{(\lambda)}$ exist, then $A_i^{(\lambda)} \nleqq A_{(\lambda)}$;*

(IV) *if $A_i = C \wedge D$, then $C \not\subseteq A$ and $D \not\subseteq A$;*

(V) *if $i, j < r$, $\lambda \in \Lambda$, and $A_i, A_j \in L_\lambda$, then $i = j$.*

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*Proof.* Part (a) is clear.

We first establish the necessity of the five conditions in part (b). Let $A$ be minimal.

Condition (I) is clearly necessary.

If condition (II) fails, that is, if, say, $A_0 \subseteq A_1 \bigvee \ldots \bigvee A_{r-1}$, then, by condition (3) of Theorem A,

$$A \subseteq A_1 \bigvee \ldots \bigvee A_{r-1} \subseteq A.$$

This implies that $A \cong A_1 \bigvee \ldots \bigvee A_{r-1}$, a shorter polynomial.

If condition (III) fails, that is, if, say, $l(A_0) > 1$ and $A_0^{(\lambda)} \leqq A_{(\lambda)}$, then, by condition (2) of Theorem A, $A_0 \subseteq A_0^{(\lambda)} \subseteq A$, and so

$$A \cong A_0^{(\lambda)} \bigvee A_1 \bigvee \ldots \bigvee A_{r-1}.$$

Since $l(A_0^{(\lambda)}) = 1 < l(A_0)$, the minimality of $A$ is contradicted.

If condition (IV) fails, we may assume that $A_0 = C \bigwedge D$ and $C \subseteq A$. Then a similar argument shows that $A \cong C \bigvee A_1 \bigvee \ldots \bigvee A_{r-1}$, and $l(C) < l(A_0)$.

If condition (V) fails, we may assume that $A_0, A_1 \in L_\lambda$. Then $A_0 \bigvee A_1 \cong A_0 \vee A_1 \in L_\lambda$; thus $A \cong (A_0 \vee A_1) \bigvee A_2 \bigvee \ldots \bigvee A_{r-1}$, and

$$l((A_0 \vee A_1) \bigvee A_2 \bigvee \ldots \bigvee A_{r-1}) = l(A) - 1,$$

again contradicting the minimality of $A$.

Thus the five conditions are necessary. We now establish their sufficiency. Let $A$ satisfy these conditions and let $B \in P(Q)$ be a *minimal* polynomial such that $A \cong B$. We shall show that $l(A) = l(B)$.

We first show that $l(B) > 1$. If $l(B) = 1$, then there is a $\lambda \in \Lambda$ such that $B \in L_\lambda$; thus, by part (d) of Theorem B, $A_{(\lambda)}$ and $A^{(\lambda)}$ exist and $A_{(\lambda)} = A^{(\lambda)} = B$. Since $A^{(\lambda)}$ exists, $A_i^{(\lambda)}$ exists for $i < r$ and $A_i^{(\lambda)} \leqq A^{(\lambda)} = A_{(\lambda)}$. By condition (V) there is an $i < r$ such that $l(A_i) > 1$; this conclusion contradicts condition (III). Thus $l(B) > 1$.

We show next that $B$ cannot be a $\bigwedge$-polynomial. If $B = B_0 \bigwedge B_1$, then, since $A \subseteq B$, we find that, for each $i < r$, $A_i \subseteq B$. Now we also have $B \subseteq A$; this cannot be derived by condition (2) of Theorem A, since $B^{(\lambda)} \leqq A_{(\lambda)}$ implies that $B^{(\lambda)} \leqq B_{(\lambda)}$, contradicting the minimality of $B$ by Lemma 2. Thus either condition (4) or (5) of Theorem A applies; either $B_j \subseteq A$ for some $j < 2$, or $B \subseteq A_{r-1}$, or $B \subseteq A_0 \bigvee \ldots \bigvee A_{r-2}$. If $\lambda \in \Lambda$ and $(A_0 \bigvee \ldots \bigvee A_{r-2})_{(\lambda)}$ exists, then $(A_0 \bigvee \ldots \bigvee A_{r-2})_{(\lambda)} \leqq A_{(\lambda)}$. Thus the quasi-inequality $B \subseteq A_0 \bigvee \ldots \bigvee A_{r-2}$ cannot be derived by condition (2) of Theorem A. Continuing in this vein we find that either $B_j \subseteq A$ for some $j < 2$, or $B \subseteq A_i$ for some $i < r$. If $B_j \subseteq A$, then $B \subseteq B_j \subseteq A \subseteq B$; thus $B \cong B_j$, contradicting the minimality of $B$. On the other hand, $B \subseteq A_0$, say, implies that $A_1 \subseteq A_0$, contradicting condition (II).

Consequently, $B$ is a $\bigvee$-polynomial, say

$$B \equiv B_0 \bigvee \ldots \bigvee B_{s-1}, \qquad s > 1,$$

where no $B_j$ is a $\bigvee$-polynomial. We observe that conditions (I)–(V) apply to $B$ since it is a minimal polynomial.

Let $i < r$ and let $l(A_i) > 1$. Now $A_i \subseteq B$. Condition (1) of Theorem A cannot apply since $A_i$ is a $\bigwedge$-polynomial and $B$ is a $\bigvee$-polynomial. Condition (2) of Theorem A implies that $A_i^{(\lambda)} \leqq B_{(\lambda)} = A_{(\lambda)}$ for some $\lambda \in \Lambda$, contradicting condition (III). Similarly, condition (4) of Theorem A contradicts condition (IV). Thus condition (5) of Theorem A must apply. Continuing this line of argument for several steps, we conclude that there is an $f(i) < s$ such that $A_i \subseteq B_{f(i)}$. Now $l(B_{f(i)}) > 1$, for if $B_{f(i)} \in L_\lambda$, $\lambda \in \Lambda$, then $A_i \subseteq B_{f(i)}$ implies that $A_i^{(\lambda)}$ exists and $A_i^{(\lambda)} \leqq B_{f(i)}$ (part (d) of Theorem B); thus $A_i^{(\lambda)} \leqq B_{f(i)} \leqq B_{(\lambda)} = A_{(\lambda)}$, contradicting (III). Since $B$ also satisfies conditions (I)–(V), we find that for each $j < s$ such that $l(B_j) > 1$ there is a $g(j) < r$ such that $B_j \subseteq A_{g(j)}$; thus $g(f(i))$ exists and $A_i \subseteq B_{f(i)} \subseteq A_{g(f(i))}$. By condition (II), $g(f(i)) = i$, and thus $A_i \cong B_{f(i)}$; by condition (I), $l(A_i) = l(B_{f(i)})$.

Thus we have established the following statement.

(*) For each $i < r$ such that $l(A_i) > 1$ there is an $f(i) < s$ such that $B_{f(i)} \cong A_i$ and $l(B_{f(i)}) = l(A_i)$, and, similarly, for each $j < s$ such that $l(B_j) > 1$ there is a $g(j) < r$ such that $A_{g(j)} \cong B_j$ and $l(A_{g(j)}) = l(B_j)$; furthermore, $g(f(i)) = i$ and $f(g(j)) = j$.

Now let $i < r$ and let $l(A_i) = 1$, that is, let $A_i \in L_\lambda$ for some $\lambda \in \Lambda$. Since $A_i \subseteq B$, part (d) of Theorem B implies that $B_{(\lambda)}$ exists, and thus there is a $j < s$ such that $(B_j)_{(\lambda)}$ exists. With no loss of generality we may assume that $0 < t \leqq s$ and that $(B_j)_{(\lambda)}$ exists if and only if $j < t$. Thus

$$B_{(\lambda)} = (B_0)_{(\lambda)} \vee \ldots \vee (B_{t-1})_{(\lambda)}.$$

If $l(B_j) > 1$ for all $j < t$, then, applying (*), $l(A_{g(j)}) > 1$ and $(A_{g(j)})_{(\lambda)} = (B_j)_{(\lambda)}$ for all $j < t$. Thus $g(j) \neq i$ for all $j < t$, and

$$(B_0)_{(\lambda)} \vee \ldots \vee (B_{t-1})_{(\lambda)} \leqq (A_0 \bigvee \ldots \bigvee A_{i-1} \bigvee A_{i+1} \bigvee \ldots \bigvee A_{r-1})_{(\lambda)}.$$

Thus $A_i \leqq (A_0 \bigvee \ldots \bigvee A_{i-1} \bigvee A_{i+1} \bigvee \ldots \bigvee A_{r-1})_{(\lambda)}$, contradicting condition (II). Consequently, there is an $f(i) < s$ such that $(B_{f(i)})_{(\lambda)}$ exists and $l(B_{f(i)}) = 1$, that is, such that $B_{f(i)} \in L_\lambda$.

Similarly, if $j < s$ and $B_j \in L_\lambda$ for some $\lambda \in \Lambda$, then there is a $g(j) < r$ such that $A_{g(j)} \in L_\lambda$. By condition (V), $g(f(i)) = i$ for any $i < r$ such that $l(A_i) = 1$ and $f(g(j)) = j$ for any $j < s$ such that $l(B_j) = 1$. Thus we have established the following statement.

(**) There are mappings

$$f\colon \{0, \ldots, r-1\} \to \{0, \ldots, s-1\}, \qquad g\colon \{0, \ldots, s-1\} \to \{0, \ldots, r-1\}$$

satisfying the conditions:

(i) if $i < r$, then $g(f(i)) = i$ and if $j < s$, then $f(g(j)) = j$;

(ii) if $i < r$ and $l(A_i) > 1$, then $B_{f(i)} \cong A_i$ and $l(B_{f(i)}) = l(A_i)$, and similarly for any $j < s$ such that $l(B_j) > 1$;

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(iii) if $i < r$ and $A_i \in L_\lambda$, $\lambda \in \Lambda$, then $B_{f(i)} \in L_\lambda$, and similarly for any $j < s$ such that $B_j \in L_\lambda$, $\lambda \in \Lambda$.

Consequently, $r = s$ and $f, g$ are permutations of $\{0, \ldots, r - 1\}$. Since $l(A_i) = l(B_{f(i)})$ for all $i < r$, it follows that $l(A) = l(B)$. Since $B$ is minimal, so is $A$.

We note that the proof of the necessity in Theorem 1 provides an algorithm for reducing any polynomial to a minimal polynomial representing the same element of $L$; by the proof of sufficiency, the resulting polynomial will be minimal.

We now present an algorithm determining when two minimal polynomials represent the same element of $L$.

THEOREM 2. *Let $A, B \in P(Q)$ be minimal polynomials. If $l(A) = 1$, then $A \cong B$ if and only if $A = B$. If $A \equiv A_0 \vee \ldots \vee A_{r-1}$, $r > 1$, where no $A_i$ is a $\vee$-polynomial, then $A \cong B$ if and only if $B$ can be written in the form $B \equiv B_0 \vee \ldots \vee B_{r-1}$ such that the following four conditions hold:*

(i) *no $B_i$ is a $\vee$-polynomial;*

(ii) *for each $i < r$ and $\lambda \in \Lambda$, $A_i \in L_\lambda$ if and only if $B_i \in L_\lambda$;*

(iii) *for each $i < r$, $l(A_i) > 1$ if and only if $l(B_i) > 1$ and in this event $A_i \cong B_i$;*

(iv) *for each $i < r$ and $\lambda \in \Lambda$, $A_i \in L_\lambda$ implies that $B_{(\lambda)}$ exists and $A_i \leqq B_{(\lambda)}$, and $B_i \in L_\lambda$ implies that $A_{(\lambda)}$ exists and $B_i \leqq A_{(\lambda)}$.*

*Proof.* If $A \cong B$, then conditions (i), (ii), and (iii) are statement (**) in the proof of Theorem 1. Condition (iv) follows since $A_{(\lambda)} = B_{(\lambda)}$ for all $\lambda \in \Lambda$ such that $A_{(\lambda)}$ exists.

If conditions (i)–(iv) hold, then, if $l(A_i) > 1$, $A_i \subseteq B_i \subseteq B$. If $A_i \in L_\lambda$, $\lambda \in \Lambda$, then $A_i^{(\lambda)} = A_i \leqq B_{(\lambda)}$; thus $A_i \subseteq B$. Thus $A_i \subseteq B$ for all $i < r$, and so $A \subseteq B$. By the symmetry of the conditions, $B \subseteq A$, and the theorem follows.

In general, an element of $L$ has several different minimal representations (see § 4). Of these we choose one, well-defined up to commutativity and associativity, which we call the *normal* representation.

*Definition* 1. If $A \in P(Q)$ and $l(A) = 1$, then $A$ is a normal polynomial. If $l(A) > 1$, then $A$ is a normal polynomial if and only if the following two conditions hold:

(i) $A$ is a minimal polynomial;

(ii) if $A$ is a $\vee$-polynomial, that is, if $A \equiv A_0 \vee \ldots \vee A_{r-1}$, $r > 1$, and no $A_i$ is a $\vee$-polynomial, then each $A_i$ is normal, and if, for some $i < r$, $A_i \in L_\lambda$ for some $\lambda \in \Lambda$, then $A_i = A_{(\lambda)}$, and dually if $A$ is a $\wedge$-polynomial.

We observe that in (ii) it is clear that if $A_i \in L_\lambda$ for some $i < r$, then $A_{(\lambda)}$ exists.

THEOREM 3. (a) *Each $x \in L$ has a normal representation.*

(b) *For each $x \in L$, its normal representation is unique up to commutativity and associativity.*

*Proof.* Let $x \in L$ and let $B \in P(Q)$ be a minimal representation of $x$. If $l(B) = 1$, then $B$ is normal. If $l(B) > 1$, we may assume, by the principle of duality, that $B \equiv B_0 \bigvee \ldots \bigvee B_{r-1}$, $r > 1$, and no $B_i$ is a $\bigvee$-polynomial. By induction on the length of polynomials, we may assume that each $B_i$ is normal. For each $i < r$ we will define $A_i \in P(Q)$. If $l(B_i) > 1$, then $A_i = B_i$. If $l(B_i) = 1$, then there is a $\lambda \in \Lambda$ such that $B_i \in L_\lambda$ and thus $B_{(\lambda)}$ exists; define $A_i = B_{(\lambda)}$. Let $A \equiv A_0 \bigvee \ldots \bigvee A_{r-1}$. By Theorem 2, $A \cong B$ and, since $l(B) = l(A)$, $A$ is minimal; thus $A$ is a normal representation of $x$.

Now let $A$ and $B$ be normal, $A \cong B$. If $l(A) = 1$, then $l(B) = 1$ and $A = B$. If $A$ is a $\bigvee$-polynomial, then Theorem 2 implies that $A \equiv B$, by a trivial induction on the length of normal representations.

**4. Canonical representations.** $L$, the free product of the family of lattices $(L_\lambda | \lambda \in \Lambda)$, is said to *admit canonical representations* if a minimal representation of each element of $L$ is unique up to commutativity and associativity, that is, if $A \cong B$, $A$, $B$ minimal, imply that $A \equiv B$. In view of Theorem 3, $L$ admits canonical representations if and only if every minimal polynomial in $P(Q)$ is normal.

LEMMA 3. *Let $|\Lambda| > 1$ and let the family $(L_\lambda | \lambda \in \Lambda)$ satisfy one of the following three conditions:*

(i) *there is a $\lambda \in \Lambda$ and incomparable $x, y \in L_\lambda$ such that $x \vee y$ is not maximal in $L_\lambda$, or dually;*

(ii) *there are distinct $\lambda, \mu \in \Lambda$ such that neither $L_\lambda$ nor $L_\mu$ is a chain;*

(iii) *$|\Lambda| \geqq 3$ and there is a $\lambda \in \Lambda$ such that $L_\lambda$ is not a chain.*

*Then the free product of the $(L_\lambda | \lambda \in \Lambda)$ does not admit canonical representations.*

*Proof.* Let condition (i) hold. Let $x \vee y = z$. Then $x$, $y$, and $z$ are all distinct and there is a $w \in L_\lambda$ such that $z < w$. Let $\mu \neq \lambda$ and let $d \in L_\mu$. Let $A = x \bigvee ((y \bigvee d) \bigwedge w)$. Applying Theorem 1, $y \bigvee d$ is minimal. Since $w \not\supseteq y \bigvee d$ ($y \bigvee d$ has no upper covers), the dual of Theorem 1 implies that $(y \bigvee d) \bigwedge w$ is minimal. Similarly, $x \not\subseteq (y \bigvee d) \bigwedge w$, and so $A$ is minimal. However, $A_{(\lambda)} = z \neq x$. Thus $A$ is not normal; indeed, the equivalent normal representation is $z \bigvee ((y \bigvee d) \bigwedge w)$.

Let condition (ii) hold. Let $x, y \in L_\lambda$ be incomparable and let $d_1, d_2 \in L_\mu$ be incomparable. Let $A = x \bigvee ((y \bigvee d_1) \bigwedge (y \bigvee d_2))$. Theorem 1 and its dual show that $A$ is minimal. However, $A_{(\lambda)} = x \vee y \neq x$; thus $A$ is minimal but not normal.

If condition (iii) holds, let $\mu_1, \mu_2 \in \Lambda$ be distinct indices different from $\lambda$. Let $x, y \in L_\lambda$ be incomparable and let $d_1 \in L_{\mu_1}$, $d_2 \in L_{\mu_2}$. Let

$$A = x \bigvee ((y \bigvee d_1) \bigwedge (y \bigvee d_2));$$

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$A$ is minimal. Since $A_{(\lambda)} = x \vee y \neq x$, $A$ is minimal but not normal.

Consequently, the free product does not admit canonical representations under any of the three conditions.

LEMMA 4. *Let $\lambda \in \Lambda$, $A_0 \in L_\lambda$, and $A' \in P(Q)$. If $A = A_0 \mathbf{V} A'$ is minimal and either $A_{(\lambda)}'$ does not exist or $A_{(\lambda)}'$ is comparable with $A_0$, then $A_{(\lambda)} = A_0$.*

*Proof.* If $A_{(\lambda)}'$ does not exist, then clearly $A_{(\lambda)} = A_0$. If $A_{(\lambda)}'$ exists and $A_0 \leqq A_{(\lambda)}'$, then $A_0 \subseteq A'$; thus $A \cong A'$, contradicting the minimality of $A$. Thus $A_0 > A_{(\lambda)}'$, and so $A_{(\lambda)} = A_0$.

COROLLARY (Sorkin [3]). *If $L_\lambda$ is a chain for each $\lambda \in \Lambda$, then the free product of the family $(L_\lambda | \lambda \in \Lambda)$ admits canonical representations.*

*Proof.* Since, for each $\lambda \in \Lambda$, any two elements of $L_\lambda$ are comparable, Lemma 4 and its dual show that any minimal polynomial is normal.

Lemma 3 and the corollary to Lemma 4 cover all non-trivial cases but one: $|\Lambda| = 2$, say $\Lambda = \{\lambda, \mu\}$, $L_\mu$ is a chain, and given incomparable $x, y \in L_\lambda$, then $x \vee y$ is maximal in $L_\lambda$ and $x \wedge y$ is minimal. Thus $L_\lambda$ has a 0 and a 1, and $L_\lambda - \{0, 1\}$ is the disjoint union of unrelated chains. We show that in this case the free product of $L_\lambda$ and $L_\mu$ admits canonical representations.

LEMMA 5. *Let $\Lambda = \{\lambda, \mu\}$ and let $L_\lambda$, $L_\mu$ be as described above. If $A \in P(Q)$ and $A_{(\mu)}$ does not exist, then $A \subseteq 1$. If $A_{(\mu)}$ exists, then $A \subseteq 1 \mathbf{V} A_{(\mu)}$.*

*Proof.* If $A_{(\mu)}$ does not exist, then, by part (c) of Theorem B, $A^{(\lambda)}$ exists. Since $A^{(\lambda)} \leqq 1 = 1_{(\lambda)}$, $A \subseteq 1$.

Now assume that $A_{(\mu)}$ exists. We prove that $A \subseteq 1 \mathbf{V} A_{(\mu)}$ by induction on $l(A)$.

If $l(A) = 1$, then $A \in L_\mu$ and $A = A_{(\mu)} \subseteq 1 \mathbf{V} A_{(\mu)}$.

If $A = B \mathbf{\Lambda} C$, then both $B_{(\mu)}$ and $C_{(\mu)}$ exist. Since $L_\mu$ is a chain we may assume that $B_{(\mu)} \leqq C_{(\mu)}$. Thus $A_{(\mu)} = B_{(\mu)}$ and, by the induction hypothesis, $B \subseteq 1 \mathbf{V} B_{(\mu)} = 1 \mathbf{V} A_{(\mu)}$. Since $A \subseteq B$, the result follows.

If $A = B \mathbf{V} C$, then one of $B_{(\mu)}$, $C_{(\mu)}$ exists. If only $B_{(\mu)}$ exists, then $A_{(\mu)} = B_{(\mu)}$ and, by induction, $B \subseteq 1 \mathbf{V} B_{(\mu)} = 1 \mathbf{V} A_{(\mu)}$. Since $C_{(\mu)}$ does not exist, $C \subseteq 1 \subseteq 1 \mathbf{V} A_{(\mu)}$; thus $A \subseteq 1 \mathbf{V} A_{(\mu)}$. If both $B_{(\mu)}$ and $C_{(\mu)}$ exist, then

$$B \subseteq 1 \mathbf{V} B_{(\mu)} \subseteq 1 \mathbf{V} A_{(\mu)}, \qquad C \subseteq 1 \mathbf{V} C_{(\mu)} \subseteq 1 \mathbf{V} A_{(\mu)}.$$

Thus $A \subseteq 1 \mathbf{V} A_{(\mu)}$.

LEMMA 6. *The free product of $L_\lambda$ and $L_\mu$ admits canonical representations.*

*Proof.* We show that any minimal polynomial is normal. We proceed by induction on the length of the minimal polynomials. By definition, any polynomial of length 1 is normal. If $A \in P(Q)$ is minimal and $l(A) > 1$ we can assume, by the principle of duality, that

$$A \equiv A_0 \mathbf{V} \ldots \mathbf{V} A_{r-1}, \qquad r > 1,$$

where no $A_i$ is a $\mathbf{V}$-polynomial and, by induction, each $A_i$ is normal.

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If $i < r$ and $A_i \in L_\mu$ then, by Lemma 4, $A_i = A_{(\mu)}$ since $L_\mu$ is a chain. If $i < r$ and $A_i \in L_\lambda$, let

$$A' = A_0 \vee \ldots \vee A_{i-1} \vee A_{i+1} \vee \ldots \vee A_{r-1}.$$

By Lemma 4 we need only consider the case where $A_{(\lambda)}'$ exists and is incomparable with $A_i$. Then $A_{(\lambda)} = 1$ and so $1 \subseteq A$. If $A_{(\mu)}'$ does not exist, then $A_{(\mu)}$ does not exist and so $A \subseteq 1$ by Lemma 5; thus $A \cong 1$, contradicting the minimality of $A$. If $A_{(\mu)}'$ exists, then $A_{(\mu)}$ exists, and so $A \subseteq 1 \vee A_{(\mu)}$. Since $1 \subseteq A$ and $A_{(\mu)} \subseteq A$, we conclude that $A \cong 1 \vee A_{(\mu)}$. Since both $A_{(\lambda)}'$ and $A_{(\mu)}'$ exist, $l(A) \geqq 3$. However $l(1 \vee A_{(\mu)}) = 2$, again contradicting the minimality of $A$. Consequently, if $A_i \in L_\lambda$, then $A_i = A_{(\lambda)}$. Thus $A$ is normal and the induction is complete.

Summarizing Lemma 3, the corollary to Lemma 4, and Lemma 6, we have the following.

THEOREM 4. *Let $L$ be the free product of the family of lattices $(L_\lambda | \lambda \in \Lambda)$. $L$ admits canonical representations if and only if one of the following three conditions holds:*

(i) $|\Lambda| = 1$;

(ii) *$L_\lambda$ is a chain for each $\lambda \in \Lambda$;*

(iii) *$|\Lambda| = 2$, say $\Lambda = \{\lambda, \mu\}$, $L_\mu$ is a chain, $L_\lambda$ has 0 and 1, and $L_\lambda - \{0, 1\}$ is a disjoint union of unrelated chains.*

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