Can. J. Math. Vol. **50** (2), 1998 pp. 356–377

SOME NORMS ON UNIVERSAL ENVELOPING ALGEBRAS

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ABSTRACT. The universal enveloping algebra, $U(\mathfrak{g})$, of a Lie algebra $\mathfrak{g}$ supports some norms and seminorms that have arisen naturally in the context of heat kernel analysis on Lie groups. These norms and seminorms are investigated here from an algebraic viewpoint. It is shown that the norms corresponding to heat kernels on the associated Lie groups decompose as product norms under the natural isomorphism $U(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \cong U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2)$. The seminorms corresponding to Green's functions are examined at a purely Lie algebra level for $\mathrm{sl}(2, \mathbb{C})$. It is also shown that the algebraic dual space U' is spanned by its finite rank elements if and only if $\mathfrak{g}$ is nilpotent.

1. **Introduction.** The present, essentially algebraic, work is motivated by some recent developments in "heat kernel analysis" on Lie groups. Suppose that G is a complex, connected, simply connected Lie group, that $\mathfrak{g}$ is its Lie algebra, that U is the universal enveloping algebra of $\mathfrak{g}$ and that U' is the algebraic dual of U. A holomorphic function u on G defines an element $\hat{u}$ of U' by means of the pairing $U \ni \beta \longmapsto \langle \hat{u}, \beta \rangle = (\beta u)(e)$, wherein β is to be interpreted as a left invariant differential operator on G. Given a Hermitian inner product on $\mathfrak{g}$ there is naturally associated to it a left invariant second order elliptic differential operator, Δ, on G, and for each real number $t > 0$ there is associated a unique probability measure μ_t on G, convolution by which gives the semigroup $e^{t\Delta/4}$. In this way the given inner product on $\mathfrak{g}$ determines a Hilbert space $L^2_{\mathrm{holo}}(G, \mu_t)$, consisting of those holomorphic functions on G which are square integrable with respect to μ_t. At the same time the inner product on $\mathfrak{g}$ determines a natural inner product on a subspace U^*_t of U' in a manner which will be described explicitly in Section 4. It is shown in [DG] that the preceding map $u \longrightarrow \hat{u}$, restricted to $L^2_{\mathrm{holo}}(G, \mu_t)$, is a unitary operator onto U^*_t. Similar unitary maps have also appeared for real Lie algebras of compact type and are closely linked with analysis over loop groups [D, DG, G1,2,3,4, GM, Ha1,2, Hij1,2].

The existence of these unitary isomorphisms raises some questions of an essentially algebraic nature concerning the subspaces U^*_t. It is the purpose of the present work to explore some of these questions. This paper will be concerned entirely with the algebraic side, U^*_t, of this isomorphism.

In Section 2 the structure of the algebraic dual U' will be investigated with the help of a kind of dual (Theorem 2.7) to the Poincaré-Birkhoff-Witt theorem. As an application it will be shown that U' is generated linearly by its finite rank elements (*cf.* Definition 2.12) if and only if $\mathfrak{g}$ is nilpotent. $\mathfrak{g}$ may be taken to be a Lie algebra over a field of characteristic

Received by the editors April 7, 1997; revised August 18, 1997.

This work was partially supported by NSF Grant DMS 9501238 and by the Humboldt Foundation.

AMS subject classification: Primary: 17B35, 16S30; secondary: 22E30.

https://doi.org/10.4153/CJM-1998-019-4 Published online by Cambridge University Press

zero in Section 2. In Section 3 it will be shown that the isomorphism in Theorem 2.7 amounts to a description of the element $\hat{u} \in U'$ given above, in terms of the element $(u \circ \exp)^{\wedge}$, which is in the dual of the universal enveloping algebra of the additive group $\mathfrak{g}$.

In Section 4 two kinds of norms (or seminorms) on U and on subspaces of U' will be discussed. The main theorem of Section 4 asserts that the usual isomorphism [B, Chapter 1, Section 2.2] between the universal enveloping algebra of a direct sum of real (or complex) Lie algebras on the one hand and the tensor product of their respective universal enveloping algebras on the other is isometric with respect to certain natural norms on the universal enveloping algebras.

It is a pleasure to thank D. Barbasch and B. Speh for useful discussions and to thank Brian Hall for communication of Lemma 4.1 to the author. The author also wishes to thank the referee for providing the proof of Theorem 4.15 and the illuminating Remark 2.10.

2. **The dual of the universal enveloping algebra.**

NOTATION 2.1. In this section $\mathfrak{g}$ will denote a finite or infinite dimensional Lie algebra over a field F of characteristic zero. For any vector space W over F denote by W' the algebraic dual space and by $W^{\otimes k}$ the algebraic tensor product, with $W^{\otimes 0} = F$. Let $T := \sum_{k=0}^{\infty} \mathfrak{g}^{\otimes k}$ be the tensor algebra over $\mathfrak{g}$. Then we may identify its algebraic dual space T' with $\prod_{k=0}^{\infty}(\mathfrak{g}^{\otimes k})'$, in the pairing

$$\langle \alpha, \beta \rangle = \sum_{k=0}^{\infty} \langle \alpha_k, \beta_k \rangle \quad \alpha \in T',\ \beta \in T \tag{2.1}$$

where

$$\alpha = \sum_{k=0}^{\infty} \alpha_k \quad \alpha_k \in (\mathfrak{g}^{\otimes k})',\ k = 0, 1, 2, \ldots \tag{2.2}$$

and

$$\beta = \sum_{k=0}^{N} \beta_k \quad \beta_k \in \mathfrak{g}^{\otimes k},\ k = 0, 1, 2, \ldots, N. \tag{2.3}$$

In particular this identifies $(\mathfrak{g}^{\otimes k})'$ as a subspace of T'. An element α given by (2.2) will be said to be *zero* in rank k if $\alpha_k = 0$ and will be said to be of *finite rank* if $\alpha_k = 0$ for all sufficiently large k. For any two tensors u and v write $u \wedge v = u \otimes v - v \otimes u$.

Denote by J the two sided ideal in T generated by

$$\{\xi \wedge \eta - [\xi, \eta] : \xi, \eta \in \mathfrak{g}\}.$$

Denote by I the two sided ideal in T generated by $\{\xi \wedge \eta : \xi, \eta \in \mathfrak{g}\}$ and let $I_k = I \cap \mathfrak{g}^{\otimes k}$, $k = 0, 1, 2, \ldots$. If S_k denotes the space of symmetric tensors in $\mathfrak{g}^{\otimes k}$ and $S = \sum_{k=0}^{\infty} S_k$ then $\mathfrak{g}^{\otimes k} = I_k + S_k$, $k = 0, 1, \ldots$, and $T = I + S$.

There is a third ideal which will be needed. Define $\mathcal{N}_k = J \cap \mathfrak{g}^{\otimes k}$ for $k = 0, 1, \ldots$ and let $\mathcal{N} = \sum_{k=0}^{\infty} \mathcal{N}_k$. If $\beta \in \mathcal{N}_k$ and $\gamma \in \mathfrak{g}^{\otimes j}$ then $\beta \otimes \gamma$ and $\gamma \otimes \beta$ are both in $\mathcal{N}_{k+j}$. So $\mathcal{N}$ is a 2-sided ideal in T. By the Poincaré-Birkhoff-Witt theorem [B, Chapter I, Lemma 7.3] $\mathcal{N}_k \subset I_k$ for all k. Hence $\mathcal{N} \subset \mathcal{I}$.

EXAMPLE 2.2. If $\mathfrak{g}$ is commutative then $J = \mathcal{I} = \mathcal{N}$.

EXAMPLE 2.3. Suppose that a, b, c, d are in $\mathfrak{g}$. Then $\beta := (a \wedge b - [a, b]) \otimes (c \wedge d) - (a \wedge b) \otimes (c \wedge d - [c, d])$ is in J. But expanding gives $\beta = -[a, b] \otimes (c \wedge d) + (a \wedge b) \otimes [c, d]$, which is a homogeneous tensor of rank 3. Hence $\beta \in \mathcal{N}_3$. Moreover if $\mathfrak{g} = su(2)$, if e_1, e_2, e_3 are elements such that $[e_i, e_j] = e_k$ for (i, j, k) cyclic, and if $a = e_1$, $b = e_2$, $c = e_3$ and $d = e_1$, then a straightforward computation shows that $\beta \neq 0$. Thus even for a simple Lie algebra $\mathcal{N}$ is not zero.

Denote by J^0, $\mathcal{I}^0$, $\mathcal{N}^0$ the annihilators of these ideals in T'. Writing $U = T/J$ for the universal enveloping algebra of $\mathfrak{g}$, it is clear that one may identify the algebraic dual space U' with J^0. It is J^0 that will actually be used, rather than U'. Note that $\mathcal{I}^0$ consists exactly of the symmetric tensors in T'.

DEFINITION 2.4. By the Poincaré-Birkhoff-Witt theorem one has a direct sum decomposition

$$T = \mathcal{S} + J \tag{2.4}$$

Define a linear map

$$V_{\mathfrak{g}} : \mathcal{I}^0 \longrightarrow J^0 \tag{2.5}$$

as follows. Let $\gamma \in \mathcal{I}^0$ and let $u \in T$. By (2.4) we may write uniquely $u = u_{\mathcal{S}} + u_J$ with $u_{\mathcal{S}} \in \mathcal{S}$ and $u_J \in J$. Define

$$(V_{\mathfrak{g}} \gamma)(u) = \gamma(u_{\mathcal{S}}) \tag{2.6}$$

Then $V_{\mathfrak{g}}\gamma \in J^0$ because if u is in J then $u_{\mathcal{S}} = 0$. Since $\mathcal{I}^0$ is naturally isomorphic to $(T/\mathcal{I})'$, $V_{\mathfrak{g}}$ may be regarded as a map from $(T/\mathcal{I})'$ to $(T/J)'$. It is in fact the adjoint of the natural map $\lambda : T/J \longrightarrow T/\mathcal{I}$ described in [V, Theorem 3.3.4]. We will need an inductive algorithm for constructing $V_{\mathfrak{g}}$. This is the goal of Theorem 2.8.

DEFINITION 2.5. Define a linear map

$$B : T \longrightarrow T/\mathcal{N}$$

as follows. If $\beta \in \mathcal{S}$ define $B\beta = 0$. If $\beta \in \mathcal{I}_k$ and there exists $\beta' \in \mathfrak{g}^{\otimes(k-1)}$ such that $\beta - \beta' \in J$ then define $B\beta = \beta' + \mathcal{N}$. Such an element β' always exists when $\beta \in \mathcal{I}_k$ because any element β in $\mathcal{I}_k$ is a sum of k-tensors of the form $u \otimes (\xi \wedge \eta) \otimes v$ with ξ and η in $\mathfrak{g}$. Since $u \otimes (\xi \wedge \eta) \otimes v - u \otimes [\xi, \eta] \otimes v \in J$, β' can be chosen to be a corresponding sum of elements $u \otimes [\xi, \eta] \otimes v$. B is well defined on $\mathcal{I}_k$ because if β' and β'' are both in $\mathfrak{g}^{\otimes(k-1)}$ while $\beta - \beta'$ and $\beta - \beta''$ are both in J then $\beta' - \beta'' \in J \cap \mathfrak{g}^{\otimes(k-1)} = \mathcal{N}_{k-1}$. so $\beta'' = \beta' \bmod \mathcal{N}$. Therefore B is well defined on all of $\mathcal{I}$ and hence on all of T.

We may and will identify $(T/\mathcal{N})'$ with $\mathcal{N}^0$. Then the adjoint map B^* may and will be regarded as a map from $\mathcal{N}^0$ into T'.

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PROPOSITION 2.6. *The maps* $B: T \longrightarrow T/\mathcal{N}$ *and* $B^*: \mathcal{N}^0 \longrightarrow T'$ *satisfy the following:*

a) $B\mathfrak{g}^{\otimes k} \subset \mathfrak{g}^{\otimes(k-1)} \bmod \mathcal{N}$.

b) $B\mathcal{N} = 0 \bmod \mathcal{N}$.

c) $B^*\mathcal{N}^0 \subset \mathcal{N}^0$.

d) $B^*\big(\mathcal{N}^0 \cap (\mathfrak{g}^{\otimes k})'\big) \subset \mathcal{N}^0 \cap (\mathfrak{g}^{\otimes(k+1)})'$.

e) The restriction $B^*\gamma|\mathcal{S} = 0$ *for all* $\gamma \in \mathcal{N}^0$.

PROOF. a) is part of the definition of B. If $\beta \in \mathcal{N}_k$ and $\beta' = 0$ then $\beta - \beta' \in J$. So $B\beta = 0 \bmod \mathcal{N}$, which is b). If $\gamma \in \mathcal{N}^0$ and $n \in \mathcal{N}$ then $\langle B^*\gamma, n\rangle = \langle \gamma, Bn\rangle = 0$ by b). So c) holds. d) follows from a). If $\gamma \in \mathcal{N}^0$ then $\langle B^*\gamma, s\rangle = \langle \gamma, Bs\rangle = 0$ for all s in $\mathcal{S}$ by Definition 2.5. This proves the assertion e). ∎

LEMMA 2.7. *Let* $\gamma \in \mathcal{I}^0$. *Define*

$$V\gamma = \sum_{k=0}^{\infty} (B^*)^k\gamma \tag{2.7}$$

Then $V\gamma \in J^0$.

PROOF. Since $\mathcal{N} \subset \mathcal{I}$, $\mathcal{I}^0 \subset \mathcal{N}^0$. So if $\gamma \in \mathcal{I}^0$ then $B^*\gamma$ is well defined, and, by Proposition 2.6c), so is $(B^*)^k$ for all k. If β is given by (2.3) then $B^k\beta = 0$ for $k > N$. So the series $\sum_{k=0}^{\infty}\langle (B^*)^k\gamma, \beta\rangle$ has only finitely many nonzero terms. Hence V is well defined. Moreover all summands in (2.7) are in $\mathcal{N}^0$. Hence $V\gamma \in \mathcal{N}^0$. Let ξ and η be in $\mathfrak{g}$ and let u and v be homogeneous elements of T. Let $w = u \otimes (\xi \wedge \eta) \otimes v$. To prove that $V\gamma \in J^0$ it suffices to show that $\langle V\gamma, w - u \otimes [\xi, \eta] \otimes v\rangle = 0$. But

$$\langle V\gamma, w - u \otimes [\xi, \eta] \otimes v\rangle = \langle V\gamma, w - Bw\rangle = \langle (I - B^*)(V\gamma), w\rangle = \langle \gamma, w\rangle$$

by collapsing the two (finite) sums that appear after the second equality. Since $w \in \mathcal{I}$, $\langle \gamma, w\rangle = 0$. ∎

THEOREM 2.8. *The map* $V: \mathcal{I}^0 \longrightarrow J^0$ *defined by (2.7) is a one to one map of* $\mathcal{I}^0$ *onto* J^0. *Moreover*

$$(V\gamma)|\mathcal{S} = \gamma|\mathcal{S} \quad \textit{for all } \gamma \in \mathcal{I}^0. \tag{2.8}$$

and $V = V_{\mathfrak{g}}$.

LEMMA 2.9. *If* $\alpha \in J^0$ *and* $\alpha|\mathcal{S} = 0$ *then* $\alpha = 0$.

PROOF. Assume $\alpha \in J^0$ and $\alpha|\mathcal{S} = 0$. Then $\alpha = 0$ in ranks zero and one. Let $n \geq 2$ and assume, for an induction proof, that $\alpha = 0$ in rank k for all $k < n$. If $\beta \in \mathcal{I}_n$ then, as noted in the definition of B, there exists an element $\beta' \in \mathfrak{g}^{\otimes(n-1)}$ such that $\beta - \beta' \in J$. Thus $\langle \alpha, \beta - \beta'\rangle = 0$. But $\langle \alpha, \beta'\rangle = 0$ by the induction assumption. Hence $\langle \alpha, \beta\rangle = 0$. So $\alpha = 0$ on both $\mathcal{I}_n$ and on $\mathcal{S}$, hence on $\mathfrak{g}^{\otimes n}$. ∎

PROOF OF THEOREM 2.8. In view of Lemma 2.7 it must be shown that V is injective and surjective. First observe that by Proposition 2.6e) $\big((B^*)^k\gamma\big)|\mathcal{S} = 0$ for $k \geq 1$.

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Therefore (2.8) follows from (2.7). Thus if $\gamma \in I^0$ and $V\gamma = 0$ then (2.8) shows that $\gamma|S = 0$. But since $\gamma|I = 0$ and $T = S + I$ it follows that $\gamma = 0$. So V is injective. To prove V is surjective suppose that $\alpha \in J^0$. Define $\gamma \in T'$ to be α on S and zero on I. Then $\gamma \in I^0$. Let $\alpha' = V\gamma$. Then $\alpha' \in J^0$ by Lemma 2.7. But $\alpha'|S = \gamma|S = \alpha|S$ by (2.8) and the definition of γ. Hence $\alpha' = \alpha$ by Lemma 2.9. So $\alpha = V\gamma$. Finally, to show that $V = V_{\mathfrak{g}}$ let $\gamma \in I^0$. Using (2.6), (2.8) and the fact that $V\gamma \in J^0$ we have $(V_{\mathfrak{g}}\gamma)(u) = \gamma(u_S) = (V\gamma)(u_S) = (V\gamma)(u)$. ∎

REMARK 2.10. The equation (2.7) can be interpreted as providing an inductive algorithm for computing $V_{\mathfrak{g}}u$, as defined in (2.6), when u is in $\mathfrak{g}^{\otimes n}$. Choose an element $u_1 \in \mathfrak{g}^{\otimes(n-1)}$ in the equivalence class Bu, an element $u_2 \in \mathfrak{g}^{\otimes(n-2)}$ in the equivalence class Bu_1, etc. Then u_{n-1} is in $\mathfrak{g}$ and is therefore symmetric. So $u_n = Bu_{n-1} = 0$. Let P_s denote the symmetrization projection on T and define $v_k = (u_k - P_s u_k) - u_{k+1}$. Since $u_k - P_s u_k \in I_k$ and $u_{k+1} = Bu_k = B(u_k - P_s u_k)$ it follows from Definition 2.5 that $v_k \in J$. Thus, writing $u_0 = u$, we have $u_k - u_{k+1} = P_s u_k + v_k$. Summing from $k = 0$ to $n-1$ we get

$$u = \sum_{k=0}^{n-1} P_s u_k + \sum_{k=0}^{n-1} v_k. \tag{2.9}$$

The first sum is in S while the second sum is in J. Therefore the unique decomposition $u = u_S + u_J$ used in Definition 2.4 may be accomplished by taking $u_S = \sum_{k=0}^{n-1} P_s u_k$. But if γ is in I^0 then $\gamma \circ P_s = \gamma$. So $(V_{\mathfrak{g}}\gamma)(u) = \sum_{k=0}^{n-1} \gamma(u_k)$, which agrees with (2.7).

COROLLARY 2.11. *If $\alpha \in J^0$ and is given by (2.2) with $\alpha_k = 0$ for $k = 0, 1, \ldots, m-1$ then α_m is symmetric. i.e. $\alpha_m \in I^0$.*

PROOF. By Theorem 2.8 there exists an element $\gamma \in I^0$ such that $\alpha = V\gamma$. Suppose that $k < m$. Then by (2.8), $\gamma_k|S_k = \alpha|S_k = 0$ and therefore $\gamma_k = 0$. Equation (2.7) now shows that $\alpha_m = \gamma_m$, which is in I^0. ∎

The following remark has been made in [G2, Remark 3.5] but bears repeating here because of the next theorem.

REMARK 2.12. If $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ then J^0 contains no elements of finite rank except those of rank zero. For if $J^0 \ni \alpha = \sum_{k=0}^m \alpha_k$ with $\alpha_k \in (\mathfrak{g}^{\otimes k})'$, and if $i+j+1 = m$, with $u \in \mathfrak{g}^{\otimes i}$, $v \in \mathfrak{g}^{\otimes j}$, ξ and $\eta \in \mathfrak{g}$ and $\zeta = [\xi, \eta]$, then $\langle \alpha_m, u \otimes \zeta \otimes v\rangle = -\langle \alpha, u \otimes (\xi \wedge \eta - [\xi, \eta]) \otimes v\rangle = 0$. Since such ζ span $\mathfrak{g}$, $\alpha_m = 0$ if $m \geq 1$. The assertion now follows by induction on m.

REMARK 2.13. In view of the preceding remark the space J^0 is a particularly complicated space in the interesting case that $\mathfrak{g}$ is semi-simple. The isomorphism of Theorem 2.8 "parametrizes" J^0 by the simpler space I^0 of symmetric tensors over $\mathfrak{g}'$. Yet it is the simple norm, (4.4), on J^0 which plays a key role in the recently developed heat kernel analysis on both compact Lie groups and complex Lie groups [D, DG, G1,2,3,4, GM, Ha1,2, Hij1,2]. The norm on I^0 induced from (4.4) by V does not seem useful. But the isomorphism V will be used as a technical tool in the next theorem. Its role in the chain rule for the exponential map: $\mathfrak{g} \longrightarrow G$ will be explained in Section 3.

DEFINITION 2.14. J^0 is generated by its finite rank elements if for any element $\alpha \in J^0$ and any nonnegative integer n there exists an element $\alpha' \in J^0$ of finite rank such that $\alpha - \alpha' = 0$ in all ranks $\leq n$.

THEOREM 2.15. *Suppose that $\mathfrak{g}$ is finite dimensional. J^0 is generated by its finite rank elements if and only if $\mathfrak{g}$ is nilpotent.*

DEFINITION 2.16. Let $\mathfrak{g}$ be a nilpotent Lie algebra. Write $\mathfrak{g}^0 = \mathfrak{g}$ and $\mathfrak{g}^n = [\mathfrak{g}, \mathfrak{g}^{n-1}]$ for $n = 1, 2, \ldots$. Let r be the largest integer such that $\mathfrak{g}^r \neq 0$. (So $\mathfrak{g}$ is an $r+1$-step nilpotent Lie algebra.) Define a *weight* function on $\mathfrak{g}$ as follows: define $w(0) = 2^{r+1}$. If $\xi \neq 0$ define

$$w(\xi) = 2^n \quad \text{if } \xi \in \mathfrak{g}^n \text{ but } \xi \notin \mathfrak{g}^{n+1}. \tag{2.10}$$

For any decomposable k tensor define its weight by

$$w(\xi_1 \otimes \cdots \otimes \xi_k) = \sum_{j=1}^{k} w(\xi_j) \quad k \geq 1. \tag{2.11}$$

LEMMA 2.17. *If $\xi \neq 0$ and $\eta \neq 0$ are in $\mathfrak{g}$ then*

$$w([\xi, \eta]) \geq w(\xi) + w(\eta). \tag{2.12}$$

PROOF. Suppose $w(\xi) = 2^n$ and $w(\eta) = 2^k$. We may assume $k \leq n \leq r$. Then $[\xi, \eta] \in \mathfrak{g}^{n+1}$. Hence $w([\xi, \eta]) \geq 2^{n+1} \geq 2^n + 2^k$. ∎

LEMMA 2.18. *Suppose $\mathfrak{g}$ is a nilpotent Lie algebra. Let $\gamma \in \mathcal{N}^0$. Let $m \geq 1$ and $k \geq 1$. Suppose that $\langle \gamma, u \rangle = 0$ for any decomposable k-tensor u of weight $\geq m$. Then $\langle (B^*\gamma), v \rangle = 0$ for any decomposable $k+1$ tensor v of weight $\geq m$.*

PROOF. Suppose $v = \xi_1 \otimes \cdots \otimes \xi_{k+1}$. If $v = 0$ then $\langle B^*\gamma, v \rangle = 0$. So suppose $v \neq 0$ and $w(v) \geq m$. Let $1 \leq i \leq k$. The permutation $\sigma = (i, i+1)$ acts on $\{1, \ldots, k+1\}$ and also acts on v. One has

$$v - \sigma v = \xi_1 \otimes \cdots \otimes \xi_{i-1} \otimes (\xi_i \wedge \xi_{i+1}) \otimes \xi_{i+2} \otimes \cdots \otimes \xi_{k+1}.$$

So, mod $\mathcal{N}$, one has

$$B(v - \sigma v) = \xi_1 \otimes \cdots \otimes \xi_{i-1} \otimes [\xi_i, \xi_{i+1}] \otimes \xi_{i+2} \otimes \cdots \otimes \xi_{k+1}.$$

This decomposable tensor has weight $\geq m$ by (2.11) and (2.12). Hence $\langle (B^*\gamma), v - \sigma v \rangle = \langle \gamma, B(v - \sigma v) \rangle = 0$. Therefore

$$\langle B^*\gamma, \sigma v \rangle = \langle B^*\gamma, v \rangle.$$

Since $\sigma v \neq 0$ and $w(\sigma v) = w(v) \geq m$ the same argument can be applied to σv to conclude, by induction, that

$$\langle B^*\gamma, \sigma_1 \cdots \sigma_j v \rangle = \langle B^*\gamma, v \rangle$$

for any set of nearest neighbor transpositions $(i, i+1)$ of $\{1, \ldots, k+1\}$. Since these generate the permutation group S_{k+1} it follows that $\langle B^*\gamma, \tau v \rangle = \langle B^*\gamma, v \rangle \ \forall \tau \in S_{k+1}$. Summing this identity over all $\tau \in S_{k+1}$ and dividing by $(k+1)!$ shows that $\langle B^*\gamma, v \rangle = \langle B^*\gamma, P_{k+1} v \rangle$ wherein P_{k+1} is the symmetrization projection on $\mathfrak{g}^{\otimes(k+1)}$. Since $B^*\gamma$ is by definition zero on symmetric tensors the lemma follows. ∎

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COROLLARY 2.19. *Suppose that* $\mathfrak{g}$ *is an* $r+1$*-step nilpotent Lie algebra and that* $\gamma \in (\mathfrak{g}^{\otimes k})' \cap N^0$. *Then*

$$(B^*)^n\gamma = 0 \quad \text{if } n \geq k(2^r - 1) + 1. \tag{2.13}$$

PROOF. Since any non-zero vector ξ in $\mathfrak{g}$ has weight at most 2^r a decomposable k-tensor u has weight at most $k2^r$ by (2.11) unless u is zero. Thus $\langle \gamma, u\rangle = 0$ if u is a decomposable k tensor of weight $\geq k2^r + 1$. By induction and Lemma 2.18 it follows that $\langle (B^*)^n\gamma, v\rangle = 0$ if v is a decomposable $n+k$ tensor of weight $w(v) \geq k2^r + 1$. But every element of $\mathfrak{g}$ has weight at least $2^0 = 1$. Hence $w(v) \geq n+k$ for any decomposable $n+k$ tensor v. So if $n+k \geq k2^r+1$ then $\langle (B^*)^n\gamma, v\rangle = 0$ for all decomposable $n+k$ tensors v and therefore for all $n+k$ tensors v. ∎

EXAMPLE 2.20. If $\mathfrak{g}$ is the Heisenberg Lie algebra then $r = 1$. In this case (2.13) asserts that $(B^*)^n\gamma = 0$ when $\gamma \in N^0 \cap (\mathfrak{g}^{\otimes k})'$ if $n \geq k+1$.

PROOF OF THEOREM 2.15. Assume first that $\mathfrak{g}$ is an $r+1$-step nilpotent Lie algebra. Suppose that $\alpha \in J^0$ and is given by (2.2). Let $k_0 = \inf\{k : \alpha_k \neq 0\}$. It suffices to show that there exists an element α' in J^0 such that α' is of finite rank while $\alpha - \alpha'$ is zero in rank $\leq k_0$. Now α_{k_0} is symmetric by Corollary 2.11. That is, α_{k_0} is in I^0. Let $\alpha' = V\alpha_{k_0}$. Then a) α' is in J^0 by Lemma 2.7 and b) the sum in (2.7) is finite because all terms are zero from $m = k_0(2^r - 1) + 1$ onward by Corollary 2.19. So α' is of finite rank. Clearly $(\alpha - \alpha')_k = 0$ for $k \leq k_0$. Thus J^0 is finitely generated.

Conversely suppose that $\mathfrak{g}$ is not nilpotent. Then there exists an integer $s \geq 1$ such that $\mathfrak{g}^n = \mathfrak{g}^s \neq (0)$ for all $n \geq s$. So $\mathfrak{g}^n \supset \mathfrak{g}^s$ for all $n \geq 0$. Choose $\eta \neq 0$ in $\mathfrak{g}^s$ and choose $\gamma \in \mathfrak{g}' \ni \langle \gamma, \eta\rangle \neq 0$. Let $\alpha = V\gamma$. By Theorem 2.8 α is in J^0. Clearly $\alpha = 0$ in rank 0 and $\alpha = \gamma$ in rank 1. In particular, $\langle \alpha, \eta\rangle \neq 0$. It suffices to show that there exists no element α' in J^0 of finite rank which agrees with α in rank one. Let $\xi_1, \ldots, \xi_n \in \mathfrak{g}$. Let

$$u_k = \{([\cdots[\xi_1, \xi_2], \ldots, \xi_k] \wedge \xi_{k+1}) \wedge \cdots \wedge \xi_n\} \quad k = 1, \ldots, n.$$

Then $u_1 = \{\cdots(\xi_1 \wedge \xi_2) \wedge \cdots\} \wedge \xi_n$ is in I_n while $u_n = [\cdots[\xi_1, \xi_2], \ldots, \xi_n]$ is in $\mathfrak{g}$. Moreover $u_k - u_{k+1}$ is in J for $k = 1, \ldots, n-1$. Therefore if α' is in J^0 and is zero in rank n then $\langle \alpha', u_1\rangle = 0$ while $\langle \alpha', u_1 - u_2\rangle = 0$. So $\langle \alpha', u_2\rangle = 0$. Continuing in this way we conclude that $\langle \alpha', u_n\rangle = 0$. Hence $\langle \alpha', \mathfrak{g}^s\rangle = 0$. But if $\alpha' = \alpha$ in rank 1 we get $\langle \alpha, \eta\rangle = 0$, which is a contradiction. ∎

REMARK 2.21. The finite dimensionality of $\mathfrak{g}$ was required only in the proof of the "only if" part of Theorem 2.15. The finite dimensionality can clearly be replaced in the converse by the hypothesis that for some s $\{0\} \neq \mathfrak{g}^s \subset \mathfrak{g}^n$ for all n.

3. **Taylor coefficients.** Consider now a Lie group G with (finite dimensional) Lie algebra $\mathfrak{g} := T_e(G)$. Suppose that W is an open neighborhood of the identity e in G and $u \in C^\infty(W)$. Let $f(\xi) = u(\exp \xi)$. It will be shown that the Taylor coefficients, γ, of f at $\xi = 0$ and the "Taylor coefficients" (see (3.1)) of u at e are related by the equation (2.6). The notation of Section 2 will be continued. The content of this section is algebraic because convergence questions will not be considered. The main theorem, Theorem 3.2, can be considered to be a systematization of the chain rule for the composition $f = u \circ \exp$.

NOTATION 3.1. Let W be an open neighborhood of e in G and let $u \in C^\infty(W)$. Define $(D^0 u)(a) = u(a)$ for $a \in W$ and, for $k \geq 1$, let $(D^k u)(a)$ be the unique element of $(\mathfrak{g}^{\otimes k})'$ such that

$$\langle (D^k u)(a), \xi_1 \otimes \cdots \otimes \xi_k \rangle = (\tilde{\xi}_1 \ldots \tilde{\xi}_k u)(a), \quad \xi_1, \ldots, \xi_k \in \mathfrak{g}, a \in W \tag{3.1}$$

wherein $\tilde{\xi}$ denotes the left invariant extension of ξ to G. The Taylor coefficients of u at a constitute the set $\{(D^k u)(a)\}_{k=0}^\infty$. It is useful to describe this set as a single element of T'. The following suggestive notation of Driver [D] will be used for this element. Define

$$(1 - D)_a^{-1} u = \sum_{k=0}^\infty (D^k u)(a) \quad u \in C^\infty(W). \tag{3.2}$$

Thus $(1-D)_a^{-1} u$ is an element of T' whose rank k component is $(D^k u)(a)$. It follows from (3.1), (3.2), (2.1) and the definition of Lie bracket that

$$(1 - D)_a^{-1} u \in J^0. \tag{3.3}$$

If $\mathfrak{g}$ is a complex Lie algebra and u is holomorphic then the right side of (3.1) is automatically complex linear. In this case the pairing on the left side of (3.1) should be taken to be the complex bilinear pairing between $\mathfrak{g}^{\otimes k}$ and its dual space as complex vector spaces. This is the case of interest in [D, DG, G4, GM]. The algebraic identities of this section are applicable in this complex case without change. However u must then be taken to be holomorphic. We will focus on the real case in the rest of this section.

$\mathfrak{g}$ is also a Lie group under addition and its Lie algebra will be identified with $\mathfrak{g}$ itself, as usual, with zero Lie bracket. So if f is a smooth function on a neighborhood of 0 in $\mathfrak{g}$ then its Taylor coefficients, at 0 say, define an element $(1 - D)_0^{-1} f$ in T' also. But since the Lie bracket is zero one has

$$(1 - D)_0^{-1} f \in I^0, \quad f \in C^\infty(\text{neighborhood of 0 in } \mathfrak{g}). \tag{3.4}$$

(That is, for a smooth function on a linear space the order of differentiations can be interchanged.)

THEOREM 3.2. *Suppose that W is an open neighborhood of e in G and that $u \in C^\infty(W)$. Let $f(\xi) = u(\exp \xi)$. Define*

$$\alpha = (1 - D)_e^{-1} u \quad \text{and} \quad \gamma = (1 - D)_0^{-1} f. \tag{3.5}$$

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Then

$$\alpha = V_{\mathfrak{g}}\gamma \tag{3.6}$$

where $V_{\mathfrak{g}}$ is defined by (2.6).

PROOF. Since an element of I^0 is determined by its values on S the equation (2.6) shows that an element $\gamma \in I^0$ can be recovered from $V_{\mathfrak{g}}\gamma$ by restricting $V_{\mathfrak{g}}\gamma$ to S. Thus it suffices to prove that if α and γ are defined by (3.5) then $\alpha|S = \gamma$. In rank 0 (3.5) gives $\alpha_0 = u(e)$ and $\gamma_0 = f(0)$, which is just $u(e)$. So $\alpha_0 = \gamma_0$. In rank $k \geq 1$ observe that S_k is spanned by $\{\xi^{\otimes k} : \xi \in \mathfrak{g}\}$ because this span contains the coefficient of $s_1 s_2 \cdots s_k$ in $(\sum_{j=1}^k s_j \xi_j)^{\otimes k}$ where all the s_i are real. This coefficient, which is the k-th derivative $\partial^k/\partial s_1 \cdots \partial s_k$ of this polynomial at $s_1 = s_2 = \cdots = s_k = 0$, is a symmetrization of $\xi_1 \otimes \cdots \otimes \xi_k$. So it suffices to show that $\langle \alpha, \xi^{\otimes k}\rangle = \langle \gamma, \xi^{\otimes k}\rangle$. But $\langle \gamma, \xi^{\otimes k}\rangle = d^k/dt^k f(t\xi)\big|_{t=0} = d^k/dt^k u(\exp t\xi)\big|_{t=0} = \langle \alpha, \xi^{\otimes k}\rangle$. ∎

EXAMPLE 3.3. Choose a neighborhood M of 0 in $\mathfrak{g}$ and a neighborhood W of e in G such that $\exp\colon M \longrightarrow W$ is a diffeomorphism. Write $\log = (\exp)^{-1}\colon W \longrightarrow M$. Let $\zeta \in \mathfrak{g}'$ and define $u(x) = \langle \zeta, \log x\rangle$ for $x \in W$. Thus if $f(\xi) = u(\exp \xi)$ then $f(\xi) = \langle \zeta, \xi\rangle$. The Taylor coefficients of f are clearly given by

$$\gamma := (1-D)_0^{-1} f = \zeta \in T'. \tag{3.7}$$

The Taylor coefficients at e of u are therefore

$$(1-D)_e^{-1}\langle \zeta, \log(\cdot)\rangle = \sum_{k=0}^{\infty} (B^*)^k \zeta. \tag{3.8}$$

The right side of (3.8) will be a finite sum, by Corollary 2.19, if $\mathfrak{g}$ is nilpotent. But in general it will be an infinite sum. A similar argument shows that the Taylor series for powers of the ζ component of the logarithm may be computed as

$$(1-D)_e^{-1}(m!)^{-1}\langle \zeta, \log(\cdot)\rangle^m = \sum_{k=0}^{\infty} (B^*)^k \zeta^{\otimes m}. \tag{3.9}$$

REMARK 3.4. There is a simple intertwining identity that is well known, but seems worthwhile repeating here in the present "local" context. For $\xi \in \mathfrak{g}$ define $R_\xi\colon T \longrightarrow T$ by $R_\xi u = u \otimes \xi$. Then $R_\xi J \subset J$. So the adjoint $R_\xi^*\colon T' \longrightarrow T'$ carries J^0 into J^0. Define $A_\xi = R_\xi^*|J^0$. A_ξ clearly lowers rank by one. It follows from the definitions (3.1) and (3.2) that

$$A_\xi (1-D)_e^{-1} = (1-D)_e^{-1}\tilde{\xi} \quad \text{for } \xi \in \mathfrak{g}. \tag{3.10}$$

See [DG, Section 7] for a discussion of how the identity (3.10) essentially determines the map $(1-D)_e^{-1}$ from functions to T' uniquely.

https://doi.org/10.4153/CJM-1998-019-4 Published online by Cambridge University Press

4. **Norms on universal enveloping algebras.** Any norm, $\| \;\|$, on the tensor algebra T of a real or complex Lie algebra induces a semi-norm, $\| \;\|_*$, on the universal enveloping algebra $U := T/J$, by the usual rule:

$$\|\beta + J\|_* = \inf\{\|\beta + j\| : j \in J\}. \tag{4.1}$$

The induced seminorm $\| \;\|_*$ might actually be a norm on U or, in the opposite extreme, might have such a large kernel that $U/\ker\| \;\|_*$ is finite dimensional. There are two naturally arising families of norms on T for which the nature of $\ker\| \;\|_*$ has been investigated. Choose a real, respectively Hermitian, inner product on a real, respectively complex, Lie algebra $\mathfrak{g}$. Define, for β given by (2.3), the norms

$$\|\beta\|_t^2 = \sum_{k=0}^{N} k!\, t^{-k} |\beta_k|^2_{\mathfrak{g}^{\otimes k}}, \quad \beta \in T,\ t > 0 \tag{4.2}$$

$$\|\beta\|_{,a}^2 = \sum_{k=0}^{N} (a^2)^{k+1} |\beta_k|^2_{\mathfrak{g}^{\otimes k}}, \quad \beta \in T,\ a > 0. \tag{4.3}$$

If $\mathfrak{g}$ is commutative then, in the notation of Section 2, $J = \mathcal{I}$ and the decomposition $T = \mathcal{S} + J$ is an orthogonal decomposition for both of the norms (4.2) and (4.3). Thus, identifying T/J with $\mathcal{S}$, it follows that the induced seminorms on T/J agree with the restrictions of the given norms to $\mathcal{S}$. Hence the induced seminorms on U are always norms when $\mathfrak{g}$ is commutative. The commutative case, with the norm (4.2), is the classical case, which has been explored thoroughly, especially in the context of quantum field theory. See *e.g.* [BSZ, Co, K, P] and their bibliographies.

If $\mathfrak{g}$ is not commutative the situation is quite different. The decomposition $T = \mathcal{S} + J$ still holds by the Poincaré-Birkhoff-Witt theorem, and in fact the corresponding projection on $\mathcal{S}$ is known explicitly [So]. But this decomposition is no longer orthogonal. So the argument of the preceding paragraph no longer applies. In fact if $\mathfrak{g}$ is the Lie algebra of a compact, simply connected Lie group, G, and the given inner product on $\mathfrak{g}$ is $\operatorname{Ad} G$ invariant, then on the one hand, O. Hijab has shown [Hij1] that the seminorms induced on U by the norms (4.2) are still norms, while on the other hand, the present author has shown [G4] that the seminorms induced on U by the norms (4.3) have such a large kernel that the quotient space $U/\ker\| \;\|_*$ is finite dimensional. The dimension in this case depends on a as well as on certain features of the representation ring of G. All of these results have been obtained using harmonic analysis over the group G. The norms (4.3) will be discussed in Section 5 in the simplest example at a purely Lie algebra level to show how $\| \;\|_*$ can have a kernel. But B. Hall has pointed out to the author that a key step in Hijab's proof of the nondegeneracy of the quotient norms induced by (4.2) is valid for a general finite dimensional Lie algebra. One has the following lemma of Hijab and Hall.

LEMMA 4.1 (HIJAB [HIJ1], HALL [HA3]). *Let $(\ ,\ )$ be an inner product on a real or complex finite dimensional Lie algebra $\mathfrak{g}$. Fix $t > 0$ and let $\bar{J}$ denote the closure of J in T with respect to the norm (4.2). Then*

$$\bar{J} = J.$$

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PROOF. Any Lie algebra representation $\varphi\colon \mathfrak{g} \longrightarrow \mathrm{End}(V)$ on a finite dimensional vector space V extends uniquely to a representation $\hat{\varphi}\colon T \longrightarrow \mathrm{End}(V)$ of the associative algebra T. Moreover $\hat{\varphi}(J) = 0$. It suffices to show that $\hat{\varphi}$ is continuous in the norm (4.2), because if $\beta \in \bar{J}$ then continuity implies that $\hat{\varphi}(\beta) = 0$, whereas by [Di, Theorem 2.5.7], if $\beta \notin J$ there exists a finite dimensional representation φ such that $\hat{\varphi}(\beta) \neq 0$. O. Hijab proved the continuity of $\hat{\varphi}$ in case $\mathfrak{g}$ is the Lie algebra of a compact group G by making use of harmonic analysis on G, [Hij1]. But for a general Lie algebra one may prove continuity as follows, [Ha3].

For any fixed inner product on V let $c = \sup\{\|\varphi(\xi)\|_{\mathrm{End}\,V} : |\xi| \leq 1\}$. It will first be shown that if $d = \dim \mathfrak{g}$ then

$$\|\hat{\varphi}(\beta)\|_{\mathrm{End}(V)} \leq c^k d^{k/2} |\beta|_{\mathfrak{g}^{\otimes k}}.$$

Indeed if $e_1, \ldots, e_d$ is an orthonormal basis of $\mathfrak{g}$ and $\beta = \Sigma a_{i_1\cdots i_k} e_{i_1} \otimes \cdots \otimes e_{i_k} \in \mathfrak{g}^{\otimes k}$ then

$$\begin{aligned}
\|\hat{\varphi}(\beta)\|_{\mathrm{End}(V)} &= \|\Sigma a_{i_1\cdots i_k} \varphi(e_{i_1}) \cdots \varphi(e_{i_k})\|_{\mathrm{End}(V)} \\
&\leq c^k \Sigma |a_{i_1\cdots i_k}| \leq c^k d^{k/2} (\Sigma |a_{i_1\cdots a_k}|^2)^{1/2} \\
&= c^k d^{k/2} |\beta|_{\mathfrak{g}^{\otimes k}}.
\end{aligned}$$

Next, if β is given by (2.3) then

$$\begin{aligned}
\|\hat{\varphi}\beta\|_{\mathrm{End}(V)} &\leq \sum_{k=0}^{N} c^k d^{k/2} |\beta_k|_{\mathfrak{g}^{\otimes k}} \\
&\leq \Big(\sum_{k=0}^{N} \{c^k d^{k/2} (t^k/k!)^{1/2}\}^2 \Big)^{1/2} \Big(\sum_{k=0}^{\infty} (k!/t^k) |\beta_k|^2_{\mathfrak{g}^{\otimes k}} \Big)^{1/2} \\
&= \exp(tc^2 d) \|\beta\|_t.
\end{aligned}$$

∎

Lemma 4.1 settles completely the conjecture made in [G2, Equation (3.4)].

Denoting now by $\bar{J}$ the closure of J in T with respect to some arbitrary norm $\|\ \|$ on T, it is elementary that the induced seminorm, $\|\ \|_{**}$, on $T/\bar{J}$ is a norm and that $T/\bar{J}$ and $U/\ker \|\ \|_*$ are isometrically isomorphic in a natural way. It is the topological dual space of $T/\bar{J}$ in the induced norm that has been a primary object of interest in [D, DG, G1,2,3,4, GM, Hij1,2] for the norms (4.2), and in [G4] for the norms (4.3). The topological dual space of $T/\bar{J}$ clearly consists of those linear functionals on T which are continuous in the given norm $\|\ \|$ and which are zero on J (and a fortiori on $\bar{J}$). The resulting subspace of J^0 can be characterized explicitly for the norms (4.2) and (4.3) with the help of the dual space norms given as follows. Denote by $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$ as an inner product space with the inner product dual to that given on $\mathfrak{g}$. In the pairing (2.1), the dual norms are then given respectively by

$$\|\alpha\|_t^2 = \sum_{k=0}^{\infty} (t^k/k!) |\alpha_k|^2_{(\mathfrak{g}^*)^{\otimes k}} \quad \alpha \in T' \tag{4.4}$$

$$\|\alpha\|_{,a}^2 = \sum_{k=0}^{\infty} (a^2)^{-(k+1)} |\alpha_k|^2_{(\mathfrak{g}^*)^{\otimes k}} \quad \alpha \in T' \tag{4.5}$$

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when α is defined by (2.2). The topological dual space of $T/\bar{J}$ is then given in the two respective cases by

$$(4.6) \qquad (J^0)_t = \{\alpha \in J^0 : \|\alpha\|_t < \infty\}$$

and

$$(4.7) \qquad J^0_{,a} = \{\alpha \in J^0 : \|\alpha\|_{,a} < \infty\}.$$

Now it is an elementary yet fundamental theorem that the universal enveloping algebra of a direct sum of Lie algebras is isomorphic to the tensor product of their universal enveloping algebras [B, Chapter I, Section 2.2]. The main result of this section will show that this algebraic isomorphism is also a Hilbert space isomorphism in the norm induced by (4.2) (for any $t > 0$.) The motivation for the product theorem in this case comes from the fact that when $\mathfrak{g}$ is complex and is the Lie algebra of a simply connected Lie group G then the Taylor map $(1-D)_e^{-1} : \{\text{holomorphic functions on } G\} \longrightarrow J^0$, described in Section 3, is actually a unitary operator onto $(J^0)_t$ when its domain is restricted to the Hilbert space of square integrable holomorphic functions on G sketched in the Introduction. This theorem is part of the reason for interest in the norms (4.2) and (4.4). It also motivates the product theorem because the heat kernel measure of the direct sum of Lie algebras (with sum inner product) is the product of the heat kernel measures, while an L^2 space of a product of measures is the tensor product of the L^2 spaces. The theorems of this section could in this way be derived from the results in [DG]. But it seems worthwhile to give an elementary combinatoric proof of this essentially algebraic theorem which avoids the heat kernel analysis used in [DG]. The comultiplication that appears in the algebraic proof (*cf.* (4.12)) can be understood as a replacement of the Leibnitz formula for derivatives of a product of two functions.

NOTATION 4.2. $\mathfrak{g}_1$ and $\mathfrak{g}_2$ will denote two real Lie algebras with real inner products $(\ ,\ )$, $i = 1, 2$ or two complex Lie algebras with Hermitian inner products. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be the direct sum of Lie algebras and of inner product spaces. Thus the Lie bracket is $[\xi_1 + \xi_2, \eta_1 + \eta_2] = [\xi_1, \eta_1] + [\xi_2, \eta_2]$ where ξ_j and η_j are in $\mathfrak{g}_j$, $j = 1, 2$. Denote by T_1, T_2 and T the respective tensor algebras over $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{g}$. J_1, J_2 and J will denote the corresponding two sided ideals defined as in Section 2 and J_1^0, J_2^0 and J^0 their annihilators in T_1', T_2' and T' respectively. There is a unique algebra homomorphism $\rho_1 : T_1 \longrightarrow T$ defined by $\rho_1(1) = 1$, $\rho_1(\xi) = \xi \oplus 0 \in \mathfrak{g}$ for ξ in $\mathfrak{g}_1$. Define $\rho_2 : T_2 \longrightarrow T$ similarly. Then there is a linear map $\rho : T_1 \otimes T_2 \longrightarrow T$ determined by

$$(4.8) \qquad \rho(\beta_1 \otimes \beta_2) = \rho_1(\beta_1)\rho_2(\beta_2) \quad \beta_i \in T_i,\ i = 1, 2.$$

Define $(J^0)_t$ as in (4.6) and define the Hilbert spaces $(J_i^0)_t$, $i = 1, 2$ similarly. Whereas tensor products have heretofore referred to the algebraic tensor product, the tensor product of Hilbert spaces such as $(J_1^0)_t \otimes (J_2^0)_t$ will mean the Hilbert space tensor product.

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THEOREM 4.3. *There is a unique linear isometry of Hilbert spaces*

$$L\colon (J_1^0)_t \otimes (J_2^0)_t \longrightarrow (J^0)_t \tag{4.9}$$

such that

$$\langle Lw, \rho\beta\rangle = \langle w, \beta\rangle \quad \forall \beta \in T_1 \otimes T_2 \text{ and } \forall w \in (J_1^0)_t \otimes (J_2^0)_t. \tag{4.10}$$

L is surjective (hence orthogonal or unitary).

REMARK 4.4. The pairing $\langle w, \beta\rangle$ on the right side of (4.10) is well defined because the map $\alpha_1 \longrightarrow \langle \alpha_1, \beta_1\rangle$ is continuous on $(J_1^0)_t$ for each element β_1 in T_1, and the same for T_2. So $\langle w, \beta\rangle$ is a finite sum of continuous linear functionals $w \longrightarrow \langle w, \beta_1 \otimes \beta_2\rangle$.

REMARK 4.5. Let $\mathfrak{g}$ be a finite dimensional Lie algebra with an inner product. Fix $t > 0$ and denote by $\overline{T}$ the completion of T in the norm (4.2). Denote by $P\colon \overline{T} \longrightarrow \overline{T} \ominus J$ the orthogonal projection. The kernel of $P|T$ is the closure of J in T, which by Lemma 4.1 is just J. Hence P induces an injective linear map $T/J \longrightarrow \overline{T} \ominus J$. In this way the universal enveloping algebra $U := T/J$ inherits the inner product in $\overline{T} \ominus J$.

Now if $\mathfrak{g}$ is a direct sum of Lie algebras as in Notation 4.2 then the map $\rho\colon T_1 \otimes T_2 \longrightarrow T$ defined in (4.8) descends to the standard algebra isomorphism $U_1 \otimes U_2 \longrightarrow U$ for universal enveloping algebras [B, Chapter I, Section 2.2]. The next theorem asserts that this isomorphism is an isometry of inner product spaces.

THEOREM 4.6. *Fix $t > 0$. Let $U_i = T_i/J_i$, for $i = 1, 2$ and $U = T/J$ be the universal enveloping algebras with the inner products induced by the norm (4.2) as in Remark 4.5. Then the map ρ of (4.8) induces a surjective isometry from the inner product space $U_1 \otimes U_2$ onto U. Specifically, if $\pi_i\colon T_i \longrightarrow U_i$, $i = 1, 2$ and $\pi\colon T \longrightarrow U$ are the canonical projections then*

$$\|\pi\rho\beta\|_U = \|(\pi_1 \otimes \pi_2)\beta\|_{U_1 \otimes U_2}, \quad \beta \in T_1 \otimes T_2. \tag{4.11}$$

Theorem 4.6 will be deduced from Theorem 4.3 by duality.

NOTATION 4.7. The comultiplication $\Delta\colon T \longrightarrow T \otimes T$ is the algebra homomorphism satisfying $\Delta 1 = 1 \otimes 1$ and $\Delta\zeta = \zeta \otimes 1 + 1 \otimes \zeta$ for $\zeta \in \mathfrak{g}$ [Sw]. The action of Δ on a decomposable tensor is then given by the shuffle product [R, Sw, p. 248]:

$$\Delta(\zeta_1 \cdots \zeta_m) = \sum_{k=0}^{m} \sum_{\substack{i_1,\ldots,i_k \\ j_1,\ldots,j_{m-k}}} (\zeta_{i_1} \cdots \zeta_{i_k}) \otimes (\zeta_{j_1} \cdots \zeta_{j_{m-k}}) \tag{4.12}$$

where in the sum, $(i_1, \ldots, i_k, j_1, \ldots, j_{m-k})$ is a permutation of $(1, \ldots, m)$ such that $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_{m-k}$.

Denote by T_{i^*} the tensor algebra over $\mathfrak{g}_i^*$ for $i = 1, 2$. Via the pairing (2.1) T_{i^*} embeds into T_i', $i = 1, 2$. $\mathfrak{g}_1^*$ embeds into $\mathfrak{g}^*$ by the rule $\langle x, \xi \oplus \eta \rangle = \langle x, \xi \rangle$ for $x \in \mathfrak{g}_1^*$, $\xi \in \mathfrak{g}_1$ and $\eta \in \mathfrak{g}_2$. Let $\sigma_1 \colon T_{1^*} \longrightarrow T'$ be the induced algebra map. Note that $\sigma_1(T_{1^*})$ annihilates the 2-sided ideal in T generated by $0 \oplus \mathfrak{g}_2$. Define $\sigma_2 \colon T_{2^*} \longrightarrow T'$ analogously. Then

$$\sigma_1 \otimes \sigma_2 \colon T_{1^*} \otimes T_{2^*} \longrightarrow T' \otimes T' \subset (T \otimes T)'. \tag{4.13}$$

Thus the map ψ given by

$$\psi = \Delta^*(\sigma_1 \otimes \sigma_2) \colon T_{1^*} \otimes T_{2^*} \longrightarrow T' \tag{4.14}$$

is well defined. Of course we have

$$\langle \psi w, \beta' \rangle = \langle (\sigma_1 \otimes \sigma_2) w, \Delta\beta' \rangle, \quad w \in T_{1^*} \otimes T_{2^*},\ \beta' \in T. \tag{4.15}$$

Finally, denote by J_0 the two sided ideal in T generated by

$$\{\xi \wedge \eta : \xi \in \mathfrak{g}_1 \oplus 0, \eta \in 0 \oplus \mathfrak{g}_2\}.$$

LEMMA 4.8. *If $u \in (\mathfrak{g}_1^*)^{\otimes r}$ and $v \in (\mathfrak{g}_2^*)^{\otimes s}$ then*

$$\psi(u \otimes v) = \sum_{\tau \in S_{r,s}} \tau\big((\sigma_1 u)(\sigma_2 v)\big) \tag{4.16}$$

where $S_{r,s}$ is the set of permutations τ of $\{1, \ldots, r+s\}$ such that $\tau(i) < \tau(i+1)$ for $i = 1, \ldots, r-1$, and for $i = r+1, \ldots, r+s-1$. The action of a permutation τ on $(\mathfrak{g}^*)^{\otimes(r+s)}$ is given by $\tau(z_1 \cdots z_{r+s}) = z_{\tau^{-1}(1)} \cdots z_{\tau^{-1}(r+s)}$.

PROOF. For simplicity of writing we will omit σ_1 and σ_2 and identify $(\mathfrak{g}_i^*)^{\otimes m}$ as a subspace of $(\mathfrak{g}^*)^{\otimes m}$ for $i = 1, 2$. Then, with u and v as above, and $w = u \otimes v$, equation (4.15) reads

$$\langle \psi(u \otimes v), \beta' \rangle = \langle u \otimes v, \Delta\beta' \rangle$$

Take $\beta' = \zeta_1 \cdots \zeta_m \in \mathfrak{g}^{\otimes m}$ and insert the expansion (4.12) for $\Delta\beta'$. One sees first that $\langle u \otimes v, \Delta\beta' \rangle = 0$ if $m \neq r+s$. Hence $\psi(u \otimes v)$ lies in $(\mathfrak{g}^*)^{(r+s)}$. Second, even if $m = r+s$ most of the terms in $\langle u \otimes v, \Delta\beta' \rangle$ arising from the expansion (4.12) are zero. Only those terms with $k = r$ and $m - k = s$ can differ from zero. For these terms we may write

$$\langle u \otimes v, \Delta\beta' \rangle = \Big\langle uv, \sum_{\tau \in S_{r,s}} \tau^{-1}\beta' \Big\rangle,$$

which is easily verified on decomposable tensors u and v. It now follows that $\langle \psi(u \otimes v), \beta' \rangle = \langle \sum_{\tau \in S_{r,s}} \tau(uv), \beta' \rangle$. This proves (4.16). ∎

LEMMA 4.9.

$$\langle \psi w, \rho\beta \rangle = \langle w, \beta \rangle \quad \forall w \in T_{1^*} \otimes T_{2^*} \text{ and } \forall \beta \in T_1 \otimes T_2. \tag{4.17}$$

$$\langle \psi w, J_0 \rangle = 0 \quad \forall w \in T_{1^*} \otimes T_{2^*} \tag{4.18}$$

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PROOF. It suffices to prove (4.17) and (4.18) in case $w = (x_1 \cdots x_r) \otimes (y_1 \cdots y_s)$ with all $x_j \in \mathfrak{g}_1$ and all $y_k \in \mathfrak{g}_2$. The straightforward and well known identity $\Delta(\zeta_1 \wedge \zeta_2) = (\zeta_1 \wedge \zeta_2) \otimes 1 + 1 \otimes (\zeta_1 \wedge \zeta_2)$ which holds for all $\zeta_i \in \mathfrak{g}$, $i = 1, 2$, together with the fact that Δ is a homomorphism shows that $\Delta J_0 \subset J_0 \otimes T + T \otimes J_0$. Since J_0 is contained in the 2-sided ideal generated by $0 \oplus \mathfrak{g}_2$ it follows that $\langle (\sigma_1 \otimes \sigma_2) w, J_0 \otimes T \rangle = 0$. Similarly $\langle (\sigma_1 \otimes \sigma_2) w, T \otimes J_0 \rangle = 0$. (4.18) then follows from (4.15).

It suffices to prove (4.17) in case $\beta = (\xi_1 \cdots \xi_j) \otimes (\eta_1 \cdots \eta_n)$ with all $\xi_i \in \mathfrak{g}_1$ and all $\eta_k \in \mathfrak{g}_2$. Suppressing ρ_1 and ρ_2 for simplicity of writing, (4.16) shows that

$$\langle \psi w, \rho\beta \rangle = \sum_{\tau \in S_{r,s}} \langle \tau(x_1 \cdots x_r y_1 \cdots y_s), \xi_1 \cdots \xi_j \eta_1 \cdots \eta_n \rangle. \tag{4.19}$$

Each term on the right is zero if $j \neq r$ or $n \neq s$ because $\mathfrak{g}_1^*$ annihilates $\mathfrak{g}_2$ and $\mathfrak{g}_2^*$ annihilates $\mathfrak{g}_1$. Similarly $\langle w, \beta \rangle$ is also zero if $j \neq r$ or $n \neq s$. If $j = r$ and $n = s$ then only the identity permutation contributes a nonzero term to the right side of (4.19), and this term is $\langle w, \beta \rangle$. ∎

COROLLARY 4.10.

$$T = \rho(T_1 \otimes T_2) \oplus J_0 \tag{4.20}$$

$$\text{Range } \psi = J_0^0 \cap T_{\mathfrak{g}^*}. \tag{4.21}$$

PROOF. T is spanned by products $\zeta_1 \cdots \zeta_m$ with each ζ_j in $\mathfrak{g}_1 \oplus 0$ or $0 \oplus \mathfrak{g}_2$. Such a product can be rearranged mod J_0 so as to have all factors from $\mathfrak{g}_1 \oplus 0$ on the left. This shows that $T = \rho(T_1 \otimes T_2) + J_0$. Moreover if $\beta \in T_1 \otimes T_2$ and $\rho(\beta) \in J_0$ then by (4.18) and (4.17) $0 = \langle \psi w, \rho(\beta) \rangle = \langle w, \beta \rangle$ for all $w \in T_{1^*} \otimes T_{2^*}$. Hence $\beta = 0$. This proves (4.20). Now ψ preserves total rank by (4.16) (and so does ρ). Therefore it suffices to prove (4.21) rank by rank. Both ψ and ρ are injective by (4.17). Denoting tensors of rank m by a subscript m one therefore has

$$\dim \psi\big((T_{1^*} \otimes T_{2^*})_m\big) = \dim \rho\big((T_1 \otimes T_2)_m\big)$$

which by (4.20) equals $\dim T_m - \dim (J_0)_m = \dim$ annihilator of $(J_0)_m$ in $(T')_m$. Since $\psi\big((T_{1^*} \otimes T_{2^*})_m\big) \subset (J_0^0)_m$ by (4.18) the assertion (4.21) follows in rank m. ∎

LEMMA 4.11. *The map $\psi\colon T_{1^*} \otimes T_{2^*} \longrightarrow T_{\mathfrak{g}^*}$ is an isometry of inner product spaces with the norm (4.4) on T_{1^*}, T_{2^*} and $T_{\mathfrak{g}^*}$ and with the cross norm inner product on the algebraic tensor product $T_{1^*} \otimes T_{2^*}$.*

PROOF. The subspaces $T_{r,s} \equiv (\mathfrak{g}_1^*)^{\otimes r} \otimes (\mathfrak{g}_2^*)^{\otimes s}$ of $T_{1^*} \otimes T_{2^*}$ are mutually orthogonal with respect to the inner product associated to the norm (4.4), for distinct pairs (r, s) and (r', s'). Moreover the images of two such subspaces under ψ are clearly mutually orthogonal if $r + s \neq r' + s'$ because the images have different total rank.

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Suppose $r+s = r'+s'$. Let $x_1, \ldots, x_r, x'_1, \ldots, x'_{r'} \in \mathfrak{g}_1^*$ and let $y_1, \ldots, y_s, y'_1, \ldots, y'_{s'} \in \mathfrak{g}_2^*$. Then by (4.16) one has

$$\begin{aligned}(4.22)\quad &\Big(\psi[(x_1 \cdots x_r) \otimes (y_1 \cdots y_s)], \psi[(x'_1 \cdots x'_{r'}) \otimes (y'_1 \cdots y'_{s'})]\Big)_{(\mathfrak{g}^*)^{\otimes(r+s)}} \\ &= \sum_{\substack{\tau \in S_{r,s} \\ \tau' \in S_{r',s'}}} \Big(\tau(x_1 \cdots x_r y_1 \cdots y_s), \tau'(x'_1 \cdots x'_{r'} y'_1 \cdots y'_{s'})\Big)_{(\mathfrak{g}^*)^{\otimes(r+s)}}.\end{aligned}$$

Any inner product on the right is a product of inner products of elements of $\mathfrak{g}^*$. At least one factor will be zero unless $r = r'$ and $s = s'$ because $\mathfrak{g}_1^* \oplus 0$ is orthogonal to $0 \oplus \mathfrak{g}_2^*$. Hence the images $\psi(T_{r,s})$ and $\psi(T_{r',s'})$ are mutually orthogonal unless $r = r'$ and $s = s'$. It suffices therefore to show that $\psi|T_{r,s}$ is isometric for each pair r, s. Take $r' = r$ and $s' = s$ in (4.22). Consider a term on the right with $\tau' \neq \tau$. Because of the monotonicity imposed on τ and τ' these two permutations are determined completely by the respective sets $\tau(\{1, \ldots, r\})$ and $\tau'(\{1, \ldots, r\})$, which are therefore distinct sets. Hence there exists an element j in the first set which is not in the second set. Thus there exists a number $p \in \{1, \ldots, r\}$ and q in $\{r+1, \ldots, r+s\}$ such that $\tau(p) = j = \tau'(q)$. In the representation of the τ, τ' term in (4.22) as a product of inner products of elements of $\mathfrak{g}^*$ the j-th factor is (x_p, y'_{q-r}), which is zero. Therefore all the terms on the right of (4.22) are zero except those for which $\tau' = \tau$. But when $\tau' = \tau$ the inner product is the same as for the identity permutation. Since the cardinality of $S_{r,s}$ is $(r+s)!/(r!\,s!)$ one therefore obtains

$$(4.23)\qquad (\psi[u \otimes v], \psi[u' \otimes v'])_{(\mathfrak{g}^*)^{\otimes(r+s)}} = (r+s)!/(r!\,s!)(u, u')_{(\mathfrak{g}_1^*)^{\otimes r}}(v, v')_{(\mathfrak{g}_2^*)^{\otimes s}}$$

for u, v, u', v' decomposable and therefore for all u and u' in $(\mathfrak{g}_1^*)^{\otimes r}$ and all v and v' in $(\mathfrak{g}_2^*)^{\otimes s}$. In view of the definition (4.4) one need only divide both sides of (4.23) by $(r+s)!$ and multiply by t^{r+s} to obtain the asserted preservation of inner products by ψ. ∎

The next elementary lemma is essentially the statement of existence of the standard isomorphism theorem among universal enveloping algebras.

LEMMA 4.12.

$$J = \rho(J_1 \otimes T_2 + T_1 \otimes J_2) + J_0.$$

PROOF. Identifying T_i with $\rho_i(T_i)$, $i = 1, 2$, the lemma asserts that $J = J_1T_2 + T_1J_2 + J_0$. But $J_i \subset J$ for $i = 0, 1, 2$. So $J \supset J_1T_2 + T_1J_2 + J_0$. Since T_1 and T_2 commute mod J_0 the right side is a 2-sided ideal. It contains the generators of J because if ξ_j and $\eta_j \in \mathfrak{g}_j$ for $j = 1, 2$ then $(\xi_1 + \xi_2) \wedge (\eta_1 + \eta_2) - [\xi_1 + \xi_2, \eta_1 + \eta_2] = (\xi_1 \wedge \eta_1 - [\xi_1, \eta_1]) + (\xi_2 \wedge \eta_2 - [\xi_2, \eta_2]) + (\xi_1 \wedge \eta_2 + \xi_2 \wedge \eta_1) \in J_1 + J_2 + J_0$. ∎

PROOF OF THEOREM 4.3. Let $\bar{T}_{i^*} = \{\alpha \in T'_i : \|\alpha\|_t < \infty\}$ for $i = 1, 2$. Then T_{i^*} is dense in $\bar{T}_{i^*}$ and $(J_i^0)_t \subset \bar{T}_{i^*}$ by (4.6). Define $\bar{T}_{\mathfrak{g}^*}$ similarly. Then $(J^0)_t \subset \bar{T}_{\mathfrak{g}^*}$. By Lemma 4.11 ψ extends to a unique isometry

$$(4.24)\qquad \bar{\psi}\colon \bar{T}_{1^*} \otimes \bar{T}_{2^*} \longrightarrow \bar{T}_{\mathfrak{g}^*}$$

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of Hilbert spaces. By (4.17) and (4.18) and continuity in w one has

$$\langle \bar{\psi} w, \rho\beta \rangle = \langle w, \beta \rangle, \quad w \in \bar{T}_{1^*} \otimes \bar{T}_{2^*},\ \beta \in T_1 \otimes T_2 \tag{4.25}$$

and

$$\langle \bar{\psi} w, J_0 \rangle = 0 \quad w \in \bar{T}_{1^*} \otimes \bar{T}_{2^*}. \tag{4.26}$$

Define

$$L = \bar{\psi} | (J_1^0)_t \otimes (J_2^0)_t. \tag{4.27}$$

Then L is an isometry into $\bar{T}_{\mathfrak{g}^*}$. Suppose that $\beta \in J_1 \otimes T_2 + T_1 \otimes J_2$. For any element w in domain L (4.25) shows that $\langle Lw, \rho\beta \rangle = \langle w, \beta \rangle$, which is zero. Hence Range L annihilates $\rho(J_1 \otimes T_2 + T_1 \otimes J_2)$. Range L also annihilates J_0 by (4.26). Therefore by Lemma 4.12 Range L annihilates J. So L maps into $(J^0)_t$. To prove surjectivity suppose that α is in $(J^0)_t$. Since $J_0 \subset J$, α annihilates J_0. Let α_m be the sum of all the homogeneous components of α up to rank m. Since the ideal J_0 is the span of its homogeneous components α_m also annihilates J_0. By (4.21) there exists an element $w_m \in T_{1^*} \otimes T_{2^*}$ such that $\psi w_m = \alpha_m$. Now α_m converges to α and $\bar{\psi}$ is isometric. Hence w_m converges to an element $w \in \bar{T}_{1^*} \otimes \bar{T}_{2^*}$. (In accordance with our conventions above, this tensor product is complete.) Clearly $\bar{\psi} w = \alpha$. For any element $\beta \in J_1 \otimes T_2 + T_1 \otimes J_2$ one has $\rho(\beta) \in J$ by Corollary 4.10. Hence $0 = \langle \alpha, \rho\beta \rangle = \langle \bar{\psi} w, \rho\beta \rangle = \langle w, \beta \rangle$ by (4.25). So w annihilates both $J_1 \otimes T_2$ and $T_1 \otimes J_2$. Therefore $w \in \big((J_1^0)_t \otimes \bar{T}_{2^*}\big) \cap \big(\bar{T}_{1^*} \otimes (J_2^0)_t\big) = (J_1^0)_t \otimes (J_2^0)_t$. This proves the surjectivity of L.

It remains to prove the uniqueness of any isometry L satisfying (4.9) and (4.10). Suppose that L' is another such isometry. If $w \in (J_1^0)_t \otimes (J_2^0)_t$ and $\alpha = Lw - L'w$ then by (4.9) α annihilates J, and in particular J_0. By (4.10) α annihilates $\rho(T_1 \otimes T_2)$. Therefore, by (4.20), α annihilates T. So $\alpha = 0$. ∎

PROOF OF THEOREM 4.6. If $f\colon T/J \longrightarrow \mathbb{R}$ or $\mathbb{C}$ is a continuous linear functional then $\alpha := f \circ \pi$ is a continuous linear functional on T in the norm (4.2), and annihilates J. Thus, in view of the pairing (2.1) and the definition (4.4), we have $\alpha \in (J^0)_t$ and moreover $\|\alpha\|_t = \|f\|_{U_t^*}$ wherein U_t^* denotes the topological dual space of U in the given inner product. Conversely, an element $\alpha \in (J^0)_t$ defines an element f in U_t^* of the same norm.

These preliminaries having been said, a standard duality argument is now applicable: for any element $\beta \in T_1 \otimes T_2$ one has

$$\begin{aligned} \|\pi\rho\beta\|_U &= \sup\{|\langle f, \pi\rho\beta \rangle| : f \in U_t^*, \|f\|_{U_t^*} = 1\} \\ &= \sup\{|\langle \alpha, \rho\beta \rangle| : \alpha \in (J^0)_t, \|\alpha\|_t = 1\} \\ &= \sup\{|\langle Lw, \rho\beta \rangle| : w \in (J_1^0)_t \otimes (J_2^0)_t, \|w\| = 1\} \end{aligned} \tag{4.28}$$

since L is a surjective isometry.

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On the other hand, since the dual space of the inner product space $U_1 \otimes U_2$ is $(J_1^0)_t \otimes (J_2^0)_t$ one has

$$\|(\pi_1 \otimes \pi_2)\beta\|_{U_1\otimes U_2} = \sup\{|\langle w, \beta\rangle| : w \in (J_1^0)_t \otimes (J_2^0)_t, \|w\| = 1\}. \tag{4.29}$$

(4.11) now follows from (4.10), (4.28) and (4.29). The surjectivity follows from (4.20) and the fact that $J_0 \subset J$. ■

REMARK 4.13. If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are commutative then, as already noted at the beginning of this section, J_1, J_2 and J have easily identifiable orthogonal complements in the norm (4.2), namely the symmetric tensors. By identifying the three quotient spaces with spaces of symmetric tensors one can prove Theorem 4.6 by direct and easy computations. This is most efficiently carried out by computing with the coherent states, which may be described as follows. Let $\exp(\zeta) = \sum_{n=0}^{\infty}(n!)^{-1}\zeta^{\otimes n}$, $\zeta \in \mathfrak{g}$. (The factors $(n!)^{-1}$ are correct for the choice of norms (4.2).) Fixing $t > 0$ and using the notation of the proof of Theorem 4.6, write P for the projection of $\overline{T}$ onto $\overline{T} \ominus J$. Similarly define $P_i \colon \overline{T_i} \longrightarrow \overline{T_i} \ominus J_i$ for $i = 1, 2$. The identity

$$\begin{aligned}\big(P\exp(\xi_1 + \xi_2), \exp(\eta_1 + \eta_2)\big)_{\overline{T}} \\ = (P_1 \exp\xi_1, \exp\eta_1)_{\overline{T_1}}(P_2\exp\xi_2, \exp\eta_2)_{\overline{T_2}} \quad \xi_i, \eta_i \in \mathfrak{g}_i,\ i = 1, 2\end{aligned} \tag{4.30}$$

is a correct identity and is easily verifiable in the commutative case because all of the projections, P, P_1, P_2, can clearly be removed in this case, and both sides can be computed explicitly. It would follow from (4.30), together with density arguments, that the map

$$(P_1 \exp\xi_1) \otimes (P_2 \exp\xi_2) \longrightarrow P\exp(\xi_1 + \xi_2)$$

extends to an isometry from $J_1^{\perp} \otimes J_2^{\perp} \longrightarrow J^{\perp}$. This is the basis for the standard algebraic proof of isometry in the commutative case. See *e.g.* the recent text [P, Proposition 19.6] for a detailed exposition of this by now classical technique. But in the noncommutative case the projections P, P_1, P_2 cannot be removed in (4.30). The only proof of (4.30) known to this author depends on the fact that the three inner products in (4.30) represent reproducing kernels. See [DG, Equation (6.1)]. Whereas the elements $\exp\xi$ are the coherent states in the commutative case, it is the elements $P\exp\xi$ which play this role in general. The difficulty of dealing with these projections seems to make a direct proof of Theorem 4.6 difficult if not infeasible, without going through the dual spaces as above.

REMARK 4.14. When $\mathfrak{g}$ is commutative the symmetric tensors of finite rank all lie in $(J^0)_t$ and are dense. For what other Lie algebras does $(J^0)_t$ contain a dense linear set of elements of finite rank? The motivation for this question is given in [DG, Section 7]. In view of Theorem 2.15 it is reasonable to conjecture that $(J^0)_t$ has a dense set of elements of finite rank if and only if $\mathfrak{g}$ is nilpotent. In the original version of this paper the author conjectured this theorem but was unable to prove either half of it. The referee, however, has provided the following proof of one half of this conjecture. It is based on Lemma 4.1.

https://doi.org/10.4153/CJM-1998-019-4 Published online by Cambridge University Press

THEOREM 4.15 (REFEREE). *Let $\mathfrak{g}$ be a real or complex finite dimensional Lie algebra with an inner product. Let $t > 0$. If the finite rank elements in $(J^0)_t$ are dense in $(J^0)_t$ then $\mathfrak{g}$ is nilpotent.*

PROOF. Suppose that $\mathfrak{g}$ is not nilpotent. Then by Theorem 2.15 there exists an integer n and an element α in J^0 such that for all β in J^0 of finite rank, α and β differ in rank $\leq n$. Denote by $T_{\leq n}$ the subspace of T consisting of elements of rank $\leq n$. Any element γ in J^0 annihilates $J \cap T_{\leq n}$ and therefore, by restriction, defines an element $\gamma|_n$ of $\big(T_{\leq n}/(J \cap T_{\leq n})\big)^*$. Moreover, every element of $\big(T_{\leq n}/(J \cap T_{\leq n})\big)^*$ arises in this way because if $u \in T_{\leq n}$ but $u \notin J$ then there exists $\gamma \in J^0$ such that $\gamma(u) \neq 0$. That is, $\{\gamma|_n : \gamma \in J^0\}$ separates from zero all non zero elements of the finite dimensional vector space $T_{\leq n}/(J \cap T_{\leq n})$ and therefore constitutes its entire dual space.

On the other hand, not every element of $\big(T_{\leq n}/(J \cap T_{\leq n})\big)^*$ arises as the restriction of a *finite rank* β in J^0, or else there would be a finite rank $\beta \in J^0$ which agrees with α up through rank n. Thus the finite rank β's in J^0 give rise to a proper subspace of $\big(T_{\leq n}/(J \cap T_{\leq n})\big)^*$. So there exists an element u in $T_{\leq n}$ such that $u \notin J$ but $\beta(u) = 0$ for all finite-rank β in J^0.

Now, by the Hijab-Hall Lemma 4.1, $u \notin \bar{J}$. So there exists $\gamma \in (J^0)_t$ such that $\gamma(u) \neq 0$. For example if T_t denotes the completion of T in the norm (4.2) and $J^\perp$ is the orthogonal complement of J in T_t then, upon identifying $(J^0)_t$ with $J^\perp$ via the pairing (2.1), we may take $\gamma = Pu$, where P is the projection onto $J^\perp$. It now follows that finite-rank β's in J^0 cannot be dense in $(J^0)_t$, since if $\beta_m \in J^0$ and is of finite rank and the β_m converge in $(J^0)_t$ norm to γ then $\gamma(u) = \lim \beta_m(u) = 0$ because $\beta_m(u) = 0$ for all m. ∎

REMARK 4.16 [PRODUCT RULE]. Take $w = w_1 \otimes w_2$ in (4.15) with $w_i \in T_{i^*}$, $i = 1, 2$ and replace β' by $\beta'\xi$ for $\xi = \xi_1 \oplus \xi_2 \in \mathfrak{g}_1 \oplus \mathfrak{g}_2$. One obtains $\langle \psi w, \beta'\xi\rangle = \langle (\sigma_1 w_1) \otimes (\sigma_2 w_2), (\Delta\beta')(\xi \otimes 1 + 1 \otimes \xi)\rangle$. But since $\sigma_1 w_1$ annihilates the two sided ideal in T generated by $0 \oplus \mathfrak{g}_2$ while $\sigma_2 w_2$ annihilates that generated by $\mathfrak{g}_1 \oplus 0$ the previous identity may be written, upon taking into account the definition (4.14)

$$\begin{aligned}\langle R_\xi^* \psi w, \beta'\rangle &= \langle (\sigma_1 w_1) \otimes (\sigma_2 w_2), (\Delta\beta')(\xi_1 \otimes 1 + 1 \otimes \xi_2)\rangle \\ &= \langle (\sigma_1 R_{\xi_1}^* w_1) \otimes (\sigma_2 w_2) + (\sigma_1 w_1) \otimes \sigma_2(R_{\xi_2}^* w_2), \Delta\beta'\rangle\end{aligned}$$

where R_ξ is defined in Remark 3.4. Therefore

$$R_\xi^* \psi w = \psi(R_{\xi_1}^* \otimes I + I \otimes R_{\xi_2}^*) w, \quad \xi_i \in \mathfrak{g}_i,\ i = 1, 2 \text{ and } w \in T_{1^*} \otimes T_{2^*}. \tag{4.31}$$

Now ψ extends to a linear map $\hat{\psi}$ from the large space $T_1' \otimes T_2'$ (algebraic tensor product) because ψ preserves total rank of tensor products of homogeneous tensors. Moreover (4.31) continues to hold for $\hat{\psi}$. Restricting then to the subspace $J_1^0 \otimes J_2^0$ (algebraic tensor product) of this large space one obtains

$$A_\xi \hat{\psi} w = \hat{\psi}(A_{\xi_1} \otimes I + I \otimes A_{\xi_2}) w, \quad \xi_i \in \mathfrak{g}_i,\ i = 1, 2,\ w \in J_1^0 \otimes J_2^0. \tag{4.32}$$

One should note here that $\hat{\psi} w \in J^0$, by Lemma 4.12, when $w \in J_1^0 \otimes J_2^0$.

The identity (4.31) is best interpreted in the context of Remark 3.4. If $u_i \in C^\infty(G_i)$ and if w_i is the set of Taylor coefficients of u_i at the identity element of G_i for $i = 1, 2$ then the definition (4.14) (and its extension to $\hat{\psi}$) simply expresses the Taylor coefficients of $u_1(x_1)u_2(x_2)$ at $(e_1, e_2) \in G_1 \times G_2$ in terms of w_1 and w_2 by means of Leibnitz' rule. Correspondingly (4.32) is exactly the product rule.

Let us finally go one step further and write the product rule (4.32) in the Hilbert space format

$$A_\xi L = L(A_{\xi_1} \otimes I + I \otimes A_{\xi_2}) \tag{4.33}$$

wherein it is desirable to interpret the three (unbounded) operators $A_\xi, A_{\xi_1}, A_{\xi_2}$ as closed operators in their respective Hilbert spaces $(J^0)_t$, $(J^0_1)_t$, and $(J^0_2)_t$. However there are serious Hilbert space domain issues for these operators. These domain issues are only understood in special cases, for example Lie algebras of compact type. See *e.g.* [DG, Section 7]. However, in the presence of sufficiently strong information on the domains of the operators A_ξ and A_{ξ_i}, $i = 1, 2$, the product rule (4.33), together with the obvious identity $L(1 \otimes 1) = 1$ completely determines the isometry L. See [DG, Section 7] for typical results of this kind.

5. **Example: Degeneracy of norms for** $\mathrm{sl}(2, \mathbb{C})$**.** The semi-norms induced on the universal enveloping algebra U, as in (4.1), by the norms (4.3) can be highly degenerate. As already noted in Section 4, $U/\ker\|\ \|_*$ is finite dimensional when one induces from the norms (4.3) if $\mathfrak{g}$ is the Lie algebra of a compact simply connected group G which leaves the given inner product on $\mathfrak{g}$ Ad invariant, [G4]. The analysis in [G4] depends on properties of heat kernels on Lie groups. It is illuminating to see, at a purely Lie algebraic level, how this degeneracy arises in the simplest example. In this section the norms $\|\beta + J\|_*$ of certain nonzero elements in U will be computed and shown to be zero if the parameter a in (4.3) is sufficiently small. This example is based on su(2). But the computations are most easily made in its complexification $\mathrm{sl}(2, \mathbb{C})$. The Ad SU(2) invariant Hermitian inner product on $\mathrm{sl}(2, \mathbb{C})$ will be used.

Let $\mathfrak{g} = \mathrm{sl}(2, \mathbb{C})$ and define $(\xi, \eta) = 2\,\mathrm{trace}\,(\eta^*\xi)$ where η^* is the Hermitian adjoint of η. Let $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then one can compute easily that $|f|^2 = 2$, $|h|^2 = 4$ and the Lie bracket $[f, h] = 2f$. Write $u \wedge v = u \otimes v - v \otimes u$ for u and v in T and $u \wedge^n h = \Big(\cdots\big((u \wedge h) \wedge h\big) \wedge \cdots \wedge h\Big)$ (n factors of h). Denote $f \otimes f \otimes \cdots \otimes f$ (k factors) by f^k. It follows by induction on k that

$$f^k \wedge h = (2k) f^k \bmod J \quad k = 1, 2, \ldots. \tag{5.1}$$

Hence

$$f^k \wedge^n h = (2k)^n f^k \bmod J \quad k \geq 1 \text{ and } n \geq 1. \tag{5.2}$$

https://doi.org/10.4153/CJM-1998-019-4 Published online by Cambridge University Press

PROPOSITION 5.1. *f^k is in the closure of J in the norm (4.3) if $a < k/2$.*

PROOF. Since $|h^j f^k h^m|_{\mathfrak{g}^{\otimes(j+k+m)}} = |h|^{j+m}|f|^k = (4^{j+m}2^k)^{1/2}$ and since $f^k \wedge^n h$ is a sum of at most 2^n terms of the form $\pm h^j f^k h^m$ with $j + m = n$ one has

$$|f^k \wedge^n h|_{\mathfrak{g}^{\otimes(k+n)}} \leq 2^n (4^n 2^k)^{1/2} = 2^{2n} 2^{k/2}. \tag{5.3}$$

By (5.2) $f^k - (2k)^{-n} f^k \wedge^n h \in J$. Hence (4.1), (4.3) and (5.3) yield

$$\begin{aligned} \|f^k + J\|_* &\leq \|f^k - (f^k - (2k)^{-n} f^k \wedge^n h)\|_{,a} \\ &= \|(2k)^{-n} f^k \wedge^n h\|_{,a} \\ &= (2k)^{-n} a^{k+n+1} \|f^k \wedge^n h\|_{\mathfrak{g}^{\otimes(k+n)}} \\ &\leq (2a/k)^n a^{k+1} 2^{k/2}. \end{aligned}$$

Taking the limit as $n \longrightarrow \infty$ shows that $\|f^k + J\|_* = 0$ if $2a < k$. ∎

COROLLARY 5.2. *Denote by $J_{,a}$ the closure of J in T in the norm (4.3). Then $f^k \in J_{,a}$ for $a < k/2$ and $f^k \notin J$ for any $k \geq 1$. In particular the norm $\| \ \|_*$ on U has an infinite dimensional kernel for all $a > 0$.*

PROOF. The statement that $f^k \in J_{,a}$ for $a < k/2$ is the content of Proposition 5.1. By [B, Chapter I, Lemma 7.3] any homogeneous tensor in J must be in the ideal $\mathcal{I}$ generated by $\{\xi \wedge \eta : \xi, \eta \in \mathfrak{g}\}$. Since no decomposable tensor is in $\mathcal{I}$, f^k is not in J. In fact no linear combination of $\{f^k\}_{k>2a}$ is in J because the highest power in such a linear combination is not in $\mathcal{I}$. Since all of these elements are in $J_{,a}$ the kernel of $\| \ \|_*$ is infinite dimensional. ∎

REMARK 5.3. Since the norm (4.3) gets stronger as a increases one always has $J_{,a} \supset J_{,b}$ if $0 < a < b$. That is, $J_{,a}$ decreases as a increases. But Corollary 5.2 suggests that the decrease may take place in jumps located at the half integers $a = k/2$ because $f^k \in J_{,a}$ if $a < k/2$ while f^k may not be in $J_{,a}$ if $a > k/2$. Actually this crude argument gives an approximately correct conclusion. $J_{,a}$ does decrease in jumps. But the jumps occur when $a^2 = (k/2)\big((k/2) + 1\big)$, $k = 1, 2, \ldots$ rather than when $a^2 = (k/2)^2$. This can be deduced from the example in [G4, Section 5] wherein it is shown that the dual space to $T/J_{,a}$ is finite dimensional and has dimension

$$\dim(T/J_{,a}) = \sum_{s(s+1)<a^2} (2s + 1)^2. \tag{5.4}$$

Here s runs over the spin values $\{k/2 : k = 0, 1, 2, \ldots\}$. In order to deduce (5.4) from [G4, Equation (5.4)] one should observe, as already noted in Section 4 above, that the topological dual space to $T/J_{,a}$ is the space $J^0_{,a}$ defined in (4.7). It is the space $J^0_{,a}$ which is studied in [G4]. A direct Lie algebra proof of (5.4) is not at present available. It is interesting to observe that multiplication in T is separately continuous in the norm (4.3) because multiplication by an element of $\mathfrak{g}$ is bounded. Consequently $J_{,a}$ is a 2-sided ideal in T and $T/J_{,a}$ is naturally a normed algebra. The example in [G4, Section 5] shows that if π_s is the representation of SU(2) of spin s and $d\pi_s$ is the map from T to operators on the representation space of π_s induced by π_s then $J_{,a}$ is in the kernel of $d\pi_s$ whenever $s(s + 1) < a^2$.

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