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EQUIVARIANT KK-THEORY FOR INVERSE LIMITS OF G-C^*-ALGEBRAS

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Abstract

The Kasparov groups $KK_*^G(A, B)$ are extended to the setting of inverse limits of G-C^*-algebras, where G is assumed to be a locally compact group. The KK-product and other important features of the theory are generalized to this setting.

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Introduction

The well-known Kasparov groups $KK_*^G(A, B)$ were defined by Kasparov for G a locally compact group and A and B G-C^*-algebras. Weidner [17, 18] preceeded by a suggestion of Phillips ([9, p. 470]), generalized the definition of the groups $KK_*(A, B)$ to the situation where A and B are inverse limits of C^*-algebras, and proved that the Kasparov pairing and other important properties of these groups extended to this setting. In this paper the equivariant groups $KK_*^G(A, B)$ are generalized to the setting of inverse limits of G-C^*-algebras where G is assumed to be a locally compact group. If G is compact then Phillips [9] has defined representable K-theory groups $RK_*^G(B)$ in this context. We show that there is a natural isomorphism $KK_*^G(\mathbb{C}, B) \cong RK_*^G(B)$ so that the KK^G groups defined here also generalize Phillips' construction.

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There are several motivations to warrant this generalization. Weidner's generalization of the (non-equivariant) KK-groups makes possible a direct and concrete homotopy-theoretic characterization of the groups $KK_*(C(X), C(Y))$ as in [14]. The analogous generalization of the equivariant KK-groups will make possible a characterization of $KK^G(C(X), C(Y))$ in terms of modern equivariant homotopy theory. (This characterization would be new even for X and Y finite G-complexes.) It is possible using this machinery to give a systematic modern treatment of the use of equivariant Spanier-Whitehead duality in KK_*^G-theory. (Added in proof: This program has been carried out. The results appear in 'On equivalent Kasparov theory and Spanier-Whitehead duality', *K-Theory* **6** (1992), 363-385.) These results also may be of use in studying index theory on the S^1-spaces $Map\,(S^1, X)$ which arise in gauge theory. In each of these cases the straight-forward proofs of the main theorems call for systematic use of topological spaces such as function spaces which are hardly ever of the G-homotopy type of finite G-complexes but are of the G-homotopy type of infinite G-complexes. These applications will appear separately. In the applications G is generally assumed to be a compact Lie group, so that the homogeneous spaces G/H are of the G-homotopy type of G-CW-complexes.

The paper is organized as follows:

Section 1. Kasparov Bimodules
Section 2. The Kasparov groups: elementary properties
Section 3. The Kasparov product
Section 4. Mapping Cones and Cofibration Exact Sequences
Section 5. Positive Splittings and Exact Sequences
Section 6. Additivity: Sums and Products
Section 7. Direct and Inverse Limits

For most of the general development it suffices to assume that G is a separable, second countable, locally compact and σ-compact group. However, in Section 5 certain exactness results seem to require that G be compact, and the results of Section 5 are used to prove the various limit results of Section 7, so there too compactness is required.

It will be clear to the reader that this paper is heavily dependent upon the work of G. G. Kasparov, N. C. Phillips, and J. Weidner, to whom we are grateful. We wish to thank Phillips especially for his continuous assistance in the evolution of this work.

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1. Kasparov Bimodules

We suppose throughout that all C^*-algebras are graded via some endomorphism ϵ, with $\epsilon^2 = 1$, that homomorphisms are graded, and that commutators are graded. All groups G are assumed to be separable, locally compact, and σ-compact, all group homomorphisms are continuous, and all subgroups are closed. A *G-algebra* A is a topological *-algebra A together with a degree-preserving action of G on A (that is, a homomorphism from G to the group of automorphisms of A) such that for each $a \in A$ the map $g \mapsto g(a)$ is continuous. (If A is an inverse limit of a sequence of G-C^*-algebras then each automorphism of A is continuous and has a continuous inverse. If not then assume these properties.) The algebra lies over the ground field $\mathbb{R}$ or $\mathbb{C}$, denoted F generically and assumed to be trivially graded. A homomorphism of G-algebras is understood to be equivariant.

A continuous C^*-seminorm x on a G-algebra A is *invariant* if

$$|g(a)|_x = |a|_x \quad \text{for all } g \in G.$$

If the group G is compact then the invariant C^*-seminorms are cofinal among all C^*-seminorms for given any C^*-seminorm x, the C^*-seminorm $[x]$ defined by

$$|a|_{[x]} = \sup\{|g(a)|_x : g \in G\}$$

which is finite and continuous (cf. [9, 5.2]) since G is compact, satisfies $x \leq [x]$. (If G is not compact then there may not be any G-invariant continuous C^*-seminorms. This is the case, for instance, for the translation action of $\mathbb{R}$ on $C(\mathbb{R})$.)

Let $\mathscr{S}(A)$ denote the partially ordered set of invariant continuous C^*-seminorms on A. For $x \in \mathscr{S}(A)$, the set

$$\{a \in A : |a|_x = 0\}$$

is a G-invariant ideal in A; let

$$A_x = A/\{a \in A : |a|_x = 0\}$$

denote the quotient G-C^*-algebra. Note that A_x is complete, by ([12, Folgerung 5.4]; see also [8, Cor. 1.12]). There are natural maps $\pi^{yx} : A_y \to A_x$ for $y \geq x$ and $\pi^y : A \to A_y$. Let

$$A_\infty = \varprojlim \{A_x : x \in \mathscr{S}(A)\}$$

with G-map $\pi^\infty : A \to A_\infty$.

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DEFINITION 1.1. A G-algebra A is a *pro-G-C^*-algebra* if one of the following equivalent conditions holds:

(1) There is an inverse system $\mathscr{B}$ of G-C^*-algebras with

$$A = \varprojlim \mathscr{B}.$$

(2) The G-map $\pi^\infty : A \to A_\infty$ is an isomorphism.

(3) The topology on A is given by a family of invariant C^*-seminorms on A and A is complete with respect to the topology.

Weidner would use 'generalized operator G-algebra', for this notion; we follow Phillips, after Voiculescu (approximately) and Arveson. An inverse limit of a sequence of G-C^*-algebras is said to be a *σ-G-C^*-algebra*. The forgetful functor takes pro-G-C^*-algebras to pro-C^*-algebras (in the sense of Weidner [17]).

If G is compact, A is a G-algebra, and the map $G \times A \to A$ is continuous then A is a pro-G-C^*-algebra, by ([9, 5.2]). If G is locally compact but not compact then many interesting examples of G-algebras do not fall under this definition. It might be possible to broaden the definition so as to include G-algebras of the form $A = \varprojlim A_j$ where each A_j is a G_j-C^*-algebra for some collection of subgroups $\{G_j\}$ of G with union G, for instance, but we have not pursued this possibility.

DEFINITION 1.2. Let B be a pro-G-C^*-algebra. A graded *pre-Hilbert G-module* E over B is a right B-module E equipped with a B-valued scalar product together with a continuous action of G on E by linear automorphisms which preserves the grading and which satisfies the compatibility conditions

$$g(eb) = g(e)g(b) \qquad \text{for} \quad e \in E,\ b \in B$$

and

$$g(\langle e, f\rangle) = \langle g(e), g(f)\rangle \qquad \text{for} \quad g \in G, \quad e, f \in E.$$

An invariant C^*-seminorm x on B determines an invariant seminorm on E by

$$|e|_x = |\langle e, e\rangle|_x{}^{1/2}.$$

Then E is a *Hilbert G-module* if E is complete with respect to the topology which is determined by these seminorms. Each

$$E_x \equiv E/\{e \in E : |e|_x = 0\}$$

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is a Hilbert G-module over B_x and

$$E \cong \varprojlim \{E_x : x \in \mathscr{S}(B)\}.$$

Given a Hilbert G-module E, one defines $\mathscr{L}(E)$ to be those B-linear operators $T : E \to E$ which possess an adjoint $T^* : E \to E$ such that

$$\langle Te, f\rangle = \langle e, T^* f\rangle$$

for all $e, f \in E$. For example, *rank one operators* $\theta_{ef} \in \mathscr{L}(E)$ are defined by

$$\theta_{ef}(z) = e\langle f, z\rangle \quad \text{for} \quad e, f, z \in E.$$

The group G acts upon $\mathscr{L}(E)$ by

$$g(T)(e) = g(Tg^{-1}(e)), \qquad T \in \mathscr{L}(E), \quad e \in E.$$

A quick check shows that $g(\theta_{ef}) = \theta_{ge,gf}$ and thus G acts upon the ideal generated by the rank one operators. Given $x \in \mathscr{S}(B)$, a seminorm is given on $\mathscr{L}(E)$ by

$$\|T\|_x = \sup\{\,|Te|_x : |e|_x \leq 1\}.$$

This seminorm is finite for each $T \in \mathscr{L}(E)$ by [17, (1.1)]. Then $\mathscr{L}(E)$ is a pro-C^*-algebra with respect to this family of seminorms and

$$\mathscr{L}(E) = \varprojlim \mathscr{L}(E_x).$$

We note that the G-action on $\mathscr{L}(E)$ is not usually continuous, just as in the C^*-algebra case, although the action is continuous for the strict topology, cf. [8, 3.13]. The inverse limit displayed is an inverse limit of G-algebras. The closed ideal $\mathscr{K}(E)$ generated by the rank one projections is closed under the action of G by the above remarks.

Next we discuss tensor products. Suppose first that A and B are pro-G-C^*-algebras. Then for each $(x, y) \in \mathscr{S}(A) \times \mathscr{S}(B)$ the G-C^*-algebra $A_x \widehat{\otimes}_{\min} B_y$ is defined, where G acts diagonally. Define

$$A \widehat{\otimes} B = \varprojlim \{A_x \widehat{\otimes}_{\min} B_y : (x, y) \in \mathscr{S}(A) \times \mathscr{S}(B)\}.$$

Similarly, if E and F are Hilbert G-modules over B then their tensor product is defined by

$$E \widehat{\otimes}_B F = \varprojlim \{E_x \widehat{\otimes}_B F_x : x \in \mathscr{S}(B)\}.$$

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It has the structure of a Hilbert G-module over B.

A more complicated situation arises given the following data. Suppose that E is a Hilbert G-module over D, F is a Hilbert G-module over B, and $\varphi : D \to \mathscr{L}(F)$ is an invariant homomorphism. If B and D were actually G-C^*-algebras then $E\widehat{\otimes}_D F$ would be defined by forming their algebraic tensor product $E \otimes_D F$ and then completing with respect to the B-valued norm

$$\langle e_1 \otimes f_1, e_2 \otimes f_2 \rangle \quad \equiv \quad \langle f_1, \varphi(\langle e_1, e_2 \rangle)(f_2) \rangle.$$

If B and D are only pro-G-C^*-algebras then one still may form the algebraic tensor product $E \otimes_D F$. Each $x \in \mathscr{S}(B)$ determines a seminorm on $E \otimes_D F$ via

$$|w|_x \equiv |\langle w, w \rangle|_x.$$

Denote by $E\widehat{\otimes}_D F$ or $E\widehat{\otimes}_\varphi F$ the completion of $E \otimes_D F$ with respect to these seminorms. It is clear that

$$E\widehat{\otimes}_D F \cong \varprojlim \{E_{\varphi^*(x)}\widehat{\otimes}_{D_x} F_x : x \in \mathscr{S}(B)\}.$$

If E is a Hilbert G-module over B_1 and $f : B_1 \to B_2$ then $E\widehat{\otimes}_{B_1} B_2$ is a Hilbert G-module over B_2, and each B_1-linear map $T \in \mathscr{L}(E)$ extends canonically to $T \otimes I_{B_2} \in \mathscr{L}(E\widehat{\otimes}_{B_1} B_2)$.

DEFINITION 1.3. A Hilbert G-module E over B is *countably generated* if E_x is a countably generated Hilbert G-module over B_x for each $x \in \mathscr{S}(B)$.

DEFINITION 1.4. Suppose that A and B are pro-G-C^*-algebras. An *A-B G-bimodule* $E = (E, \varphi)$ is a graded countably generated Hilbert G-module E over B together with an invariant homomorphism $\varphi : A \to \mathscr{L}(E)$. Given an A-B G-bimodule E, define $Q_A(E)$ to be those $T \in \mathscr{L}(E)$ whose graded commutator $[T, \varphi(a)]$ is in $\mathscr{K}(E)$ and whose associated map $g \mapsto g(\varphi(a)T)$ is continuous for all $a \in A$. Further, define $I_A(E)$ to be those $T \in Q_A(E)$ such that $T\varphi(a)$ and $\varphi(a)T$ lie in $\mathscr{K}(E)$ for all $a \in A$.

DEFINITION 1.5. A *Kasparov A-B G-bimodule* (E, φ, T) is an A-B G-bimodule (E, φ) together with an operator $T \in Q_A(E)$ of degree 1 such that

$$(T - T^*), \quad (T^2 - I), \quad \text{and} \quad (g(T) - T)$$

lie in $I_A(E)$ for all $g \in G$.

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The collection of all Kasparov A-B G-bimodules is denoted $\mathscr{E}^G(A, B)$. Under the obvious operation this is an associative commutative monoid with identity. A Kasparov A-B G-bimodule is said to be *degenerate* if

$$(T - T^*)\varphi(a) = 0, \qquad (T^2 - I)\varphi(a) = 0,$$
$$[T, \varphi(a)] = 0 \quad \text{and} \quad (g(T) - T)\varphi(a) = 0$$

for all $a \in A$, $g \in G$.

2. The Kasparov groups: elementary properties

DEFINITION 2.1. An *operator homotopy* from (E, T^0, φ) to (E, T^1, φ) is a norm-continuous path $\{T^t\}$ with $0 \le t \le 1$ which agrees at $t = 0, 1$ with the given operators and which yields a Kasparov A-B G-bimodule for each t.

DEFINITION 2.2. If A and B are G-C^*-algebras then the Kasparov group $KK^G(A, B)$ is defined to be the quotient of $\mathscr{E}^G(A, B)$ by the equivalence relation given by operator homotopy.

A more general version of a Kasparov A-B G-bimodule is needed in the setting of pro-G-C^*-algebras. We follow Weidner:

DEFINITION 2.3. Suppose given a pro-G-C^*-algebra B. An (*ordered*) *seminorm simplex* σ in $\mathscr{S}(B)$ is a list $(x_0, \dots, x_n)$ with $x_i \in \mathscr{S}(B)$ and $x_0 \le x_1 \le \dots \le x_n$. The seminorm simplex is *nondegenerate* if $x_0 < x_1 < \dots < x_n$. Given an ordered seminorm simplex σ in $\mathscr{S}(B)$, its i'th face $\sigma\epsilon^i$ is obtained by deleting the i'th entry on the list.

Let
$$\Delta_n = \{\tau = (\tau^0, \dots, \tau^n) \in \mathbb{R}^{n+1} : \tau^i \ge 0, \sum \tau^i = 1\}$$
denote the usual geometric n-simplex.

DEFINITION 2.4. Let A and B be pro-G-C^*-algebras. A *Kasparov pro-A-B G-bimodule* $(E, \varphi, \{T_\sigma\})$ is an A-B G-bimodule (E, φ) together with a collection of norm-continuous maps

$$T_\sigma : \Delta_n \to \mathscr{L}(E_{x_0})$$

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for each nondegenerate seminorm simplex σ which satisfy the following conditions:

(1) $(E_{x_0}, \varphi_{x_0}, \{T_\sigma(\tau)\}) \in \mathscr{E}(A, B_{x_0})$ for each $\tau \in \Delta_n$, where $\varphi_y = \pi_*^y \varphi$.

(2) coherence: for each nondegenerate seminorm simplex $\sigma = (x_0, \dots, x_n)$,

$$T_\sigma \epsilon^0 = \pi_*^{x_1 x_0}(T_{\sigma\epsilon^0})$$

and

$$T_\sigma \epsilon^i = T_{\sigma\epsilon^i} \qquad \text{for} \quad i > 0.$$

We denote by $\mathscr{E}^G(A, B)$ the collection of Kasparov pro-A-B G-bimodules. Note that a coherent collection of maps $\{T_\sigma\}$ defined for nondegenerate seminorm simplices may be extended canonically to all seminorm simplices (cf. [17, p. 62].) With this understood, it is easy to see that $\mathscr{E}^G(A, B)$ is a contravariant functor in A and a covariant functor in B.

DEFINITION 2.5. Two Kasparov pro-A-B G-bimodules $\eta^0, \eta^1 \in \mathscr{E}^G(A, B)$ are *homotopic* if there is some $\eta \in \mathscr{E}^G(A, B[0, 1])$ restricting to η^0, η^1 at endpoints.

Just as in the non-equivariant case, the homotopy relation is transitive and symmetric but not obviously reflexive; however Weidner's argument proving reflexivity extends verbatim, so that homotopy is an equivalence relation. Further, the obvious equivariant extension of Weidner's argument implies [17, p. 62-63] that two Kasparov pro-A-B G-bimodules are homotopic if they are operator homotopic.

DEFINITION 2.6. For two pro-G-C^*-algebras A, B, the *Kasparov group* $KK^G(A, B)$ is defined to be the quotient of $\mathscr{E}^G(A, B)$ modulo the homotopy relation 2.5.

PROPOSITION 2.7. (1) *For each A and B pro-G-C^*-algebras, $KK^G(A, B)$ is an abelian group, with the group operation given by direct sum and identity element the homotopy class of degenerate elements.*

(2) *The assignment $(A, B) \mapsto KK^G(A, B)$ is a bifunctor on the category of pro-G-C^*-algebras to the category of abelian groups which is contravariant in A and covariant in B and which extends the Kasparov groups defined for G-C^*-algebras.*

(3) *The bifunctor is homotopy-invariant in each variable.*

(4) *If B is a G-C^*-algebra then $KK^G(A, B)$ is isomorphic to the set of homotopy classes of Kasparov A-B G-bimodules.*

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PROOF. Only the last sentence requires comment. If B is a G-C^*-algebra then in $\mathscr{S}(B)$ the C^*-norm $| \ |_B$ is a maximal element. A Kasparov pro-A-B G-bimodule with respect to $| \ |_B$ is precisely a Kasparov A-B G-bimodule, so pro-bimodules may be replaced by bimodules uniquely up to homotopy.

3. The Kasparov Product

We turn next to the definition of the Kasparov product. Assume that A, B, and D are pro-G-C^*-algebras. The basic KK^G -pairing is a map

$$KK^G(A, D) \otimes KK^G(D, B) \longrightarrow KK^G(A, B)$$

which is denoted

$$(\alpha, \beta) \longmapsto \alpha \otimes_D \beta.$$

(Note that we follow Kasparov's notation; Weidner would write $\beta \otimes_D \alpha$.) We assume that A is *separable*; that is, that each A_x is a separable G-C^*-algebra. (This unconventional terminology follows Weidner.) Suppose given

$$(E, \varphi, \{S_\sigma\}) \in \mathscr{E}^G(A, D)$$

and

$$(F, \psi, \{T_\sigma\}) \in \mathscr{E}^G(D, B).$$

Given $x \in \mathscr{S}(B)$, let $\varphi^*(x)$ denote the induced seminorm on D. Similarly, given a seminorm simplex $\sigma = (x_0, \dots, x_n) \in \mathscr{S}(B)$, let $\varphi^*(\sigma) = (\varphi^*(x_0), \dots, \varphi^*(x_n)) \in \mathscr{S}(D)$.

DEFINITION 3.1. A Kasparov pro-A-B G-bimodule

$$(E \widehat{\otimes}_D F, \psi_* \circ \varphi, \{R_\sigma\})$$

is called a *Kasparov product* of $(E, \varphi, \{S_\sigma\})$ and $(F, \psi, \{T_\sigma\})$ if there are operator homotopies $(E, \varphi, \{S_\sigma^t\})$ and $(F, \psi, \{T_\sigma^t\})$ for $0 \le t \le 1$ such that

(1) $S_\sigma^0 = S_\sigma, \quad T_\sigma^0 = T_\sigma.$

(2) For each seminorm simplex $\sigma \in \mathscr{S}(B)$ and $\tau \in \Delta$,

$$R_\sigma(\tau) \in S^1_{\varphi^*(\sigma)}(\tau) \sharp T^1_\sigma(\tau)$$

where $\sharp$ denotes the Kasparov KK^*-pairing $(\ ) \otimes_{\varphi^*(x_0)} (\ )$ for G-C^*-algebras.

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We require here that the graded commutator $[R_\sigma, \varphi_*(T_\sigma)]$ can be decomposed into the sum of two norm continuous functions $p + j$ such that $p \geq 0$ and for $\tau \in \Delta$, $j(\tau) \in Q_A(E\widehat{\otimes}_D F_{x_0})$.

THEOREM 3.2. *Suppose given A, B, D pro-G-C^*-algebras with A separable, and suppose given*

$$\alpha \in \mathscr{E}^G(A, D), \qquad \beta \in \mathscr{E}^G(D, B).$$

Then there is a Kasparov product $\alpha \otimes_D \beta \in \mathscr{E}^G(A, B)$. This pairing passes to a unique well-defined Kasparov pairing

$$\otimes_D : \; KK^G(A, D) \otimes KK^G(D, B) \longrightarrow KK^G(A, B)$$

which extends the KK-pairing for G-C^*-algebras. If B is also separable, B' is a pro-G-C^*-algebra, and $\gamma \in \mathscr{E}^G(B, B')$, then the product is associative:

$$(\alpha \otimes_D \beta) \otimes_B \gamma = \alpha \otimes_D (\beta \otimes_B \gamma).$$

PROOF. Suppose that α is represented by $(E, \varphi, \{S_\sigma\})$ and β is represented by $(F, \psi, \{T_\sigma\})$. We must show that a Kasparov product $\alpha \otimes_D \beta \in \mathscr{E}^G(A, B)$ exists and that its equivalence class in $KK^G(A, B)$ is independent of choices of representatives. The proof of existence in [17, pages 71-72] translates easily. The idea is to construct $R_\sigma : \Delta_n \to \mathscr{L}((E \otimes_D F)_{x_0})$ for each seminorm simplex $\sigma = (x^0, \dots, x^n)$ such that

(1) $(E \otimes_D F, R_\sigma, \varphi_* \circ \psi) \in \mathscr{E}^G(A, B)$, and

(2) $R_\sigma(\tau) \in S^1_{\varphi^*(\sigma)}(\tau) \sharp T^1_\sigma(\tau)$ for each σ.

The construction is by induction on the number of vertices in σ and uses certain fixed homotopies on Δ_n for induction. These constructions preserve the relevant G-structures and produce an element in $\mathscr{E}^G(A, B)$. The uniqueness and associativity arguments carry through by similar reasoning.

DEFINITION 3.3. A pro-G-C^*-algebra A is *countable at infinity* if each associated G-C^*-algebra A_x has a countable approximate unit.

THEOREM 3.4. *The Kasparov product defined in* (3.2) *has the following properties:*

(1) *The product is natural in each variable.*

(2) *If B is countable at infinity then a G-map $\varphi : A \to B$ defines an element $[\varphi] \in KK^G(A, B)$, and $[\varphi] = \varphi_*[1_A]$.*

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(3) *The product generalizes composition of maps.*
(4) *Kasparov product with a KK^G-invertible element is an isomorphism.*
(5) *If D is countable at infinity then the natural map*

$$\mathrm{Hom}_G(A, B) \longrightarrow \mathrm{Hom}_G(A\widehat{\otimes}D, B\widehat{\otimes}D)$$

induces a homomorphism of groups

$$\sigma_D : KK^G(A, B) \longrightarrow KK^G(A\widehat{\otimes}D, B\widehat{\otimes}D)$$

and similarly for $(D\widehat{\otimes}A, D\widehat{\otimes}B)$.

The proof is direct from [15].

DEFINITION 3.5. The Kasparov product

$$\otimes_D : KK^G(A_1, B_1\widehat{\otimes}D) \otimes KK^G(D\widehat{\otimes}A_2, B_2) \longrightarrow KK^G(A_1\widehat{\otimes}A_2, B_1\widehat{\otimes}B_2)$$

is defined by the formula

$$\alpha_1 \otimes_D \alpha_2 \equiv \sigma_{A_2}(\alpha_1) \otimes_{B_1\otimes D\otimes A_2} \sigma_{B_1}(\alpha_2)$$

provided that A_1 is separable and that B_1 and A_2 are countable at infinity.

Henceforth *we assume that all operator algebras are countable at infinity when necessary to form the requisite products.*

THEOREM 3.6. *Let A_1 and A_2 be separable. Then the Kasparov KK^G product*

$$\otimes_D : KK^G(A_1, B_1\widehat{\otimes}D) \otimes KK^G(D\widehat{\otimes}A_2, B_2) \longrightarrow KK^G(A_1\widehat{\otimes}A_2, B_1\widehat{\otimes}B_2)$$

has the following properties:
(1) *The product is natural in each variable.*
(2) *The product is natural in D in the following sense: if $\varphi : D_1 \to D_2$ then*

$$\varphi_*(\alpha_1) \otimes_{D_2} \alpha_2 = \alpha_1 \otimes_{D_1} \varphi^*(\alpha_2).$$

(3) *The product is associative.*
(4) *If $D = F$ then the resulting cup product is commutative.*
(5) *The element $1_A \in KK^G(A, A)$ is a unit for the product.*
(6) *The ring $R(G) = KK^G(F, F)$ is commutative and for A separable each $KK^G(A, B)$ is a two-sided $R(G)$-module.*

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(7) *If A_1, A_2, D_2 are separable and D_1 is σ-unital then*

$$\sigma_{D_2}(\alpha) \otimes_{D_1 \widehat{\otimes} D \widehat{\otimes} D_2} \sigma_{D_1}(\beta) = \alpha \otimes_D \beta$$

for $\alpha \in KK^G(A_1, B_1 \widehat{\otimes} D_1 \widehat{\otimes} D)$, $\beta \in KK^G(D \widehat{\otimes} D_2 \widehat{\otimes} A_2, B_2)$.

(8) *If A_1, A_2, and D_1 are separable then*

$$\sigma_{D_1}(\alpha \otimes_D \beta) = \sigma_{D_1}(\alpha) \otimes_{D \otimes D_1} \sigma_{D_1}(\beta)$$

for $\alpha \in KK^G(A_1, B_1 \otimes D)$, $\beta \in KK^G(D \otimes A_2, B_2)$.

These proofs are the same as in the nonequivariant case [15]. We note that if G is compact, then $R(G)$ is the usual representation ring. This identification is realized as follows. Suppose that $(E, T) \in \mathscr{E}^G(F, F)$. Then $1 \in F$ acts upon E as a projection $P \in \mathscr{L}(E)$. Let Q be defined by

$$Q(e) = \int_G g(T)e\, dg.$$

Then $Q \in \mathscr{L}(E)$ and $Q - T \in I_F(E)$. Let

$$M = PQ : P(E^0) \to P(E^1)$$

be the restriction of PQ to $P(E^0)$. Then M is a G-Fredholm operator, and its G-index

$$\operatorname{ind}_G(M) = [\operatorname{Ker}(M)] - [\operatorname{Ker}(M^*)]$$

lies in $R(G)$. The construction $(E, T) \mapsto \operatorname{ind}_G(M)$ induces the isomorphism

$$KK^G(F, F) \to R(G).$$

Suppose that V and W are linear G-spaces with associated Clifford algebras C_V and C_W. (The isometric action of G extends naturally to an action of G on the Clifford algebras.) The associated graded Kasparov groups are then defined by

$$K_V K^{W\,G}(A, B) = KK^G(A \widehat{\otimes} C_V, B \widehat{\otimes} C_W)).$$

As in the classical case, the groups depend only upon the class $a = [V] - [W] \in R(G)$, so write

$$KK_a^G(A, B) = K_V K^{W\,G}(A, B) \qquad \text{for } a = [V] - [W].$$

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If the action of G on V is trivial then $C_V \cong C_n$, the standard Clifford algebra, for some n (cf. [4, Section 2]) and

$$KK_V^G(A, B) \cong KK_n^G(A, B).$$

The isomorphism is fixed by the choice of an orientation of C_V (cf. [4, p. 543-544].)

If $V \cong \mathbb{C}^n$ with real coordinate subspace $\mathbb{R}^n$ then

$$\mathrm{Spin}(V) = \{g \in C_V : g^*g = 1, \quad gxg^* \in \mathbb{R}^n \; for \; x \in \mathbb{R}^n\}.$$

There is a natural double (for $n > 2$) covering map $\mathrm{Spin}(V) \to SO(\mathbb{R}^n)$ where $SO(\mathbb{R}^n)$ denotes the group of orientation-preserving isometries of $\mathbb{R}^n$. An orientation-preserving action of the group G on $\mathbb{R}^n$ 'lifts to a spin representation' if the natural map $G \to SO(n)$ factors through $\mathrm{Spin}(V)$. This should be regarded as an orientability assumption upon the action which is slightly more stringent than the usual assumption.

THEOREM 3.9. BOTT PERIODICITY (KASPAROV) *If the action of the group G on $\mathbb{R}^n$ lifts to a spin representation then the Bott element*

$$\beta_n \in KK^G(F, C_0(\mathbb{R}^n)\widehat{\otimes}C_n)$$

is KK^G-invertible with KK^G-inverse

$$\alpha_n \in KK^G(C_0(\mathbb{R}^n)\widehat{\otimes}C_n, F).$$

Kasparov product with the Bott element defines an isomorphism

$$KK_{a+n}^G(A\widehat{\otimes}C_0(\mathbb{R}^n), B) \cong KK_a^G(A, B) \cong KK_{a-n}^G(A, B\widehat{\otimes}C_0(\mathbb{R}^n))$$

in the real and complex cases.

PROOF. The Bott element has a KK^G-inverse by [4, p. 547], and hence induces an isomorphism under pairing.

REMARK 3.10. The classical Thom isomorphism theorem for equivariant K-theory generalizes to G-C^*-algebras, as in [4, Theorem 8, p. 549]. Since the isomorphism is given by KK^G-pairing with a KK^G-invertible element, this isomorphism also extends immediately to our setting. We omit the statements for brevity and refer the reader to [4].

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THEOREM 3.11. *The Kasparov product* 3.5 *extends to a unique product*

$$\otimes_D : KK_a^G(A_1, B_1 \widehat{\otimes} D) \otimes KK_b^G(D \widehat{\otimes} A_2, B_2) \longrightarrow KK_{a+b}^G(A_1 \widehat{\otimes} A_2, B_1 \widehat{\otimes} B_2)$$

for $a, b \in R(G)$ *with the obvious graded extensions of the properties which are listed in* 3.6. *The product respects all periodicity maps and Thom maps.*

PROOF. Only the last sentence requires comment. The point is that the periodicity and Thom maps are given by KK^G pairing with appropriate elements, so that the general associativity properties of the pairing translate into the desired properties for periodicity and Thom maps.

4. Mapping Cone and Cofibration Exact Sequences

We begin by introducing a bit of notation. Given an operator algebra A, define

$$IA = C[0,1] \widehat{\otimes} A;$$
$$CA = \{\xi \in IA : \xi(0) = 0\} \quad \text{the } \textit{cone} \text{ on A;}$$
$$SA = \{\xi \in CA : \xi(1) = 0\} \cong C_o(\mathbb{R}) \widehat{\otimes} A \quad \text{the } \textit{suspension} \text{ of A.}$$

The group G acts trivially upon $C[0, 1]$ and acts upon A by the given action.

If $\varphi : A \to B$ then the *mapping cone* of φ is defined by

$$C\varphi = \{(\xi, a) \in CB \oplus A : \ \xi(1) = \varphi(a)\}.$$

There is a natural mapping cone sequence

(4.1) $$0 \longrightarrow SB \xrightarrow{\iota} C\varphi \xrightarrow{\pi} A \longrightarrow 0$$

defined by $\pi(\xi, a) = a$ and $\iota(\xi) = (\xi, 0)$. This sequence induces mapping cone sequences of the form

(4.2) $$KK_n^G(D, C\varphi) \xrightarrow{\pi_*} KK_n^G(D, A) \; o \xrightarrow{\varphi_*} KK_n^G(D, B)$$

and

(4.3) $$KK_n^G(C\varphi, D) \xleftarrow{\pi^*} KK_n^G(A, D) \xleftarrow{\varphi^*} KK_n^G(B, D)$$

with $\varphi_* \pi_* = 0$ and $\pi^* \varphi^* = 0$.

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LEMMA 4.4. (WEIDNER)

(1) *In the sequence* (4.2), $\operatorname{Ker}\varphi_* = \operatorname{Im}\pi_*$ *provided that* D *is separable.*

(2) *In the sequence* (4.3), $\operatorname{Ker}\pi^* = \operatorname{Im}\varphi^*$ *provided that* A *and* B *are separable.*

PROOF. Weidner's proof holds verbatim. The basic fact is that the version of Bott periodicity required may be realized as a KK^G-pairing with an appropriate Bott element which is invertible by classical equivariant Bott periodicity (cf. 3.9).

THEOREM 4.5. MAPPING CONE SEQUENCES *Let* $\varphi : A \to B$ *be a morphism of pro-*G*-*C^**-algebras.*

(1) *If* D *is separable then there is a long exact sequence of the form*

$$\cdots \to KK^G_{n+1}(D,B) \xrightarrow{\partial} KK^G_n(D,C\varphi) \xrightarrow{\pi_*} KK^G_n(D,A) \xrightarrow{\varphi_*} \ldots KK^G_n(D,B) \to \ldots \tag{4.6}$$

The boundary homomorphism ∂ *in* (4.6) *is given by the composite*

$$KK^G_{n+1}(D,B) \cong KK^G_n(D,SB) \xrightarrow{\iota_*} KK^G_n(D,C\varphi).$$

(2) *If* A *and* B *are separable then there is a long exact sequence of the form*

$$\ldots KK^G_n(B,D) \xrightarrow{\varphi^*} KK^G_n(A,D)\, o \xrightarrow{\pi^*} KK^G_n(C\varphi,D) \xrightarrow{\partial} KK^G_{n+1}(B,D) \ldots \tag{4.7}$$

The boundary map ∂ *in* (4.7) *is given by the composite*

$$KK^G_n(C\varphi,D) \xrightarrow{\iota^*} KK^G_n(SB,D) \cong KK^G_{n+1}(B,D).$$

PROOF. This is a formal consequence of Lemma 4.4 and standard mapping cone arguments.

The mapping cone exact sequences (4.6) and (4.7) are precisely those expected for generalized homology/cohomology theories.

REMARK 4.8. The usual Mayer-Vietoris sequences associated to the mapping cones of a commutative square of pro-G-C^*-algebras hold in this generality as well. In the presence of excision these reduce to the very well-known Mayer-Vietoris sequences. For details, see [18, p. 83-84], [13].

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THEOREM 4.9. *Suppose that $\varphi : A \to A/J$ is a cofibration in the sense of* [13] *(called 'fibration' by Weidner). Then:*

(1) *The natural inclusion $\xi : J \to C\varphi$ given by $\xi(j) = (0, j)$ is an equivalence.*

(2) *If D is separable then there is a long exact sequence of the form*

$$\ldots \to KK_n^G(D,J) \to KK_n^G(D,A) \xrightarrow{\varphi_*} KK_n^G(D,A/J) \longrightarrow KK_{n-1}^G(D,J) \to \ldots \tag{4.10}$$

(3) *If A is separable then there is a long exact sequence of the form*

$$\ldots KK_n^G(A/J,D) \xrightarrow{\varphi^*} KK_n^G(A,D) \longrightarrow KK_n^G(J,D) \longrightarrow KK_{n+1}^G(A/J,D) \ldots \tag{4.11}$$

PROOF. If $\varphi : A \to A/J$ is a cofibration then the natural inclusion $J \to C\varphi$ is an equivalence by [13, 2.4]. The rest of the Theorem is immediate from Theorem 4.5.

The cofibration assumption in Theorem 4.9 is strong enough to imply exactness properties for any generalized homology theory. It is possible for applications within KK-theory to get by with much less. One needs only that the map $J \to C\varphi$ act like an equivalence with respect to Kasparov theory. Theorem 4.13 and Corollary 4.16 pursue this theme.

NOTATION 4.12. There are many exact sequences which arise below. For ease of exposition, we abbreviate as follows. When the conclusion of (4.10) holds we shall say that *the theory $KK_*^G(D, -)$ is exact for $(J, A, A/J)$*. Similarly, when the conclusion of (4.11) holds we shall say that *the theory $KK_*^G(-, D)$ is exact for $(J, A, A/J)$*.

THEOREM 4.13. *Suppose given a short exact sequence*

$$0 \longrightarrow J \longrightarrow A \xrightarrow{\pi} A/J \longrightarrow 0 \tag{4.14}$$

of pro-G-C^*-algebras, with natural associated sequence

$$0 \longrightarrow J \xrightarrow{\xi} C\pi \longrightarrow CA \longrightarrow 0. \tag{4.15}$$

Then:

(1) *Suppose that D is a separable pro-G-C^*-algebra. Then the theory $KK_*^G(D, -)$ is exact for $(J, A, A/J)$ if and only if the map $\xi : J \to C\pi$ is a $KK_*^G(D, -)$-equivalence.*

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(2) *Suppose that A is separable. Then the theory $KK_*^G(-, D)$ is exact for $(J, A, A/J)$ if and only if the map $\xi : J \to C\pi$ is a $KK_*^G(-, D)$-equivalence.*

PROOF. This follows directly from the exactness of the mapping cone sequences (4.6) and (4.7) and general mapping cone arguments.

COROLLARY 4.16. *Suppose given the short exact sequence* (4.14) *and suppose that the associated map $\xi : J \to C\pi$ is KK^G-invertible. Then:*

(1) *If D is a separable pro-G-C^*-algebra, then the theory $KK_*^G(D, -)$ is exact for $(J, A, A/J)$.*

(2) *If A is separable then the theory $KK_*^G(-, D)$ is exact for $(J, A, A/J)$.*

PROOF. Any KK^G-equivalence induces an isomorphism in KK^G-theory in either variable.

Corollary 4.16 shifts attention to conditions that imply that the map $\xi : J \to C\pi$ is KK^G-invertible. This is the subject of Section 5.

5. Positive Splittings and Exact Sequences

The results in this section are equivariant generalizations of standard exactness arguments and owe much to the treatment of [2].

DEFINITION 5.1. If A is a pro-G-C^*-algebra then $|a|_\beta$ is defined for $a \in A$ by

$$|a|_\beta = \sup\{|a|_x : x \in \mathcal{S}(A)\}.$$

An element $a \in A$ is said to be *bounded* if $|a|_\beta$ is finite.

DEFINITION 5.2. The exact sequence $(J, A, A/J)$ of pro-G-C^*-algebras is *positively split* if there is a continuous linear graded G-map $\sigma : A/J \to A$ such that

(1) The map σ is a section for $\pi : A \to A/J$.
(2) The map σ is completely positive.
(3) The map σ is norm-decreasing on bounded elements; that is,

$$|\sigma(w)|_\beta \le |w|_\beta \quad \text{for all } w \in A/J.$$

Such a map σ is a *positive splitting* for $(J, A, A/J)$ and is denoted $\sigma : A/J \Rightarrow A$.

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PROPOSITION 5.3. *Let G be a compact group. Suppose that*

$$0 \longrightarrow J \longrightarrow A \xrightarrow{\pi} A/J \longrightarrow 0$$

is an exact sequence of pro-G-C^*-algebras, A is a separable σ-G-C^*-algebra, and $\sigma : A/J \Rightarrow A$ is a positive splitting. Then the associated sequence

$$0 \longrightarrow SJ \longrightarrow CA \xrightarrow{\psi} C\pi \longrightarrow 0$$

is positively split.

PROOF. (Sketch—see [18, p. 87-88], for details.) Since A is a separable σ-G-C^*-algebra, there is a countable approximate unit $(h_n)_{n\in\mathbb{N}}$ in J with $h_1 = 0$ which is quasi-central for A. By averaging, we may assume that each h_n is fixed by the action of G. (This is the place in the paper where compactness of G seems to be necessary. If 5.3 can be established without the compactness assumption then so can all of Section 5.) Define $h : (0, 1] \to J$ by

$$h(t) = (2^n t - 1)h_n + (2 - 2^n t)h_{n+1} \qquad \text{for} \quad 2^{-n} \le t \le 2^{1-n}.$$

Then h is norm-continuous. Write

$$C\pi = \{(\zeta, a) \in C(A/J) \oplus A : \zeta(1) = \pi(a)\}$$

and define $\hat{\sigma} : C\pi \to CA$ by setting $\hat{\sigma}(\zeta, a)(t) = 0$ for $t = 0$ and, for $t > 0$, setting

$$\hat{\sigma}(\zeta, a)(t) = (1 - h(t))^{1/2}\sigma(\zeta(t))(1 - h(t))^{1/2} + h(t)^{1/2}(1 - t)ah(t)^{1/2}.$$

Then $\hat{\sigma} : C\pi \Rightarrow CA$ is a positive splitting for ψ.

THEOREM 5.4. [2, Theorem 2.1] *Let G be a compact group. Suppose that*

$$0 \longrightarrow J \longrightarrow A \xrightarrow{\pi} A/J \longrightarrow 0$$

is an exact sequence of pro-G-C^*-algebras with a positive splitting, A is separable, and $\xi : J \to C\pi$ is the natural inclusion. Then the associated element $\xi_*(1_J) \in KK_*^G(J, C\pi)$ is KK^G-invertible.

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PROOF. Let $\alpha \in KK^G(SC_1, F)$ be the Bott inverse element (3.9). The extension

$$0 \longrightarrow SJ \longrightarrow CA \longrightarrow C\pi \longrightarrow 0.$$

is positively split, by Proposition 5.3. Such a positively split extension gives rise to a canonical element

$$\delta \in KK^G(C\pi, SJ \otimes C_1)$$

by the equivariant version of the construction of Skandalis [16, p. 196-197]. (Note that this construction makes use of the equivariant Stinespring stabilization theorem (cf. [5, p. 151]).) Define $\omega \in KK^G(C\pi, J)$ by $\omega = -\delta \widehat{\otimes}_{SC_1} \alpha$. Then

$$\xi^*(\omega) = 1_J \in KK^G(J, J) \tag{5.5}$$

by the argument of [2] 2.3, so that ω is a left inverse to $\xi_*(I_J) \in KK_*^G(J, C\pi)$.

Next consider the exact sequence

$$0 \longrightarrow C\xi \longrightarrow C(C\pi) \xrightarrow{\varphi} C(A/J) \longrightarrow 0 \tag{5.6}$$

where φ is the natural composite

$$C(C\pi) \to C\pi \to C(A/J).$$

The sequence (5.6) admits a positive splitting since each factor does. Applying the argument of (5.5) to φ (rather than to π) implies that the map

$$KK^G(B, C\xi) \longrightarrow KK^G(B, C\varphi)$$

is injective for any B. However, the mapping cone sequence

$$KK_{n-1}^G(B, C(A/J)) \longrightarrow KK_n^G(B, C\varphi) \longrightarrow KK_n^G(B, C(C\pi))$$

is exact and the functor $KK_*^G(B, -)$ is homotopy-invariant, and hence $KK_*^G(B, C\varphi) = 0$. This implies that $KK_*^G(B, C\xi) = 0$. Using the mapping cone sequence for $C\xi$, this implies in turn that the map

$$\xi_* : KK^G(B, J) \longrightarrow KK^G(B, C\pi)$$

is an isomorphism. Taking $B = C\pi$ and using the fact (5.5) that ω is a left inverse to $\xi_*(1_J)$, it is immediate that ω is an inverse to $\xi_*(1_J)$.

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THEOREM 5.7. *Let G be a compact group. Suppose that*

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

is a positively split short exact sequence of pro-G-C^*-algebras and that A is separable. Then:

(1) *If D is separable then the theory $KK_*^G(D, -)$ is exact for $(J, A, A/J)$.*
(2) *The theory $KK_*^G(-, D)$ is exact for $(J, A, A/J)$.*

PROOF. This follows from combining Theorem 4.13 and Theorem 5.4.

We say that a pro-G-C^*-algebra A is *nuclear* if each A_x is nuclear.

THEOREM 5.8. *Let G be a compact group. Suppose that*

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

is a short exact sequence of pro-G-C^*-algebras and that A is separable and nuclear. Then:

(1) *If D is separable then the theory $KK_*^G(D, -)$ is exact for $(J, A, A/J)$.*
(2) *The theory $KK_*^G(-, D)$ is exact for $(J, A, A/J)$.*

PROOF. The fact that A is nuclear implies that there is always a positive splitting, by inductive use of the Choi-Effros theorem [1]. Then invoke Theorem 5.7.

The following consequence of Theorem 5.7 is useful in applications.

THEOREM 5.9. *Let G be a compact group. Suppose that*

$$0 \longrightarrow J\widehat{\otimes}B \longrightarrow A\widehat{\otimes}B \xrightarrow{\pi\widehat{\otimes}1_B} (A/J)\widehat{\otimes}B \longrightarrow 0 \tag{5.10}$$

is a short exact sequence of pro-G-C^*-algebras and that A is a separable nuclear σ-G-C^*-algebra and B is separable. Then:

(1) *If D is separable then the theory $KK_*^G(D, -)$ is exact for $(J\widehat{\otimes}B, A\widehat{\otimes}B, (A/J)\widehat{\otimes}B)$.*
(2) *The theory $KK_*^G(-, D)$ is exact for $(J\widehat{\otimes}B, A\widehat{\otimes}B, (A/J)\widehat{\otimes}B)$.*

PROOF. In this situation π has a positive splitting $\sigma : A/J \Rightarrow A$. Then $\sigma \otimes 1_B : (A/J)\widehat{\otimes}B \Rightarrow A\widehat{\otimes}B$ is a positive splitting for $\pi\widehat{\otimes}1_B$, so Theorem 5.7 applies.

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6. Additivity: sums and products

If $(A_\lambda)_{\lambda\in\Lambda}$ is a family of pro-G-C^*-algebras, let $\mathscr{F}$ denote the collection of families $x = (x_\lambda)_{\lambda\in\Lambda}$ where x_λ is a continuous C^*-seminorm on A_λ. For $x \in \mathscr{F}$, define

(6.1) $$|(a_\lambda)|_x = \sup\{|a_\lambda|_{x_\lambda} : \lambda \in \Lambda\}.$$

DEFINITION 6.2. The *sum* $\oplus_n A_n$ of a sequence of pro-G-C^*-algebras is defined by

$$\oplus_n A_n = \{(a_n) \in \prod_{n=1}^{\infty} A_n : \lim_{n\to\infty} |a_n|_{x_n} = 0 \quad \text{for each } (x_n) \in \mathscr{F}\}.$$

The group G acts on $\oplus_n A_n$ by $g(a_n) = (ga_n)$. For each x, the seminorm (6.1) is a G-C^*-seminorm on $\oplus_n A_n$. Give $\oplus_n A_n$ the topology which is induced by these seminorms.

Note that if each A_n is a σ-G-C^*-algebra then $\oplus_n A_n$ is a pro-G-C^*-algebra, but it is almost never the inverse limit of a *sequence* of G-C^*-algebras, since $\mathscr{F}$ generally has no countable cofinal subset.

Let $\iota_k : A_k \to \oplus_n A_n$ denote the natural universal map.

PROPOSITION 6.3. FINITE ADDITIVITY *Let A, B, A_1, A_2, B_1, B_2 be pro-G-C^*-algebras. Then there are natural isomorphisms induced by $\{\iota_k\}$ of the form*

$$KK_*^G(A_1 \oplus A_2, B) \cong KK_*^G(A_1, B) \oplus KK_*^G(A_2, B)$$

and

$$KK_*^G(A, B_1 \oplus B_2) \cong KK_*^G(A, B_1) \oplus KK_*^G(A, B_2).$$

PROOF. This is immediate from elementary considerations.

PROPOSITION 6.4. COUNTABLE ADDITIVITY *Let $A_1, A_2, \ldots$ be a sequence of pro-G-C^*-algebras and let B be a pro-G-C^*-algebra. Then the natural map*

$$KK_*^G(\oplus_n A_n, B) \longrightarrow \prod_n KK_*^G(A_n, B)$$

is an isomorphism.

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PROOF. This is identical to [18, 4.2-4.3].

Countable additivity in the second variable fails in general even if the group G is trivial and only C^*-algebras are involved, though it does hold when $A = F$: cf. the discussion in [11, Remark 7.12].

DEFINITION 6.5. If $(A_\lambda)_{\lambda\in\Lambda}$ is a family of pro-G-C^*-algebras then the *product* $A = \prod_\lambda A_\lambda$ with the product topology is again a pro-G-C^*-algebra. Let $\pi^\lambda : A \to A_\lambda$ denote the canonical map.

Note that if Λ is infinite then A is almost never a C^*-algebra except in trivial cases.

DEFINITION 6.6. Given a finite set $\{\lambda_1, \dots, \lambda_m\} \subseteq \Lambda$ and $x_i \in \mathscr{S}(A_{\lambda_i})$ for $1 \le i \le m$, define a new seminorm $x = (x_1|\dots|x_m)$ on A by

$$|(a^\lambda)|_x = \max\{|a^\lambda|_{x_i} : 1 \le i \le m\}$$

Then

$$\mathscr{S}(A) = \{(x_1|\dots|x_m) : \; x_i \in \mathscr{S}(A^{\lambda_i}), \; 1 \le i \le m\}.$$

where $\{\lambda_1, \dots, \lambda_m\} \subseteq \Lambda$ ranges over all finite subsets of Λ.

DEFINITION 6.7. Suppose that E is a Hilbert G-module over $A = \prod_\lambda A_\lambda$, the product of pro-$G$-$C^*$-algebras. For each $\lambda \in \Lambda$, let

$$E_\lambda = E\widehat{\otimes}_A A_\lambda.$$

For each $x \in \mathscr{S}(A_\lambda)$, let $\hat{x} \in \mathscr{S}(A)$ be the seminorm which is obtained by the composite

$$A \xrightarrow{\pi^\lambda} A_\lambda \xrightarrow{|\,|_x} \mathbb{R}^+.$$

Then there are natural maps $E \to E_{\hat{x}}$ for each x, and

$$E_\lambda \cong \varprojlim\{E_{\hat{x}} : x \in \mathscr{S}(A_\lambda)\},$$

so there are natural maps $E \to E_\lambda$ and $\psi : E \to \prod_\lambda E_\lambda$.

PROPOSITION 6.8. (1) *The map $\psi : E \to \prod_\lambda E_\lambda$ is an isomorphism of Hilbert G-modules.*

(2) *The induced map*

$$\mathscr{L}(E) \longrightarrow \prod_\lambda \mathscr{L}(E\widehat{\otimes}_A A_\lambda) = \prod_\lambda \mathscr{L}(E_\lambda)$$

is an isomorphism.

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PROOF. The map ψ is an isometry, since $\mathcal{S}(A)$ is given by seminorms $x = (x_1|\dots|x_m)$ with $x_i \in \mathcal{S}(A_{\lambda_i})$. The map ψ is onto the dense subspace $\oplus_n E_n$ and both E and $\prod_\lambda E_\lambda$ are complete, so ψ is an isomorphism. This proves (1); part (2) follows immediately from (1).

THEOREM 6.9. COUNTABLE ADDITIVITY FOR PRODUCTS *Let A and $\{B_\lambda : \lambda \in \Lambda\}$ be pro-G-C^*-algebras. Then the natural map*

$$\pi = (\pi_*^\lambda) : KK_*^G(A, \prod_\lambda B_\lambda) \longrightarrow \prod_\lambda KK_*^G(A, B_\lambda)$$

is an isomorphism.

PROOF. Let $B = \prod_\lambda B_\lambda$. Define an inverse to π as follows. For each $\lambda \in \Lambda$, let $\eta^\lambda = (E_\lambda, T_\sigma^\lambda, \varphi^\lambda)$ be a Kasparov pro-A-B_λ-G-bimodule. Then $E = \prod_\lambda E_\lambda$ is a countably generated Hilbert G-module over B, and

$$\varphi \equiv (\varphi^\lambda) : A \longrightarrow \prod \mathcal{L}(E_\lambda) \cong \mathcal{L}(E)$$

by Proposition 6.8. An n-simplex $\sigma = (\sigma^1|\dots|\sigma^m) \in \mathcal{S}(B)$ has the form

$$\sigma^i = (x_0^i, \dots, x_n^i) \in \mathcal{S}(B_{\lambda_i}) \quad 0 \le i \le m.$$

Define

$$T_\sigma : \Delta_n \longrightarrow \mathcal{L}(E_{x_0^1} \times \dots \times E_{x_0^m}) = \mathcal{L}(E_{(x_0^1|\dots|x_0^m)})$$

by $t \longmapsto (T_{\sigma^1}(t), \dots, T_{\sigma^m}(t))$. Then $\eta = (E, T_\sigma, \varphi)$ is a Kasparov pro-A-B G-bimodule. The map

$$(\eta^\lambda)_{\lambda\in\Lambda} \longmapsto \eta$$

defines a G-map which is inverse to π.

THEOREM 6.10. *If (A_λ) is a family of pro-G-C^*-algebras and B is a G-C^*-algebra, then the natural map*

$$\pi = (\pi_\lambda^*) : \quad \oplus_\lambda KK_*^G(A_\lambda, B) \longrightarrow KK_*^G(\prod_\lambda A_\lambda, B)$$

is an isomorphism with inverse induced by the natural maps

$$A_\mu \xrightarrow{\iota_\mu} \oplus_n A_n \longrightarrow \prod_\lambda A_\lambda.$$

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PROOF. Let $A = \prod_\lambda A_\lambda$ and let $\eta = (E, T_\sigma, \varphi)$ be a Kasparov A-B G-bimodule. Since $\mathcal{L}(E)$ is a G-C^*-algebra, there are finitely many $\{\lambda_1, \dots, \lambda_n\}$ such that $\varphi : A \to \mathcal{L}(E)$ factors over $A_{\lambda_1} \oplus \dots \oplus A_{\lambda_n}$. Therefore, for each η, $\iota_\lambda^*(\eta) = 0$ for all but finitely many λ, so that $\sum_\lambda \iota_\lambda^*(\eta)$ is a finite sum. Thus $\iota^* = (\iota_\lambda^*)$ is well-defined as a map

$$\iota^* : KK_*^G(\prod_\lambda A_\lambda, B) \longrightarrow \oplus_\lambda KK_*^G(A_\lambda, B)$$

and it is easy to see that ι^* is inverse to π.

REMARK 6.11. (WEIDNER). If B is not a C^*-algebra then the result is false in general; it fails for $B = \prod_{n=1}^\infty \mathbb{C}$.

7. Direct and Inverse Limits

The presence of suitable additivity and long exactness axioms imply pleasant behavior with respect to limits, as is well-known, since (at a categorical level) direct limits are formed from direct sums and cokernels, and inverse limits are formed from products and kernels. Since the primary exactness results in Section 5 were proved under the added assumption that G was compact, the same will be true in the present section. Here is the first result.

THEOREM 7.1. *Let G be a compact group. Let*

$$\dots \longleftarrow A_n \xleftarrow{\alpha^n} A_{n+1} \longleftarrow \dots$$

be a sequence of pro-G-C^*-algebras with

$$A = \varprojlim A_n.$$

Suppose that either all maps α^n are cofibrations or that A is a separable nuclear σ-G-C^*-algebra. Then:

(1) *For all separable G-C^*-algebras D, there is a natural short exact sequence*

$$0 \to \varinjlim{}^1 KK_{a+1}^G(D, A_n) \to KK_a^G(D, A) \to \varprojlim KK_a^G(D, A_n) \to 0.$$

(2) *If each A_n is separable then for each G-C^*-algebra D, the natural map*

$$\varinjlim KK_*^G(A_n, D) \longrightarrow KK_*^G(A, D)$$

is an isomorphism.

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PROOF. This follows from the usual telescope argument (cf. [18, 91-92], [6]). Note that the assumption that each α^n is a cofibration or that A is a separable nuclear σ-G-C^*-algebra is necessary to assure exactness, and the assumption in b) that D is a G-C^*-algebra is necessary to assure additivity of $KK_*^G(-, D)$.

Theorem 7.1 represents a new phenomenon, since for C^*-algebras the question would be almost vacuous.

The remaining limit questions deal with $KK_*^G(\varinjlim A_n, B)$. Of course for C^*-algebras this leads to the familiar $\varinjlim^1$ sequence. For pro-G-C^*-algebras, one must first define $\varinjlim A_n$.

DEFINITION 7.2. (WEIDNER). Let $\{A_\lambda\}$ be a directed system of pro-G-C^*-algebras, with

$$\alpha_{\lambda\mu} : A_\lambda \to A_\mu \qquad for\, \lambda \leq \mu.$$

The direct limit A of the system is defined as follows. Let

$$\mathscr{X} = \{x = (x_\lambda) \in \prod_\lambda \mathscr{S}(A_\lambda) : \alpha^*_{\lambda\mu}(x_\mu) \leq x_\lambda \quad \text{for all} \quad \lambda \leq \mu\}.$$

Each $x \in \mathscr{X}$ determines a directed system of G-C^*-algebras

$$\{(A_\lambda)_{x_\lambda} : \lambda \in \Lambda\}.$$

Let

$$A_x = \varinjlim (A_\lambda)_{x_\lambda}$$

with C^*-norm $|\ |_x$. Then $\{A_x : x \in \mathscr{X}\}$ is an inverse system of G-C^*-algebras; let

$$A = \varprojlim \{A_\lambda\}$$

be its inverse limit. It is a pro-G-C^*-algebra. By construction there are natural morphisms $\iota_\lambda : A_\lambda \to A$.

Weidner comments that this construction corresponds to taking inverse limits of compactly generated spaces which are completely Hausdorff. Note that in general A will not be a σ-G-C^*-algebra even if Λ is countable, since $\mathscr{X}$ has no countable cofinal subset.

Phillips [8] notes that this is the categorical direct limit in the category of pro-G-C^*-algebras. He also gives examples to show how poorly behaved this construction really is. The most striking ([8, 3.9]) is an example of a countable

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direct system in which every map is injective and no algebra is zero, but for which the direct limit is zero.

Recall that a map $f : X \to Y$ of locally compact spaces is *proper* if $f^{-1}(K)$ is compact for every compact subset K of Y. Such a map induces a map on one-point compactifications. If X is compact then every map $X \to Y$ is proper. Proper maps of spaces induce proper maps of function spaces, and conversely. Weidner generalizes to operator algebras as follows.

DEFINITION 7.3. $\alpha : A \to B$ between pro-G-C^*-algebras is *proper* if for each $x \in \mathscr{S}(A)$, the map

$$a \mapsto |a|_{\alpha_*(x)} \equiv \sup\{|a|_y : \alpha^*(y) \leq x\}$$

defines a finite, continuous C^*-seminorm on B.

If B is a G-C^*-algebra or α is onto then α is proper. If α is proper then $\alpha_*(\mathscr{S}(A))$ is a cofinal subset of $\mathscr{S}(B)$. If $\{A_n\}$ is a direct sequence of σ-G-C^*-algebras and each map $A_n \to A_{n+1}$ is proper then $\varinjlim A_n$ is a σ-G-C^*-algebra.

THEOREM 7.4. *Let G be a compact group. Let*

$$A_1 \xrightarrow{\alpha_1} A_2 \longrightarrow \dots$$

be a direct sequence of separable nuclear σ-G-C^*-algebras with

$$A = \varinjlim A_n.$$

Suppose that each α_n is proper. Then for each separable pro-G-C^*-algebra D there is a natural short exact sequence

$$0 \to \varinjlim{}^1 KK^G_{a-1}(A_n, D) \to KK^G_a(A, D) \to \varprojlim KK^G_a(A_n, D) \to 0.$$

PROOF. This follows from the additivity and exactness properties of the cohomology theory $KK^G_*(-, D)$, cf. [18, 93-94], [13].

REMARK 7.5. As noted previously, the natural map

$$\iota_* : \varinjlim KK^G_*(A, B_j) \longrightarrow KK^G_*(A, \varinjlim B_j)$$

is not an isomorphism in general, even for G-C^*-algebras with G trivial. However, the class $\mathscr{D}$ of pro-G-C^*-algebras A for which ι_* *is* an isomorphism is fairly interesting. For example, $\mathscr{D}$ contains all $C(X)$, where X is a G-CW-complex or a G-ENR. The class $\mathscr{D}$ is closed under certain bootstrap operations; see [18, p. 95] and [11, (7.13)] for further information on this subject.

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We apply the results of this section to compare the one-variable functor $KK_*^G(\mathbb{C}, -)$ to the representable K-theory $RK_*^G(-)$ defined by Phillips [9] for G compact. For B a unital σ-G-C^*−algebra, let $M(\mathcal{K} \otimes B)$ be the associated multiplier algebra and let

$$Q(B) = M(\mathcal{K} \otimes B)/(\mathcal{K} \otimes B)$$

be the associated outer multiplier algebra. In this setting $\mathcal{K}$ is understood to be the algebra of compact operators on $L^2(G) \otimes \ell^2$ and has the inner action of G associated to the tensor product of the left regular representation of G on $L^2(G)$ and the trivial representation on ℓ^2. For any G-algebra A, write A^G for the set of elements fixed by the G-action and $U^G(A)$ for the unitary elements of A^G. Then $RK_0^G(B)$ is defined by

$$RK_0^G(B) = U^G(Q(B))/U_o^G(B)$$

where $U_o^G(B)$ denotes the connected component of the identity. This extends as usual to non-unital B and to $RK_*^G(B)$.

THEOREM 7.6. *Let G be a compact group. Then there is a natural isomorphism*

$$\Gamma_B : RK_*^G(B) \to KK_*^G(\mathbb{C}, B)$$

for all separable nuclear σ-G-C^*-algebras.

PROOF. It suffices to verify the Theorem for B unital. Let $\alpha \in RK_0^G(B)$ be represented by an element $\alpha \in M(\mathcal{K} \otimes B)$ whose image in $Q(B)$ is an invariant unitary element. The element α may be taken to be invariant by [9, Lemma 5.5]. Let H_B be the Hilbert G-module over B given by

$$H_B = L^2(G) \otimes \ell^2 \otimes B.$$

Identify

$$M(\mathcal{K} \otimes B) \cong \mathcal{L}(H_B),$$
$$\mathcal{K} \otimes B \cong \mathcal{K}(H_B)$$

by [3, p. 137]. Then we are left with an invariant element $\alpha \in \mathcal{L}(H_B)$ which projects to an invariant unitary $\hat{\alpha} \in Q(B)$. Define $S(\alpha) \in \mathcal{L}(H_B \oplus H_B)$ by

$$S(\alpha) = \begin{bmatrix} 0 & \alpha \\ \alpha^* & 0 \end{bmatrix}.$$

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Then $S(\alpha)$ is invariant, $S(\alpha) \in Q_{\mathbb{C}}(H_B)$, $(S(\alpha) - S(\alpha^*)) \in I_{\mathbb{C}}(H_B)$, and $(S(\alpha)^2 - 1) \in I_{\mathbb{C}}(H_B)$. Thus $(H_B \oplus H_B, 1, S(\alpha))$ defines a Kasparov pro-$\mathbb{C}$-B-G-bimodule, denoted $\Gamma_B(\alpha)$. If $\alpha \in U_o^G(B)$ then $\Gamma_B(\alpha)$ is degenerate. The map Γ_B is well-defined, by [3, Section 6, Theorem 3] and it is a homomorphism by standard arguments.

The map Γ_B extends the map of [7, pp. 402-403], hence commutes with Bott periodicity and yields a natural transformation

$$\Gamma_B : RK_*^G(B) \to KK_*^G(\mathbb{C}, B)$$

which is an isomorphism when B is a unital G-C^*-algebra by [7]. For the general case, write $B = \varprojlim B_n$ as the inverse limit of a sequence of G-C^*-algebras. Then there is a commuting diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \lim^1 RK_{a+1}^G(B_n) & \longrightarrow & RK_a^G(B) & \longrightarrow & \varprojlim RK_a^G(B_n) & \longrightarrow & 0 \\ & & {\scriptstyle \{\Gamma_{B_n}\}} \downarrow & & {\scriptstyle \Gamma_B} \downarrow & & {\scriptstyle \{\Gamma_{B_n}\}} \downarrow & & \\ 0 & \longrightarrow & \lim^1 KK_{a+1}^G(\mathbb{C}, B_n) & \longrightarrow & KK_a^G(\mathbb{C}, B) & \longrightarrow & \varprojlim KK_a^G(\mathbb{C}, B_n) & \longrightarrow & 0. \end{array}$$

The upper row is exact, by [9, p. 473]. The lower row is exact by Theorem 7.1(1). Each Γ_{B_n} is an isomorphism, so Γ_B is an isomorphism by the Five Lemma.

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