SOLUTION OF THE WORD PROBLEM FOR CERTAIN TYPES OF GROUPS II

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The purpose of this paper is to prove a theorem which concerns the normal subgroup of a free product Π generated by a given subset Ω. This theorem was stated in the first paper of this series (Britton [**1**]) and an application was made to the word problem. The present work is, however, independent.

Let the group Π be the free product of the set of subgroups $\{G_\tau ;\ \tau \epsilon \Gamma\}$. These subgroups will be called the *constituent* groups of Π. Let Ω be a subset of Π which meets none of the constituent groups, *i.e.*, such that $\Omega \cap G_\tau$ is empty for each $\tau \epsilon \Gamma$. The theorem gives information about the elements of the normal subgroup generated by Ω when Ω satisfies conditions which restrict cancellations between certain conjugates of the elements of Ω and their inverses.

Notation

We denote elements of Π by capital letters, the identity being denoted by I, and write $X \,.\, Y$ for the product of the elements X and Y. It is convenient, however, to denote elements which are known to have unit length by small letters ; we write

$$x \sim y \quad \text{or} \quad x \sim' y$$

according to whether x and y belong to the same constituent group or not.

If $X = X_1 \,.\, X_2 \,.\, \ldots \,.\, X_m$, where the length of X equals the sum of the lengths of the factors X_i, we omit the dots and write $X = X_1X_2 \ldots X_m$. Every element Y of Π except I has a unique representation $Y = y_1y_2 \ldots y_n$ and we write $l(Y) = n$, $\text{In}\,(Y) = y_1$ and $\text{Fin}\,(Y) = y_n$. Thus $l(Y)$ denotes the length of Y. Finally, if X, Y are elements of Π different from I, we define $\beta(X, Y)$ and $\varepsilon(X, Y)$ as the numbers of cancellations and amalgamations respectively in the product $X \,.\, Y$, and write $\alpha(X, Y) = \beta(X, Y) + \varepsilon(X, Y)$. Thus

$$l(X \,.\, Y) = l(X) + l(Y) - 2\beta(X, Y) - \varepsilon(X, Y)$$

and $\varepsilon(X, Y)$ is either 0 or 1.

We may assume that every element W of Ω satisfies the conditions

$$l(W) \geqslant 2, \quad \text{In}\,(W) \sim' \text{Fin}\,(W).$$

If an element U satisfying these conditions has normal form $U = a_1a_2 \ldots a_n$, then by the *cyclic arrangements of* U we understand the n elements

$$a_\lambda a_{\lambda+1} \ldots a_n a_1 a_2 \ldots a_{\lambda-1} \quad (\lambda = 1, 2, \ldots, n).$$

Let Ω^* consist of the cyclic arrangements of all elements of Ω and their inverses. If $U \epsilon \Omega^*$, we define the integer $\alpha(U)$ by

$$\alpha(U) = \text{Max}\, \alpha(U'^{\kappa}, V),$$

where U' is a cyclic arrangement of U, κ is ± 1 and V is an element of Ω^* such that

$$U'^{\kappa} \,.\, V \neq I.$$

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The conditions we shall impose on Ω are :

(1) If $U, V \epsilon \Omega^*$ and $U . V \neq I$, then $6\alpha(U, V) < \text{Min}\,(l(U), l(V))$.

(2) If $U \epsilon \Omega^*$, then $\alpha(U) \neq 0$.

THEOREM. *Let Π be a free product of groups and let Ω be a subset of Π in which every element W satisfies the conditions $l(W) \geqslant 2$, $\text{In}\,(W) \sim' \text{Fin}\,(W)$. Further, assume that* (1) *and* (2) *hold.*

Then, if U_0 is any element, different from the identity, of the normal subgroup of Π generated by Ω,

(i) *U_0 has length at least l_0, where $l_0 = \underset{W \epsilon \Omega}{\text{Min}}\, l(W)$,*

(ii) *if U_0 has length exactly l_0, then $U_0 \epsilon \Omega^*$,*

(iii) *the normal form of U_0 can be written in the form XKZ, where K is such that an element V of Ω^* exists with normal form $K'K$, say, and*

$$l(K) \geqslant l(V) - 3\alpha(V) - 1,$$

and equality implies that $\text{Fin}\,(K') \sim \text{Fin}\,(X)$ *and* $\text{In}\,(K') \sim \text{In}\,(Z)$.

COROLLARY. *The element $X . K'^{-1} . Z$ belongs to the normal subgroup and has length strictly less than $l(U_0)$.*

The corollary is easily proved. For $l(V) > 6\alpha(V)$ and hence either $l(K) > \frac{1}{2}l(V)$ or $l(K) \geqslant \frac{1}{2}(l(V) - 1)$, $\text{Fin}\,(K') \sim \text{Fin}\,(X)$ and $\text{In}\,(K') \sim \text{In}\,(Z)$.

Note. To prepare the way for a later paper, in which different conditions will be imposed on Ω, the proof of the theorem has been arranged so that most of it remains valid when the inequality in (1) is replaced by

(1′) $$4\alpha(U, V) + 1 < \text{Min}\,(l(U), l(V)).$$

In fact there is only one point where it is necessary to use (1) instead of (1′). (This is at the end of § 4.)

1. Three basic lemmas. The three lemmas proved in this section are the main tools used in the proof of the theorem. We require some preliminary definitions.

If $Y \epsilon \Pi$ and $Y \neq I$, we define the subsets $\mathscr{L}(Y)$ and $\mathscr{R}(Y)$ of Π as follows. Let $Y = y_1 y_2 \dots y_n$; then $\mathscr{L}(Y)$ consists of the $n-1$ elements

$$y_1 y_2 \dots y_i \quad (i = 1, 2, \dots, n-1)$$

and $\mathscr{R}(Y)$ consists of the $n-1$ elements

$$y_j y_{j+1} \dots y_n \quad (j = 2, 3, \dots, n).$$

If $n = 1$, then both subsets are empty. If Z is also an element of Π different from I, we write

$$\mathscr{L}(Y) \frown \mathscr{L}(Z) = \mathscr{L}(Y, Z).$$

If $B \epsilon \mathscr{L}(Y, Z)$ and if, further, $\text{In}\,(B^{-1} . Y) \sim \text{In}\,(B^{-1} . Z)$, we write

$$B \,\hat{\epsilon}\, \mathscr{L}(Y, Z).$$

$\mathscr{R}(Y, Z)$ is defined similarly, and if $C \epsilon \mathscr{R}(Y, Z)$ and $\text{Fin}\,(Y . C^{-1}) \sim \text{Fin}\,(Z . C^{-1})$, we write $C \,\hat{\epsilon}\, \mathscr{R}(Y, Z)$.

If a number of small letters (usually two or three) representing components are enclosed by round brackets, we mean that the components all belong to the same constituent group and their product is not the identity. We give this convention priority over the "dot

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convention", so that dots can be omitted inside the brackets. Thus $a(bc)de$ denotes an element of length 4 and $a'b'$ $(c'd'e')$ an element of length 3.

Definition 1.1. *A* ***chain*** *is a finite sequence of at least two elements of Π each of which has length at least two.*

The *normal form* of a chain $\tilde{C} = \langle F_1, F_2, \ldots, F_n\rangle$ means the normal form of $F_1 . F_2 . \ldots . F_n$ It is denoted by C.

A *subchain* of the chain $\tilde{C}$ means a chain of the form $\langle F_p, F_{p+1}, \ldots, F_q\rangle$, where $1 \leqslant p < q \leqslant n$.

The *sum* of two subchains,

$$\tilde{C}_1 = \langle F_p, F_{p+1}, \ldots, F_q\rangle \quad \text{and} \quad \tilde{C}_2 = \langle F_r, F_{r+1}, \ldots, F_s\rangle$$

is only defined when $q = r$ and in this case

$$\tilde{C}_1 + \tilde{C}_2 = \langle F_p, F_{p+1}, \ldots, F_s\rangle.$$

Definition 1.2. *The chain $\tilde{S} = \langle F_i, F_{i+1}, \ldots, F_j\rangle$ is* ***simple*** *if its normal form S involves only components of F_i and F_j, in the following way. If $F_i = a_1 a_2 \ldots a_u$ and $F_j = b_1 b_2 \ldots b_v$, then*

either (i) $S = a_1 \ldots a_p (a_{p+1} b_q) b_{q+1} \ldots b_v$

or (ii) $S = a_1 \ldots a_p b_{q+1} \ldots b_v$,

where, in each case, $1 \leqslant p < u$ and $1 \leqslant q < v$, and in (ii) *we have*

$$a_{p+1} \sim' b_{q+1} \text{ and } a_p \sim' b_q. \qquad (1.21)$$

The conditions (1.21) are satisfied if $a_{p+1} \sim b_q$. If a simple chain satisfies this stronger condition, it is called *naturally simple.*

It is easy to see that the normal form of a simple chain has a unique decomposition of the kind occurring in Definition 1.2. We shall write

$$F_i^l = a_1 \ldots a_p \quad \text{and} \quad F_j^r = b_{q+1} \ldots b_v,$$

so that

$$S = F_i^l c^\varepsilon F_j^r,$$

where $\varepsilon = 1$ in case (i) and $\varepsilon = 0$ in case (ii). Clearly $F_i^l \in \mathscr{L}(F_i, S)$. Also, $F_i^l \hat{\in} \mathscr{L}(F_i, S)$ if and only if $\varepsilon = 1$. Similar results hold for F_j^r.

The first basic lemma deals with chains in any free product and may have other applications.

Lemma 1A. *Let the chain $\tilde{S}$ be the sum of n subchains :*

$$\tilde{S} = \tilde{S}_1 + \tilde{S}_2 + \ldots + \tilde{S}_n,$$

where $n \geqslant 2$ and

$$\tilde{S}_\nu = \langle F_{k(\nu-1)}, F_{k(\nu-1)+1}, \ldots, F_{k(\nu)}\rangle \quad (\nu = 1, 2, \ldots, n).$$

For each ν, let there be a factorization

$$S_\nu = A_{k(\nu-1)} B_\nu C_{k(\nu)}$$

satisfying the following conditions.

(i) $A_{k(\nu-1)} \in \mathscr{L}(F_{k(\nu-1)})$, $C_{k(\nu)} \in \mathscr{R}(F_{k(\nu)})$.

(ii) *If $B_\nu = I$, then $\tilde{S}_\nu$ is simple.*

(iii) $l(C_{k(\nu)}) + l(A_{k(\nu)}) \geqslant l(F_{k(\nu)}) \quad (\nu = 1, 2, \ldots, n-1)$,

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and equality implies that either

$$C_{k(\nu)} \hat{\epsilon} \mathscr{R}(F_{k(\nu)}, S_\nu) \quad or \quad A_{k(\nu)} \hat{\epsilon} \mathscr{L}(F_{k(\nu)}, S_{\nu+1}).$$

Then $\tilde{S}$ *has normal form*

$$S = A_{k(0)} B_1 J_1 B_2 J_2 \ldots B_{n-1} J_{n-1} B_n C_{k(n)},$$

where $J_\nu = C_{k(\nu)} \cdot F_{k(\nu)}^{-1} \cdot A_{k(\nu)}$. *Further, if* $H_\nu = B_\nu J_\nu B_{\nu+1}$, *then* $H_\nu \neq I$, $\text{In}(H_\nu) \sim \text{In}(B_\nu C_{k(\nu)})$ *and* $\text{Fin}(H_\nu) \sim \text{Fin}(A_{k(\nu)} B_{\nu+1})$.

Proof. 1°. For $1 \leqslant \nu \leqslant n-1$, we have

$$S_\nu = A_{k(\nu-1)} B_\nu C_{k(\nu)} \quad \text{and} \quad S_{\nu+1} = A_{k(\nu)} B_{\nu+1} C_{k(\nu+1)}.$$

Thus the product of the terms in the chain $\tilde{S}_\nu + \tilde{S}_{\nu+1}$ equals

$$A_{k(\nu-1)} B_\nu \cdot J_\nu \cdot B_{\nu+1} C_{k(\nu+1)}. \quad \ldots\ldots (1.22)$$

If there is strict inequality in (iii), then $J_\nu \neq I$, $\text{In}(J_\nu) = \text{In}(C_{k(\nu)})$ and $\text{Fin}(J_\nu) = \text{Fin}(A_{k(\nu)})$; so the dots can be removed from (1.22). If there is equality in (iii), then $F_{k(\nu)} = A_{k(\nu)} C_{k(\nu)}$ and $J_\nu = I$. Suppose first that $C_{k(\nu)} \hat{\epsilon} \mathscr{R}(F_{k(\nu)}, S_\nu)$. Then

$$\text{Fin}(A_{k(\nu-1)} B_\nu) = \text{Fin}(S_\nu \cdot C_{k(\nu)}^{-1}) \sim \text{Fin}(F_{k(\nu)}) \cdot C_{k(\nu)}^{-1}) = \text{Fin}(A_{k(\nu)}) \sim' \text{In}(B_{\nu+1} C_{k(\nu+1)}).$$

Now suppose that $A_{k(\nu)} \hat{\epsilon} \mathscr{L}(F_{k(\nu)}, S_{\nu+1})$. Then

$$\text{Fin}(A_{k(\nu-1)} B_\nu) \sim' \text{In}(C_{k(\nu)}) = \text{In}(A_{k(\nu)}^{-1} \cdot F_{k(\nu)}) \sim \text{In}(A_{k(\nu)}^{-1} \cdot S_{\nu+1}) = \text{In}(B_{\nu+1} C_{k(\nu+1)}).$$

In either case, therefore, the dots can be removed from (1.22).

2°. It will now be shown that

$$C_{k(\nu+1)} \hat{\epsilon} \mathscr{R}(S_{\nu+1}, P_\nu), \quad \ldots\ldots (1.23)$$

where P_ν denotes the expression (1.22) with the dots removed. This is trivial if $B_{\nu+1} \neq I$. If $B_{\nu+1} = I$ and $J_\nu \neq I$, then, by 1°, $\text{Fin}(J_\nu) = \text{Fin}(A_{k(\nu)})$ and (1.23) follows. Finally, if $B_{\nu+1} = J_\nu = I$, then $F_{k(\nu)} = A_{k(\nu)} C_{k(\nu)}$ and one of the alternatives of (iii) holds. If

$$C_{k(\nu)} \hat{\epsilon} \mathscr{R}(F_{k(\nu)}, S_\nu),$$

then $\text{Fin}(A_{k(\nu)}) \sim \text{Fin}(A_{k(\nu-1)} B_\nu)$, which is just the required result. The other alternative cannot hold, since $\tilde{S}_{\nu+1}$ is simple when $B_{\nu+1} = I$.

3°. It now follows that $H_\nu \neq I$. For $H_\nu = I$ implies that $B_\nu = J_\nu = B_{\nu+1} = I$, which, by 2°, implies that $\text{Fin}(A_{k(\nu)}) \sim \text{Fin}(A_{k(\nu+1)})$ and therefore that $C_{k(\nu)} \hat{\epsilon} \mathscr{R}(F_{k(\nu)}, S_\nu)$. This contradicts the fact that $\tilde{S}_\nu$ is simple when $B_\nu = I$.

4°. The results of 2° and 3° combine to give $\text{Fin}(H_\nu) \sim \text{Fin}(A_{k(\nu)} B_{\nu+1})$. The other result, namely $\text{In}(H_\nu) \sim \text{In}(B_\nu C_{k(\nu)})$, follows by symmetry.

5°. Finally, it will be proved by induction on m that if $2 \leqslant m \leqslant n$, then $\tilde{S}_1 + \tilde{S}_2 + \ldots + \tilde{S}_m$ has normal form $A_{k(0)} B_1 J_1 \ldots B_{m-1} J_{m-1} B_m C_{k(m)}$. By taking $\nu = 1$ in 1°, we obtain the result for $m = 2$. Assume the result true for m. Then $\tilde{S}_1 + \tilde{S}_2 + \ldots + \tilde{S}_{m+1}$ has normal form

$$E B_m C_{k(m)} \cdot F_{k(m)}^{-1} \cdot A_{k(m)} B_{m+1} C_{k(m+1)},$$

i.e.,

$$E B_m \cdot J_m \cdot B_{m+1} C_{k(m+1)}, \quad \ldots\ldots (1.24)$$

where $E = A_{k(0)} B_1 \ldots J_{m-1}$. But $\tilde{S}_m + \tilde{S}_{m+1}$ has normal form $A_{k(m-1)} B_m J_m B_{m+1} C_{k(m+1)}$, so that the dots in (1.24) can be removed if $\text{Fin}(E B_m) = \text{Fin}(A_{k(m-1)} B_m)$, that is, if

$$\text{Fin}(B_{m-1} J_{m-1} B_m) \sim \text{Fin}(A_{k(m-1)} B_m),$$

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which is so, by 4°.

This completes the proof of the lemma.

Considering now the situation of the theorem, let U_0 be an arbitrary but henceforth fixed element of the normal subgroup and let $U_0 \neq I$.

Thus

$$U_0 = F_1 . F_2 . \dots . F_h, \quad \dots\dots(1.25)$$

where $h \geqslant 1$ and each F_i is a conjugate either of an element of Ω or of the inverse of an element of Ω, so that F_i has normal form

$$e_1 e_2 \dots e_{\delta_i} x_1 x_2 \dots x_{\sigma} {}_i e_{\sigma_i}^{-1} \dots e_2^{-1} e_1^{-1}, \quad \dots\dots(1.26)$$

where either (i) $x_1 x_2 \dots x_{\sigma_i} \in \Omega^*$ or (ii) $(x_{\sigma_i} x_1) x_2 \dots x_{\sigma_i - 1} \in \Omega^*$. In both cases we denote the element of Ω^* by W_i. Let λ_i and θ_i be defined by

$$\lambda_i = l(W_i) \quad \text{and} \quad \theta_i = \sigma_i - \lambda_i,$$

so that $\theta_1 = 0$ in case (i) and $\theta_i = 1$ in case (ii).

Definition 1.3. *The* ***kernel*** *of F_i is the sequence of components*

$$x_{1+\theta_i}, x_{2+\theta_i}, \dots, x_{\lambda_i}.$$

Evidently these are components of W_i.

The representation (1.25) is not of course unique. From the many possible representations, we select a particular one as follows :

Take those representations (1.25) *for which h is minimal, and from them select one for which* $\sum_{i=1}^{h} l(F_i)$ *is minimal.*

Henceforth we shall assume that (1.25) is this fixed minimal representation of U_0.

Defining α_i and α_{ij} by $\alpha_i = \alpha(W_i)$ and $\alpha_{ij} = \text{Min}(\alpha_i, \alpha_j)$ $(i, j = 1, 2, \dots, h)$, we have $\alpha_i \geqslant 1$ and $\alpha_{ij} \geqslant 1$.

From (1.26), we have

$$l(F_i) = 2\delta_i + \sigma_i \quad (i = 1, 2, \dots, h).$$

δ_i may of course be zero.

In what follows, the letters i, j, k, l, m, when used as suffixes, will denote integers in the range 1 to h.

The second basic lemma gives an upper bound for the number of cancellations and amalgamations between a product $B = F_i . F_{i+1} . \dots . F_{j-1}$ and a neighbouring single factor F_j, when $l(B) < l(F_j)$.

Lemma 1B. *In the representation* (1.25), *let* $h \geqslant 2$ *and let* $B = F_i . F_{i+1} . \dots . F_{j-1}$, *where* $1 \leqslant i \leqslant j-1 < h$. *Let* $l(B) < l(F_j)$. *Denote the normal forms of B and* F_j *by* $b_1 b_2 \dots b_n$ *and* $c_1 c_2 \dots c_{\delta_j} z_1 z_2 \dots z_{\sigma_j} c_{\delta_j}^{-1} \dots c_2^{-1} c_1^{-1}$, *respectively,*† *and denote by* α, β *and* ε *the integers* $\alpha(B, F_j)$, $\beta(B, F_j)$ *and* $\varepsilon(B, F_j)$, *respectively. Then we have the following results :*

(i) *If* $\beta \leqslant \delta_j$, *then* $\beta \leqslant \frac{1}{2}n$ *and* $\alpha < 1 + \frac{1}{2}n$.

(ii) *If* $\beta > \delta_j$, *then* $\beta < \delta_j + \sigma_j$, $\beta \leqslant n - \delta_j$ *and* $\alpha \leqslant n - \delta_j - \phi_j(\varepsilon)$, *where* $\phi_j(\varepsilon) = \theta_j(1 - \varepsilon)$.

Note. $\phi_j(\varepsilon)$ retains this meaning throughout the paper.

Proof. 1°. We prove first that

† $B \neq I$, for if B were equal to I, there would be a representation of U_0 with less than h factors.

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$$l(B\,.\,F_j\,.\,B^{-1}) \geqslant l(F_j). \quad \text{.....................................(1.31)}$$

Write $\bar{F} = B.\,F_j\,.\,B^{-1}$. Then $\bar{F}\,.\,B = B\,.\,F_j$ and

$$U_0 = (F_1\,.\,\ldots\,.\,F_{i-1})\,.\,\bar{F}\,.\,(F_i\,.\,\ldots\,.\,F_{j-1})\,.\,(F_{j+1}\,.\,\ldots\,.\,F_h).$$

This is a representation of U_0 because $\bar{F}$ is a conjugate of F_j and hence a conjugate either of an element of Ω or of the inverse of an element of Ω. Since the number of factors in this representation is h, the sum of the lengths of the factors is not less than $\sum_{\nu=1}^{h} l(F_\nu)$ and so $l(\bar{F}) \geqslant l(F_j)$.

2°. Write $\delta = \delta_j$ and $\sigma = \sigma_j$. We prove (1). Now

$$B\,.\,F_j = b_1 \ldots b_{n-\beta}\,.\,c_{\beta+1} \ldots c_\delta z_1 \ldots z_\sigma c_\delta^{-1} \ldots c_1^{-1}$$

and $$B\,.\,F_j\,.\,B^{-1} = b_1 \ldots b_{n-\beta}\,.\,c_{\beta+1} \ldots c_\delta z_1 \ldots z_\sigma c_\delta^{-1} \ldots c_{\beta+1}^{-1}\,.\,b_{n-\beta}^{-1} \ldots b_1^{-1}.$$

Hence $l(B\,.\,F_j\,.\,B^{-1}) \leqslant 2(n-\beta) + 2(\delta-\beta) + \sigma - \varepsilon$, where ε cannot be replaced by 2ε because we allow $\beta = \delta$. Thus $l(B\,.\,F_j\,.\,B^{-1}) - l(F_j) \leqslant 2(n-\beta) - 2\beta - \varepsilon$. By (1.31), we obtain

$$\beta \leqslant \tfrac{1}{2}n - \tfrac{1}{4}\varepsilon.$$

Thus $\beta \leqslant \frac{1}{2}n$. It remains to prove that $\alpha < 1 + \frac{1}{2}n$. This is trivial if $\beta < \frac{1}{2}n$. So assume that $\beta = \frac{1}{2}n$. This gives $\frac{1}{4}\varepsilon \leqslant 0$, *i.e.*, $\varepsilon = 0$, so that $\alpha = \beta < 1 + \frac{1}{2}n$.

3°. Finally, we prove (ii). Suppose that $\beta \geqslant \delta + \sigma$. Then $B\,.\,F_j = b_1 \ldots b_u\,.\,c_\delta^{-1} \ldots c_1^{-1}$, where $u = n - \delta - \sigma$, and $B\,.\,F_j\,.\,B^{-1} = b_1 \ldots b_u\,.\,b_{n-\delta}^{-1} \ldots b_1^{-1}$. Therefore

$$l(B\,.\,F_j\,.\,B^{-1}) - l(F_j) \leqslant u + n - \delta - (2\delta + \sigma) = 2(n - 2\delta - \sigma) < 0,$$

in contradiction to (1.31).

Therefore $\delta < \beta < \delta + \sigma$ and

$$B\,.\,F_j = b_1 \ldots b_{n-\beta}\,.\,z_{\nu+1} \ldots z_\sigma c_\delta^{-1} \ldots c_1^{-1}, \quad \text{.........................(1.32)}$$

where $\nu = \beta - \delta$.

Also $b_{n-\beta+1} \ldots b_n\,.\,c_1 \ldots c_\delta z_1 \ldots z_\nu = I$, so that

$$B\,.\,F_j\,.\,B^{-1} = b_1 \ldots b_{n-\beta}\,.\,z_{\nu+1} \ldots z_\sigma\,.\,z_1 \ldots z_\nu\,.\,b_{n-\beta}^{-1} \ldots b_1^{-1}. \quad \text{...............(1.33)}$$

The length p of the last expression is not greater than $2(n-\beta) + \sigma$. By (1.31), $2\delta + \sigma \leqslant p$. Hence $\beta \leqslant n - \delta$. It remains to prove that $\alpha \leqslant n - \delta - \phi_j(\varepsilon)$ or, equivalently, that

$$\beta \leqslant n - \delta - (\theta_j - \theta_j \varepsilon + \varepsilon). \quad \text{....................................(1.34)}$$

The bracketed expression is either 0 or 1; so (1.34) certainly holds if $\beta \leqslant n - \delta - 1$. But when $\beta = n - \delta$, we have $p = 2(n-\beta) + \sigma$ or, in other words, the dots in (1.33) can be removed. Therefore the dot in (1.32) can be removed (giving $\varepsilon = 0$) and also $z_\sigma \sim' z_1$ (giving $\theta_j = 0$). (1.34) follows, and this completes the proof of the lemma.

Note 1.4. By one of the assumptions of the theorem, we have $\lambda_i > 6\alpha_i$ $(i = 1, 2, \ldots, h)$. However, most of the arguments in the proof of the theorem require only that

$$\lambda_i \geqslant 4\alpha_i + 1 \quad (i = 1, 2, \ldots, h). \quad \text{.................................(1.41)}$$

This inequality will be used freely, but whenever a stronger inequality is required, the fact will be mentioned explicitly. (See, *e.g.*, Lemma 2.2 (vi).)

The third basic lemma does not use the full minimal hypothesis for the representation (1.25) but only the hypothesis that h is minimal.

LEMMA 1C. *In the representation* (1.25), *suppose that* $h \geqslant 2$ *and let*

$$B = F_i . F_{i+1} . \dots . F_{j-1},$$

where $1 \leqslant i \leqslant j-1 < h$. *Let the normal forms of* F_i *and* F_j *be*

$$d_1 \dots d_s x_1 \dots x_{\sigma_i} d_s^{-1} \dots d_1^{-1} \quad \text{and} \quad e_1 \dots e_p z_1 \dots z_{\sigma_j} e_p^{-1} \dots e_1^{-1},$$

respectively, so that $s = \delta_i$ *and* $p = \delta_j$.

If there exists an integer μ *such that*

$$d_1 \dots d_s x_1 \dots x_\mu \in \mathscr{L}(B) \quad \text{and} \quad 1 + \alpha_{ij} \leqslant \mu \leqslant \lambda_i,$$

let $B = d_1 \dots d_s x_1 \dots x_\mu c_0 c_1 \dots c_\tau$ $(\tau \geqslant 0)$. *Then we have the following results :*

(i) *If* $p + 1 + \theta_j \leqslant \tau + 2 \leqslant p + \sigma_j - \alpha_{ij}$, *then* $\alpha \leqslant \tau + 1 + \alpha_{ij}$.

(ii) *If* $\tau + 2 < p + 1 + \theta_j \leqslant \tau + \mu + 1 - \alpha_{ij}$, *then* $\alpha \leqslant p + \theta_j + \alpha_{ij} - \phi_j(\varepsilon)$.

(Here, as in Lemma 1B, α, β and ε denote $\alpha(B, F_i)$, $\beta(B, F_i)$ and $\varepsilon(B, F_i)$, respectively.)

Proof. 1°. To prove (i) we observe first that the result is trivial if $\beta < \tau + 1$. So we assume that $\beta \geqslant \tau + 1$. Then

$$B . F_j = d_1 \dots d_s x_1 \dots x_\mu . z_q \dots z_{\sigma_j} e_p^{-1} \dots e_1^{-1} \quad \text{.....(1.42)}$$

and

$$c_0 c_1 \dots c_\tau . e_1 \dots e_p z_1 \dots z_{q-1} = I, \quad \text{.....(1.43)}$$

where $q = \tau + 2 - p$. Thus

$$1 + \theta_j \leqslant q \leqslant \sigma_j - \alpha_{ij}. \quad \text{.....(1.44)}$$

Now define X' and Z' by

$$X' = x_{\mu+1} \dots x_{\sigma_i} . x_1 \dots x_\mu \quad \text{and} \quad Z' = z_q \dots z_{\sigma_j} . z_1 \dots z_{q-1}. \quad \text{.....(1.45)}$$

X' is obviously in Ω^* when $\theta_i = 0$. But the same is true when $\theta_i = 1$, because then there is at least one component on each side of the dot (since $\mu \leqslant \lambda_i = \sigma_i - 1 < \sigma_i$). Similarly $Z' \in \Omega^*$, by (1.44).

We shall prove that $X' . Z' \neq I$ by showing that the supposition that $X' . Z' = I$ implies that the products

$$D = F_{i+1} . F_{i+2} . \dots . F_{j-1} \quad \text{and} \quad E = F_i . F_{i+1} . \dots . F_j$$

are equal. (This is inconsistent with the minimal hypothesis.)

Suppose that $X' . Z' = I$. Since

$$D = F_i^{-1} . B = d_1 \dots d_s x_{\sigma_i}^{-1} \dots x_{\mu+1}^{-1} . c_0 c_1 \dots c_\tau,$$

we have, by (1.43),

$$D = d_1 \dots d_s x_{\sigma_i}^{-1} \dots x_{\mu+1}^{-1} . [e_1 \dots e_p z_1 \dots z_{q-1}]^{-1}.$$

Consequently

$$\begin{aligned} E . D^{-1} &= (B . F_j) . D^{-1} \\ &= d_1 \dots d_s x_1 \dots x_\mu . z_q \dots z_{\sigma_j} e_p^{-1} \dots e_1^{-1} . e_1 \dots e_p z_1 \dots z_{q-1} . x_{\mu+1} \dots x_{\sigma_i} d_s^{-1} \dots d_1^{-1} \\ &= d_1 \dots d_s x_1 \dots x_\mu . Z' . x_{\mu+1} \dots x_{\sigma_i} d_s^{-1} \dots d_1^{-1}. \end{aligned}$$

The last expression is a conjugate of $X' . Z'$. Hence $E . D^{-1} = I$.

This completes the proof that $X' . Z' \neq I$.

Since $X', Z' \in \Omega^*$ and $X' . Z' \neq I$, we have $\alpha(X', Z') \leqslant \alpha_{ij}$. Let

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$$X' = X''x_2 \dots x_\mu \quad \text{and} \quad Z' = z_q \dots z_{\sigma_j - 1}Z''.$$

Then $$l(x_2 \dots x_\mu) \geqslant \alpha_{ij} \quad \text{and} \quad l(z_q \dots z_{\sigma_j-1}) \geqslant \alpha_{ij}.$$

Moreover Fin $(X'') \sim x_1$ and In $(Z'') \sim z_{\sigma_j}$. So, by (1.42) and (1.45), the number of cancellations and amalgamations which can occur in (1.42) is not greater than α_{ij}. Therefore

$$\alpha \leqslant \tau + 1 + \alpha_{ij}.$$

2°. The proof of (ii) is similar, although an additional argument is necessary when $\phi_j(\varepsilon) = 1$. We may suppose that $\beta \geqslant p + \theta_j$. Then

$$B \,.\, F_j = d_1 \dots d_s x_1 \dots x_\nu \,.\, z_{1+\theta_j} \dots z_{\sigma_j} e_p^{-1} \dots e_1^{-1} \quad \text{.......................(1.46)}$$

and $$x_{\nu+1} \dots x_\mu c_0 c_1 \dots c_\tau \,.\, e_1 \dots e_p z_1 \dots z_{\theta_j} = I, \quad \text{..........................(1.47)}$$

where $\nu = \mu + \tau + 1 - p - \theta_j$, so that $1 + \alpha_{ij} \leqslant \nu \leqslant \mu - 1$. Define X' and Z' by

$$X' = x_{\nu+1} \dots x_{\sigma_i} \,.\, x_1 \dots x_\nu \quad \text{and} \quad Z' = z_{1+\theta_j} \dots z_{\sigma_j} \,.\, z_1 \dots z_{\theta_j}.$$

These are elements of Ω^*. If $X' \,.\, Z' = I$, then, by (1.47),

$$E = B \,.\, F_j = d_1 \dots d_s x_1 \dots x_\nu \,.\, z_{1+\theta_j} \dots z_{\sigma_j} \,.\, [z_1 \dots z_{\theta_j} \,.\, x_{\nu+1} \dots x_\mu c_0 c_1 \dots c_\tau]$$

and $$D^{-1} = B^{-1} \,.\, F_i = c_\tau^{-1} \dots c_1^{-1} c_0^{-1} \,.\, x_{\mu+1} \dots x_{\sigma_i} d_s^{-1} \dots d_1^{-1}.$$

Hence $$E \,.\, D^{-1} = d_1 \dots d_s x_1 \dots x_\nu \,.\, Z' \,.\, x_{\nu+1} \dots x_{\sigma_i} d_s^{-1} \dots d_1^{-1},$$

giving the same contradiction $D = E$ as before. It is straightforward to deduce that

$$\alpha \leqslant p + \theta_j + \alpha_{ij}.$$

3°. It remains to be proved that if $\phi_j(\varepsilon) = 1$, then $\alpha \leqslant p + \theta_j + \alpha_{ij} - 1$. Suppose that this is false, *i.e.*, suppose that $\alpha = p + \theta_j + \alpha_{ij}$. Now $\phi_j(\varepsilon) = 1$ implies that $\theta_j = 1$ and $\varepsilon = 0$. Therefore $\beta = p + \theta_j + \alpha_{ij}$; so, from (1.46),

$$x_{\nu+1-t} \dots x_\nu \,.\, z_2 \dots z_{1+t} = I \quad (t = \alpha_{ij}),$$

and, from (1.47), $x_{\nu+1} \,.\, z_1 = I$.

Now consider the product $Z_1 \,.\, X_1$, where

$$Z_1 = z_{2+t} \dots z_{\sigma_j - 1}(z_{\sigma_j} z_1) z_2 \dots z_{1+t}$$

and $$X_1 = x_{\nu+1-t} \dots x_{\nu+1} \dots x_{\sigma_i} \,.\, x_1 \dots x_{\nu-t}.$$

It is easily verified that Z_1 and X_1 are cyclic arrangements of W_j and W_i, respectively. But $\alpha(Z_1, X_1) = \alpha_{ij} + 1$, so that $Z_1 \,.\, X_1 = I$. This implies that $(z_{\sigma_j} z_1) \,.\, x_{\nu+1} = I$. We have already seen that $x_{\nu+1} \,.\, z_1 = I$, so that $z_{\sigma_j} = I$.

This contradiction completes the proof of the lemma.

2. Some special chains. Until § 6 we shall neglect the trivial case in which $h = 1$ in the representation (1.25). It is clear that $\langle F_1, F_2, \dots, F_h \rangle$ is a chain; we denote it by $\mathfrak{U}_0$, and in what follows the word " chain " will mean a subchain of $\mathfrak{U}_0$.

DEFINITION 2.1. *A chain $\langle F_i, F_{i+1}, \dots, F_j \rangle$ is* ***left-closed*** *if the following conditions are satisfied.*

(i) *The chain is naturally simple, so that its normal form is, say, $F_i^l c^\varepsilon F_j^r$.*

(ii) $l(F_i^l) \geqslant \delta_i + \lambda_i - \alpha_i + \phi_i(\varepsilon)$.

(iii) $l(F_j^r) \geqslant \delta_j + \mathfrak{c}_j(\varepsilon)$, *where*

$$\mathfrak{c}_j(\varepsilon) = [\tfrac{1}{2}(\sigma_j - \varepsilon + 1)] - \alpha_j, \quad \text{.................................(2.11)}$$

square brackets denoting the integral part of the number concerned.

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A *right-closed* chain is defined similarly. It will be shown later that a chain with two terms is in general either left-closed or right-closed.

It is easily verified that

$$\mathfrak{c}_j(0) \geqslant \mathfrak{c}_j(1) \geqslant \mathfrak{c}_j(0) - 1$$

and

$$\mathfrak{c}_j(\varepsilon) \geqslant \tfrac{1}{2}(\lambda_j - 1) - \alpha_j,$$

so that

$$\mathfrak{c}_j(\varepsilon) \geqslant \alpha_j.$$

For the next definition we require two more integers $\mathfrak{b}_k(\varepsilon)$ and $\mathfrak{b}_k^*(\varepsilon)$, defined, like $\mathfrak{c}_k(\varepsilon)$, when the suffix is in the range 1, 2, ..., h and $\varepsilon = 0$ or 1. The special property of $\mathfrak{b}_k(\varepsilon)$ is that

$$\mathfrak{b}_k(\varepsilon_1) + \mathfrak{c}_k(\varepsilon_2) \geqslant \sigma_k, \qquad (2.12)$$

where equality implies that $\varepsilon_1 = 1$ or $\varepsilon_2 = 1$. We define $\mathfrak{b}_k(\varepsilon)$ by

$$\mathfrak{b}_k(\varepsilon) = [\tfrac{1}{2}(\sigma_k - \varepsilon)] + \alpha_k + 1. \qquad (2.13)$$

To prove (2.12), we observe that the left-hand side is an integer and is not less than

$$\sigma_k + \tfrac{1}{2} - \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2),$$

which is not less than $\sigma_k - \frac{1}{2}$ and, in the case in which $\varepsilon_1 = \varepsilon_2 = 0$, is not less than $\sigma_k + \frac{1}{2}$.

We define $\mathfrak{b}_k^*(\varepsilon)$ by

$$\mathfrak{b}_k^*(\varepsilon) = \text{Max}\,(\lambda_k - 2\alpha_k + \phi_k(\varepsilon),\ \mathfrak{b}_k(\varepsilon)). \qquad (2.14)$$

Lemma 2.2.

(i) *If* $\lambda_k > 6\alpha_k$, *then* $\mathfrak{b}_k^*(\varepsilon) = \lambda_k - 2\alpha_k + \phi_k(\varepsilon)$.
If $\lambda_k \leqslant 6\alpha_k$, *then* $\mathfrak{b}_k^*(\varepsilon) = \mathfrak{b}_k(\varepsilon)$.

(ii) $\mathfrak{b}_k^*(0) \geqslant \mathfrak{b}_k^*(1) \geqslant \mathfrak{b}_k^*(0) - 1$.

(iii) $\mathfrak{b}_k^*(\varepsilon) \geqslant \mathfrak{b}_k(\varepsilon) \geqslant \frac{1}{2}\lambda_k + \alpha_k$.

(iv) $\mathfrak{b}_k^*(\varepsilon) \geqslant \lambda_k - 3\alpha_k + \theta_k - 1$.

(v) $\mathfrak{b}_k^*(\varepsilon_1) + (\lambda_k - \alpha_k + \phi_k(\varepsilon_2)) - \sigma_k \geqslant \lambda_k - 3\alpha_k - 1$,
where equality implies that $\theta_k = \varepsilon_1 = \varepsilon_2 = 1$.

(vi) $\lambda_k - \alpha_k \geqslant \mathfrak{b}_k^*(0)$. [$\lambda_k > 4\alpha_k + 1$ *is required here.*]

(vii) $\lambda_k - \alpha_k + \phi_k(\varepsilon) \geqslant \mathfrak{b}_k^*(\varepsilon)$.

Proof. We only prove (iv) and (v); the other results are trivially verified. We note that (v) can be written in the form $\mathfrak{b}_k^*(\varepsilon_1) \geqslant \lambda_k - 2\alpha_k - 1 + \theta_k\varepsilon_2$, so that (v) implies (iv). To prove (v), we assume first that $\lambda_k > 6\alpha_k$. By (i), we have to prove that $\phi(\varepsilon_1) \geqslant -1 + \theta_k\varepsilon_2$. This is trivially proved, and equality implies that $\theta_k = \varepsilon_1 = \varepsilon_2 = 1$. Now suppose that $\lambda_k \leqslant 6\alpha_k$; then $\mathfrak{b}_k^*(\varepsilon_1) = \mathfrak{b}_k(\varepsilon_1)$ and it is easy to verify that (v) holds (with strict inequality).

Definition 2.3. *A chain* $\tilde{S} = \langle F_i, F_{i+1}, \ldots, F_j \rangle$ *has* **double barriers** *if its normal form has a factorization*

$$S = A_i B C_j$$

satisfying the following conditions.

(i) $A_i \in \mathscr{L}(F_i)$ *and* $C_j \in \mathscr{R}(F_j)$.

(ii) *If* $B = I$, *then the chain is simple (but not necessarily naturally simple).*

(iii) *Either* $l(A_i) = \delta_i + \mathfrak{b}_i^*(0)$, *or* $l(A_i) = \delta_i + \mathfrak{b}_i^*(1)$ *and* $A_i \mathrel{\hat{\in}} \mathscr{L}(F_i, S)$.

(iv) *Either* $l(C_j) = \delta_j + \mathfrak{b}_j^*(0)$, *or* $l(C_j) = \delta_j + \mathfrak{b}_j^*(1)$ *and* $C_j \mathrel{\hat{\in}} \mathscr{R}(F_j, S)$.

Lemma 2.4. *The sum of two chains with double barriers has itself double barriers.*

Proof. Let the two chains be

$$\langle F_k, \ldots, F_l\rangle \quad \text{and} \quad \langle F_l, \ldots, F_m\rangle,$$

with normal forms

$$A_k B C_l \quad \text{and} \quad A_l B' C_m,$$

respectively. Then $l(C_l) + l(A_l) > l(F_l)$, because

$$l(C_l) \geqslant \delta_l + \mathfrak{b}_l^*(1) \geqslant \delta_l + \tfrac{1}{2}\lambda_l + \alpha_l$$

(by Lemma 2.2 (iii)) and similarly for $l(A_l)$. By Lemma 1A, the sum $\langle F_k, \ldots, F_m\rangle$ has normal form $A_k BJB' C_m = A_k H C_m$, where $H \neq I$, In $(H) \sim$ In (BC_l) and Fin $(H) \sim$ Fin $(A_l B')$. This shows that the sum has double barriers.

Complementary to the chains with double barriers are the *open chains*.

DEFINITION 2.5. *A chain $\langle F_i, F_{i+1}, \ldots, F_j\rangle$ satisfying the condition $\alpha(F_t, F_{t+1}) \neq 0$ $(t = i, i+1, \ldots, j-1)$ and such that no subchain has double barriers, is called an* **open chain.**

(We can express the first condition by saying that there is at least an amalgamation between each adjacent pair of terms.)

LEMMA 2.6. *Any chain $\tilde{S}$ in which at least an amalgamation occurs between each adjacent pair of terms has a decomposition into a sum of subchains,*

$$\tilde{S} = \tilde{S}_1 + \tilde{S}_2 + \ldots + \tilde{S}_p \quad (p \geqslant 1)$$

in which the subchains are alternately open chains and chains with double barriers.

Proof. If $\tilde{S}$ has only two terms, it is, by Definition 2.5, either an open chain or a chain with double barriers. Thus we have a basis for induction on the number of terms. Assuming the result true for n terms, let

$$\tilde{S} = \langle F_k, F_{k+1}, \ldots, F_{k+n}\rangle,$$

so that $\tilde{S}$ has $n+1$ terms. If $\tilde{S}$ is either an open chain or a chain with double barriers, there is nothing to prove. If not, there is a *proper* subchain $\tilde{B} = \langle F_l, F_{l+1}, \ldots, F_m\rangle$ with double barriers. If $k < l$, there is a decomposition

$$\langle F_k, F_{k+1}, \ldots, F_l\rangle = \tilde{S}_1 + \tilde{S}_2 + \ldots + \tilde{S}_q,$$

and if $m < k+n$, there is a decomposition

$$\langle F_m, F_{m+1}, \ldots, F_{k+n}\rangle = \tilde{S}_1' + \tilde{S}_2' + \ldots + \tilde{S}_r'.$$

Hence

$$\tilde{S} = \tilde{S}_1 + \tilde{S}_2 + \ldots + \tilde{S}_q + \tilde{B} + \tilde{S}_1' + \tilde{S}_2' + \ldots + \tilde{S}_r',$$

where now one, but not both, of q and r may be zero. This is not yet necessarily a decomposition of the required kind, because $\tilde{S}_q$ or $\tilde{S}_1'$ may have double barriers, but such a decomposition clearly exists, by Lemma 2.4. This completes the proof.

3. Open chains with two terms. In this section it will be proved that an open chain $\langle F_i, F_{i+1}\rangle$ with two terms is either left-closed or right-closed. It is convenient to write $j = i+1$, $s = \delta_i$ and $p = \delta_j$. There is no loss of generality in assuming that

$$s + \theta_i \leqslant p + \theta_j\,; \quad \text{(3.01)}$$

and we do so. We denote the normal forms of F_i and F_j by

$$d_1 \ldots d_s x_1 \ldots x_{\sigma_i} d_s^{-1} \ldots d_1^{-1} \quad \text{and} \quad e_1 \ldots e_p z_1 \ldots z_{\sigma_j} e_p^{-1} \ldots e_1^{-1},$$

respectively, and we write $\alpha = \alpha(F_i, F_j)$, with similar definitions for β and ε.

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LEMMA 3.1. *For an open chain with two terms,*

$$\alpha \leqslant p + \theta_j + \alpha_{ij} - \phi_j(\varepsilon) \quad \text{......(3.11)}$$

and

$$\alpha \leqslant s + \tfrac{1}{2}(\sigma_i + \theta_j + \alpha_{ij} - \phi_j(\varepsilon). \quad \text{......(3.12)}$$

Proof. 1°. From Lemma 1C with $\mu = \lambda_i$ and $\tau = \theta_i + s - 1$ it follows that

(i) if $p + 1 + \theta_j \leqslant \theta_i + s + 1 \leqslant p + \sigma_j - \alpha_{ij}$, then $\alpha \leqslant \theta_i + s + \alpha_{ij}$;
and (ii) if $\theta_i + s + 1 < p + 1 + \theta_j \leqslant \theta_i + s + \lambda_i - \alpha_{ij}$, then $\alpha \leqslant p + \theta_j + \alpha_{ij} - \phi_j(\varepsilon)$.

2°. We prove (3.11). Equality cannot hold in (3.01). For, assuming equality, (i) is satisfied, giving $\alpha \leqslant \theta_i + s + \alpha_{ij}$ and hence also $\alpha \leqslant p + \theta_j + \alpha_{ij}$. It is easily verified from (vi) of Lemma (2.2) that the chain has double barriers in this case, contrary to the hypothesis that the chain is open.

Therefore (ii) and hence (3.11) hold, unless $p + 1 + \theta_j > \theta_i + s + \lambda_i - \alpha_{ij}$. But the last inequality implies that $l(F_j) > l(F_i)$, so that Lemma 1B is available. Moreover, this inequality combined with $\alpha \leqslant l(F_i) - \delta_j - \phi_j(\varepsilon)$ implies that (3.11) holds. So we may assume that $\beta \leqslant \delta_j$, *i.e.*, that $\beta \leqslant p$. But then $\alpha \leqslant p + 1$, and (3.11) follows trivially.

3°. Finally, we prove (3.12). The case in which $\alpha < 1 + \frac{1}{2} l(F_i)$ is trivial. So we assume that $\alpha \geqslant 1 + \frac{1}{2} l(F_i)$. Then, by (3.11), we have $l(F_j) > l(F_i)$, so that Lemma 1B is applicable; we find that $\beta > p$ and $\alpha \leqslant (2s + \sigma_i) - p - \phi_j(\varepsilon)$. The last inequality combined with (3.11) leads at once to (3.12).

This completes the proof of the lemma.

LEMMA 3.2. *An open chain of two terms is either left-closed or right-closed.*

Proof. Using the notation of the previous lemma, we shall show that the chain $\langle F_i, F_j \rangle$ is right-closed when (3.01) is satisfied.

The chain will be naturally simple if $\alpha < l(F_i)$ and $\alpha < l(F_j)$. These inequalities are simple consequences of (3.12) and (3.11), respectively. Moreover, it is easily verified that

$$l(F_j) - \alpha \geqslant \delta_j + \lambda_j - \alpha_j + \phi_j(\varepsilon)$$

and

$$l(F_i) - \alpha \geqslant \delta_i + c_i(\varepsilon).$$

So the chain is right-closed.

4. The sum of a left-closed chain and a right-closed chain. This section is devoted to proving the following lemma.†

LEMMA 4.1. *An open chain $\tilde{C}$ which is the sum of a left-closed chain $\tilde{C}_1$ and a right-closed chain $\tilde{C}_2$ (so that $\tilde{C} = \tilde{C}_1 + \tilde{C}_2$) is either left-closed or right-closed.*

Notation. Let $\tilde{C}_1 = \langle F_k, F_{k+1}, \ldots, F_m \rangle$ and $\tilde{C}_2 = \langle F_m, F_{m+1}, \ldots, F_l \rangle$. Then

$$\tilde{C} = \langle F_k, F_{k+1}, \ldots, F_l \rangle.$$

Define D by

$$D = F_{k+1} \cdot F_{k+2} \cdot \ldots \cdot F_{l-1}.$$

Then $D \neq I$. In the notation introduced for simple chains, let

$$C_1 = F_k^l c_1^{\varepsilon_1} F_m^r, \qquad C_2 = F_m^l c_2^{\varepsilon_2} F_l^r,$$

where

$$\begin{cases} F_k = a_1 a_2 \ldots a_u, & F_l = b_1 b_2 \ldots b_v, \\ F_k^l = a_1 a_2 \ldots a_p, & F_l^r = b_{q+1} b_{q+2} \ldots b_v, \\ F_m = f_1 f_2 \ldots f_w, & \\ F_m^r = f_\xi f_{\xi+1} \ldots f_w, & F_m^l = f_1 f_2 \ldots f_\eta. \end{cases}$$

† It may help the reader if he postpones the proof and passes now to § 5.

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LEMMA 4.2. *$l(F_m^r)+l(F_m^l)-l(F_m)$ is either -1 or 0. In the first case,*

$$\eta = \xi-2,\ \varepsilon_1 = \varepsilon_2 = 1 \quad and \quad c_1^{\varepsilon_1}\,.\,f_{\xi-1}^{-1}\,.\,c_2^{\varepsilon_2} = I. \qquad \text{(4.21)}$$

In the second case

$$\eta = \xi-1 \quad and \quad \varepsilon_1 = \varepsilon_2 = 0. \qquad \text{(4.22)}$$

Proof. We shall assume that neither (4.21) nor (4.22) holds and prove that $C=F_k^l A F_l^r$, where $l(A)\geqslant 1$. This is a contradiction to the assumption that $\tilde{C}$ is open, because

$$l(F_k^l)\geqslant \delta_k+\lambda_k-\alpha_k+\phi_k(\varepsilon)\geqslant \delta_k+\mathfrak{b}_k^*(0)$$

and similarly $l(F_l^r)\geqslant\delta_l+\mathfrak{b}_l^*(0)$, by Lemma 2.2 (vi), so that $\tilde{C}$ has double barriers.

Let $X=F_k^l c_1^{\varepsilon_1}$ and $Z=c_2^{\varepsilon_2}F_l^r$; then

$$C = XF_m^r\,.\,F_m^{-1}\,.\,F_m^l Z = X\,.\,f_{\xi-1}^{-1}\dots f_1^{-1}\,.\,f_1\dots f_\eta\,.\,Z.$$

If $\eta<\xi-1$, then $C=X\,.\,f_{\xi-1}^{-1}\dots f_{\eta+1}^{-1}\,.\,Z$. Since $\tilde{C}_1$ is simple, $f_{\xi-1}\neq \text{Fin}\,(X)$ and $f_{\xi-1}\sim\text{Fin}\,(X)$ if and only if $\varepsilon_1=1$. Similarly, $f_{\eta+1}\sim\text{In}\,(Z)$ if and only if $\varepsilon_2=1$. Thus C is of the required form unless (4.21) holds. On the other hand, if $\eta\geqslant\xi-1$, we can apply Lemma 1A to the sum $\tilde{C}_1+\tilde{C}_2$ and hence C has the required form.

LEMMA 4.3. *$F_k\,.\,D$ has normal form $F_k^l E$, where $l(E)\geqslant 1$ and* In $(E)\neq a_{p+1}$.

Proof. $C=F_k^l\,.\,F_l^r$, because if (4.21) holds, then $C=X\,.\,f_{\xi-1}^{-1}\,.\,Z$ and if (4.22) holds, then $C=X\,.\,Z$. Thus

$$F_k\,.\,D = C\,.\,F_l^{-1} = a_1a_2\dots a_p\,.\,b_q^{-1}\dots b_1^{-1}.$$

The dot in the last expression can be removed, and $a_{p+1}\neq b_q^{-1}$. For in the first case of Lemma 4.2 we have $c_1=a_{p+1}\,.\,f_{\xi-1}$, $c_2=f_{\eta+1}\,.\,b_q$, $c_1\,.\,f_{\xi-1}^{-1}\,.\,c_2=I$ and $\xi-1=\eta+1$, so that

$$b_q^{-1} = c_2^{-1}\,.\,f_{\eta+1} = c_1 = a_{p+1}\,.\,f_{\xi-1}\sim a_{p+1}\sim' a_p.$$

In the second case we use the fact that $\tilde{C}_1$ and $\tilde{C}_2$ are naturally simple chains, so that $a_{p+1}\sim f_{\xi-1}$ and $f_{\eta+1}\sim b_q$. But $\eta=\xi-1$; hence $a_{p+1}\sim' b_q$, so that $a_{p+1}\neq b_q^{-1}$. Finally, since $\varepsilon_1=0$, we have that $a_p\sim' f_\xi=f_{\eta+1}\sim b_q\sim b_q^{-1}$.

This proves the lemma.

COROLLARY 4.31. $\qquad l(C)<l(F_k^l)+l(F_l^r).$

Proof. By the proof of the lemma, $C=F_k^l\,.\,F_l^r$; so it is required to prove that

$$\alpha(F_k^l, F_l^r)>0.$$

Suppose that this is not the case. Then it is easily seen that $l(F_k^l)\geqslant\delta_k+\mathfrak{b}_k^*(0)$ and

$$l(F_l^r)\geqslant\delta_l+\mathfrak{b}_l^*(0).$$

We show that $\tilde{C}$ is simple. Now $C=a_1a_2\dots a_pb_{q+1}b_{q+2}\dots b_v$, so we wish to prove that $a_{p+1}\sim' b_{q+1}$ and $a_p\sim' b_q$. The proof of the lemma shows that $a_p\sim' b_q$. Moreover in the first case, $b_q^{-1}\sim a_{p+1}$, so that $a_{p+1}\sim' b_{q+1}$. In the second case, since $\varepsilon_2=0$, we have

$$b_{q+1}\sim' f_\eta = f_{\xi-1}\sim a_{p+1}.$$

Thus we have proved that $\tilde{C}$ has double barriers. This contradiction completes the proof.

COROLLARY 4.32. *Let $\alpha'=\alpha(F_k, D)$ and $\alpha''=\alpha(D, F_l)$. Then*

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(i) $\alpha' = l(F_k) - l(F_k^l)$ *and* $\alpha'' = l(F_l) - l(F_l^r)$,

(ii) $l(D) = \alpha' + \alpha'' - \varepsilon_1$,

(iii) $l(F_k . D) = l(F_k^l) + \alpha''$,

(iv) $\varepsilon(F_k, D) = \varepsilon_1$.

Proof. Since $F_k . D = a_1 \dots a_p b_q^{-1} \dots b_1^{-1}$, we have $D = a_u^{-} \dots a_{p+1}^{-1} . b_q^{-1} \dots b_1^{-1}$, where, by the proof of the lemma, the dot can be removed in the second case but in the first case there is a single amalgamation because $a_{p+1}^{-1} . b_q^{-1} = f_{t-1}$.

Thus $l(D) = (u - p) + q - \varepsilon_1$ and $\varepsilon(F_k, D) = \varepsilon_1$. By the lemma, $l(F_k) - \alpha' = l(F_k^l)$ and so, by symmetry, $l(F_l) - \alpha'' = l(F_l^r)$. Thus $\alpha' = u - p$, $\alpha'' = q$ and (ii) follows. Finally,

$$l(F_k . D) = p + q = l(F_k^l) + \alpha''.$$

Corollary 4.33. *In the notation of the lemma,*

$$\alpha(F_k . D, F_l) > l(E).$$

Proof. Using an obvious notation, we have

$$l(C) = l(F_k . D) + l(F_l) - 2\alpha + \varepsilon = l(F_k^l) + l(F_l^r) + 2\alpha'' - 2\alpha + \varepsilon.$$

By Corollary 4.31, we obtain $2\alpha'' - 2\alpha + \varepsilon < 0$. Thus $\alpha'' < \alpha$. By Corollary 4.32 (iii) and the lemma, we have $\alpha'' = l(E)$, so that $\alpha > l(E)$ as required.

In the remainder of § 4, we shall be concerned mainly with cancellations between $F_k . D$ and F_l. We use a notation conforming to that of Lemma 1C and write

$$F_k = d_1 \dots d_s x_1 \dots x_{\sigma_k} d_s^{-1} \dots d_1^{-1}, \quad F_l = e_1 \dots e_p z_1 \dots z_{\sigma_l} e_p^{-1} \dots e_1^{-1},$$

so that $s = \delta_k$ and $p = \delta_l$.

By Lemma 4.3, the normal form of $F_k . D$ starts with F_k^l. Now $\tilde{C}_1$ is left-closed, so that

$$l(F_k^l) \geqslant \delta_k + \lambda_k - \alpha_k + \phi_k(\varepsilon_1). \quad \dots\dots(4.34)$$

Thus the normal form of F_k^l starts with $d_1 \dots d_s x_1 \dots x_\mu$, where $\mu = \text{Min}\,(l(F_k^l) - s, \lambda_k)$. Clearly

$$\lambda_k - \alpha_k + \phi_k(\varepsilon_1) \leqslant \mu \leqslant \lambda_k. \quad \dots\dots(4.35)$$

Since $\tilde{C}_2$ is a simple chain, $l(F_l^r) < l(F_l)$ and therefore, by Corollary 4.32 (i) and (iii), we may write

$$F_k . D = d_1 \dots d_s x_1 \dots x_\mu c_0 c_1 \dots c_\tau \quad (\tau \geqslant 0).$$

Bounds for τ are given in the next lemma.

Lemma 4.4. $\theta_k - 1 \leqslant \tau - s \leqslant \sigma_k + \alpha_l - \mu - 1.$

Proof. By Corollary 4.32, $l(F_k . D) = l(F_k) - \alpha' + \alpha''$, *i.e.*,

$$\tau - s = \alpha'' - \alpha' + \sigma_k - \mu - 1.$$

We first prove that $\theta_k - 1 \leqslant \tau - s$, *i.e.*, that $\mu - \lambda_k \leqslant \alpha'' - \alpha'$. This is obvious if $\alpha' \leqslant \alpha''$; so assume that $\alpha'' < \alpha'$. Then $l(D) < l(F_k)$; for, by Corollary 4.32,

$$l(D) - l(F_k) < l(F_k) - 2l(F_k^l) - \varepsilon_1,$$

and the right-hand side of this inequality is non-positive, by (4.34). Lemma 1B is therefore applicable, and if $\beta > s$, we obtain $\alpha' \leqslant l(D) - s - \phi_k(\varepsilon_1)$. But the other case, $\beta \leqslant s$, cannot hold. For if $\beta \leqslant s$, then $\beta \leqslant \tfrac{1}{2} l(D)$ and $\alpha' < 1 + \tfrac{1}{2} l(D)$; hence

$$2\alpha' \leqslant 1 + l(D) = 1 + \alpha' + \alpha'' - \varepsilon_1 \leqslant 2\alpha' - \varepsilon_1,$$

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which implies that $\varepsilon_1 = 0$, $2\alpha' = 1 + l(D)$, *i.e.*, that $2\beta = 1 + l(D)$, which is a contradiction.

The result just proved can be written in the form $s + \varepsilon_1 + \phi_k(\varepsilon_1) \leqslant \alpha''$, and the required result will follow if

$$\alpha' + \mu - \lambda_k \leqslant s + \varepsilon_1 + \phi_k(\varepsilon_1).$$

This is so because

$$\alpha' + \mu - \lambda_k = l(F_k) - l(F_k^l) + \mu - \lambda_k \leqslant (2s + \sigma_k) - (\mu + s) + \mu - \lambda_k = s + \theta_k \leqslant s + \varepsilon_1 + \phi_k(\varepsilon_1).$$

The other part of the lemma is easily proved. For $\mu \geqslant \lambda_k - \alpha_k$ and hence $-\alpha_k \leqslant \alpha'' - \alpha'$. By symmetry, $-\alpha_l \leqslant \alpha' - \alpha''$, which is equivalent to the required inequality.

Henceforth we shall assume that $l(F_k) \leqslant l(F_l)$, so that

$$2s + \sigma_k \leqslant 2p + \sigma_l. \qquad \text{.......(4.41)}$$

This involves no loss of generality, but the symmetry of our assumptions is now destroyed.

We wish to study cancellations between $F_k . D$ and F_l, that is, between

$$d_1 \ldots d_s x_1 \ldots x_\mu c_0 c_1 \ldots c_\tau \quad \text{and} \quad e_1 \ldots e_p z_1 \ldots z_{\sigma_l} e_p^{-1} \ldots e_1^{-1}.$$

Let β and ε be the numbers of cancellations and amalgamations, respectively, and let $\alpha = \beta + \varepsilon$, so that $\alpha = \alpha(F_k . D, F_l)$.

Lemma 1C is available because $1 + \alpha_{kl} \leqslant \mu \leqslant \lambda_k$; so it is natural to consider the four cases

$$\begin{cases} \tau - p < \alpha_{kl} + \theta_l - \mu, & \text{.......(A)} \\ \alpha_{kl} + \theta_l - \mu \leqslant \tau - p < \theta_l - 1, & \text{.......(B)} \\ \theta_l - 1 \leqslant \tau - p \leqslant \sigma_l - \alpha_{kl} - 2, & \text{.......(C)} \\ \sigma_l - \alpha_{kl} - 2 < \tau - p. & \text{.......(D)} \end{cases}$$

In case (B), $\alpha \leqslant p + \theta_l + \alpha_{kl} - \phi_l(\varepsilon)$, and in case (C), $\alpha \leqslant \tau + 1 + \alpha_{kl}$.

In fact, case (D) cannot occur. For if it did, then from the equation

$$(p - \tau) + (s - p) + (\tau - s) = 0$$

in conjunction with (D), (4.41) and Lemma 4.4, it would follow that

$$\tfrac{1}{2}(\sigma_l - \sigma_k) + \mu - \alpha_{kl} - \alpha_l \leqslant 0.$$

But $\lambda_k - \alpha_k \leqslant \mu$ and $\alpha_{kl} \leqslant \frac{1}{2}(\alpha_k + \alpha_l)$, so that a contradiction would arise.

It will be shown eventually that in cases (A) and (B), the chain $\bar{C}$ is right-closed. First we have the following lemma.

Lemma 4.5. *In cases* (A) *and* (B), α *has the following bounds :*

$$\alpha \leqslant p + \theta_l + \alpha_{kl} - \phi_l(\varepsilon), \qquad \text{.......(4.51)}$$
$$\alpha \leqslant \tau + 1 + \tfrac{1}{2}(\mu + \alpha_{kl} + \theta_l - \theta_k - \phi_l(\varepsilon)). \qquad \text{.......(4.52)}$$

Proof. First suppose that $l(F_l) \leqslant l(F_k . D)$, that is, that $2p + \sigma_l \leqslant s + \mu + \tau + 1$. Then, by Lemma 4.4,

$$p - \tau \leqslant \tfrac{1}{2}(2 - \theta_k + \mu - \sigma_l). \qquad \text{.......(4.53)}$$

It is not difficult to show that case (A) cannot hold. Therefore case (B) holds, and (4.51) follows immediately. (4.52) follows from (4.51) and (4.53).

Now suppose that $l(F_k . D) < l(F_l)$. By Lemma 1B,

if $\beta \leqslant p$, then $\alpha < 1 + \frac{1}{2}(s + \mu + \tau + 1)$

and if $\beta > p$, then $\alpha \leqslant (s + \mu + \tau + 1) - p - \phi_l(\varepsilon)$. ....(4.54)

If $\beta \leqslant p$, then, since $\theta_k - 1 \leqslant \tau - s$, we have

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$$2\alpha \leqslant 3 + 2\tau + \mu - \theta_k. \qquad \text{.................................(4.55)}$$

(4.52) now follows because $1 \leqslant \alpha_{kl} + \theta_l - \phi_l(\varepsilon)$. (4.51) holds in case (B), as we have seen, and in case (A) it is a consequence of (4.55).

If $\beta > p$, then case (A) cannot hold. For

$$p + 1 \leqslant \alpha \leqslant (s + \mu + \tau + 1) - p - \phi_l(\varepsilon) \leqslant ((\tau + 1 - \theta_k) + \mu + \tau + 1) - p - \phi_l(\varepsilon)$$

and on using (A) we obtain an inequality independent of p and τ, which contradicts the assumption that $\lambda_k \geqslant 4\alpha_k + 1$.

Thus (B) holds, giving (4.51) at once. Combining (4.51) with (4.54), we obtain

$$2\alpha \leqslant (s + \mu + \tau + 1) - 2\phi_l(\varepsilon) + \theta_l + \alpha_{kl}.$$

Now $s \leqslant \tau + 1 - \theta_k$ and $-\phi_l(\varepsilon) \leqslant 0$, so that (4.52) follows.

This proves the lemma.

Corollary 4.56. *In cases* (A) *and* (B), $\tilde{C}$ *is a right-closed chain.*

Proof. It follows from (4.51) and (4.52), respectively, that

$$l(F_l) - \alpha \geqslant p + \lambda_l - \alpha_l + \phi_l(\varepsilon)$$

and

$$l(F_k \,.\, D) - \alpha \geqslant s + c_k(\varepsilon).$$

In view of Lemma 4.3, $\tilde{C}$ will be naturally simple if $\alpha > l(E) = l(F_k \,.\, D) - l(F_k^l)$, $\alpha < l(F_k \,.\, D)$ and $\alpha < l(F_l)$. But the last two of these inequalities are trivial consequences of the inequalities just proved, while the first was proved in Corollary 4.33.

Therefore $\tilde{C}$ is right-closed.

Lemma 4.6. *In case* (C), *if* $\mu = \lambda_k$ *and either* $\phi_k(\varepsilon) = 0$ *or* $\alpha < \tau + 1 + \alpha_k$, *then* $\tilde{C}$ *is left-closed.*

Proof. It is sufficient to prove the two inequalities

$$l(F_k \,.\, D) - \alpha \geqslant s + \lambda_k - \alpha_k + \phi_k(\varepsilon),$$
$$l(F_l) - \alpha \geqslant p + c_l(\varepsilon).$$

The first inequality is easily proved, since it can be written in the form

$$\alpha \leqslant (\mu - \lambda_k) + (\tau + 1 + \alpha_k) - \phi_k(\varepsilon).$$

The second inequality can be written in the form

$$(p - s) + (s - \tau) + (\tau - \alpha) + (\sigma_l - c_l(\varepsilon)) \geqslant 0. \qquad \text{.......................(4.61)}$$

Now, in case (C), $\alpha \leqslant \tau + 1 + \alpha_{kl}$; so, using (4.41), Lemma 4.4 and the definition of $c_l(\varepsilon)$, we find that the left-hand side of (4.61) is not less than

$$(\tfrac{1}{2}\sigma_k - \tfrac{1}{2}\sigma_l) + (\mu + 1 - \sigma_k - \alpha_l) + (-1 - \alpha_{kl}) + (\tfrac{1}{2}\sigma_l - \tfrac{1}{2} + \alpha_l),$$

which is equal to $\tfrac{1}{2}\sigma_k - \theta_k - \alpha_{kl} - \tfrac{1}{2}$; so (4.61) follows. This proves the lemma.

To complete the proof of Lemma 4.1, we have to show that $\tilde{C}$ is left-closed or right-closed in case (C) when the conditions of Lemma 4.6 do not apply. Thus we now assume that

$$\left.\begin{array}{l} \text{(i) } \theta_l - 1 \leqslant \tau - p \leqslant \sigma_l - \alpha_{kl} - 2, \\ \text{and (ii) } \textit{Either } \mu < \lambda_k \\ \qquad \textit{or } \mu = \lambda_k,\ \phi_k(\varepsilon) = 1 \text{ and } \alpha = \tau + 1 + \alpha_k. \end{array}\right\} \qquad \text{......................(C}'\text{)}$$

We note that, since (C′) implies (C), $\alpha \leqslant \tau + 1 + \alpha_{kl}$, and further that, if $\mu = \lambda_k$, then $\theta_k = 1$, so that $\mu + 1 \leqslant \sigma_k$ in any case.

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LEMMA 4.7. *Under assumption* (C′), *if* $\alpha = \tau + 1 + \alpha_{kl}$, $p \leqslant \tau$ *and* $\varepsilon = 0$, *then* $\varepsilon_1 = \varepsilon_2 = 0$ *and* F_k^l *has length* $s + \mu$.

Proof. Since $\mu + 1 \leqslant \sigma_k$, the normal form of F_k contains the component $x_{\mu+1}$. We shall prove that this component does not belong to the same constituent group as the component c_0 in the normal form of $F_k \cdot D$. This will imply that $\varepsilon(F_k, D) = 0$ and $l(F_k^l) = s + \mu$, and hence, by Lemma 4.2, that $\varepsilon_1 = \varepsilon_2 = 0$.

Suppose then that $c_0 \sim x_{\mu+1}$; we shall obtain a contradiction. Since $\varepsilon = 0$, we have $\beta = \tau + 1 + \alpha_{kl}$, so that

$$x_i \ldots x_\mu c_0 c_1 \ldots c_\tau \cdot e_1 \ldots e_p z_1 \ldots z_j = I,$$

where $i = \mu + 1 - \alpha_{kl}$ and $j = \tau + 1 + \alpha_{kl} - p$.

Define W_l' and W_k' by

$$W_l' = z_{j+1} \ldots z_{\sigma l} \cdot z_1 \ldots z_j,$$
$$W_k' = x_i \ldots x_\mu \ldots x_{\sigma_k} \cdot x_1 \ldots x_{i-1}.$$

These are cyclic arrangements of W_l and W_k, respectively, because in both expressions there is at least one component on each side of the dot. (*E.g.*, $j + 1 \leqslant \sigma_l$ follows from (i) of (C′).) Also $\alpha(W_l', W_k') \geqslant \alpha_{kl} + 1$. For $c_0 \cdot z_f = I$, where $f = \tau + 1 - p$, so that if a is the component next to the right of x_μ in the normal form of W_k' and b is the component next to the left of z_{f+1} in W_l', we have $a \sim x_{\mu+1}$ and $b \sim z_f$. But $x_{\mu+1} \sim c_0 \sim z_f$; hence $a \sim b$, and the result follows, since $x_i \ldots x_\mu$ has length α_{kl}.

Therefore $W_l' \cdot W_k' = I$. But In $(W_l') \sim z_{j+1}$ and Fin $(W_k') \sim x_{i-1}$, so that $z_{j+1} \sim x_{i-1}$, in contradiction to the assumption that $\varepsilon = 0$.

This proves the lemma.

LEMMA 4.8. *Under assumption* (C′), F_k^l *has length* $s + \mu$. *Thus* $F_k^l = d_1 \ldots d_s x_1 \ldots x_\mu$.

Proof. If $\mu < \lambda_k$, the result follows by the definition of μ. Now let $\mu = \lambda_k$, so that $\phi_k(\varepsilon) = 1$ and $\alpha = \tau + 1 + \alpha_k$. Since $\alpha \leqslant \tau + 1 + \alpha_{kl}$, we have $\alpha_{kl} = \alpha_k$; so if $p \leqslant \tau$, the result follows from Lemma 4.7. But if $\tau < p$, we are led to a contradiction as follows:

$$\theta_l - 1 \leqslant \tau - p < 0;$$

hence $\theta_l = 0$ and $\tau - p = -1$. Therefore $\alpha \leqslant \tau + 1 + \alpha_{kl} \leqslant p + \alpha_l$ and

$$l(F_l) - \alpha \geqslant p + \sigma_l - \alpha_l \geqslant p + \mathfrak{b}_l^*(0),$$
$$l(F_k \cdot D) - \alpha = s + \lambda_k - \alpha_k \geqslant s + \mathfrak{b}_k^*(0),$$

so that $\tilde{C}$ has double barriers ($\tilde{C}$ is simple, even naturally simple, by Corollary 4.33). This proves the lemma.

COROLLARY 4.81.

(i) $\tau + 2 \leqslant \alpha \leqslant \tau + 1 + \alpha_{kl}$.
(ii) $p + \theta_l + 1 \leqslant \alpha \leqslant p + \theta_l + 2\alpha_l - \phi_l(\varepsilon_2)$.
(iii) $l(F_l^\tau) = 2p + \sigma_l - \tau - 1$.
(iv) $\tau + 1 \leqslant p + \theta_l + \alpha_l - \phi_l(\varepsilon_2)$.
(v) $\tilde{C}$ *is a naturally simple chain.*

Proof. $l(F_k^l) = s + \mu$; so, by Corollary 4.33, $\alpha > \tau + 1$. But $\theta_l - 1 \leqslant \tau - p$, so that

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$$p+\theta_l+1\leqslant\alpha.$$

By Corollary 4.32,

$$l(F_k\,.\,D)-l(F_k^l) = \alpha'' = l(F_l)-l(F_l^r),$$

and (iii) follows at once. Now $\tilde{C}_2$ is right-closed. Thus $l(F_l^r)\geqslant p+\lambda_l-\alpha_l+\phi_l(\varepsilon_2)$ and (iv) follows. (i) and (iv) imply the remaining inequality in (ii). Finally, (v) follows by the usual argument (see, *e.g.*, the proof of Corollary 4.56).

LEMMA 4.9.

(i) *$\tilde{C}$ is left-closed if*

$$l(F_k\,.\,D)-\alpha\geqslant s+\lambda_k-\alpha_k+\phi_k(\varepsilon)\,;$$

(ii) *$\tilde{C}$ is right-closed if*

$$l(F_l)-\alpha\geqslant p+\lambda_l-\alpha_l+\phi_l(\varepsilon).$$

Proof. For (i), it is only necessary to prove that $l(F_l)-\alpha\geqslant p+\mathfrak{c}_l(\varepsilon)$, *i.e.*, that

$$\alpha\leqslant p+\sigma_l-\mathfrak{c}_l(\varepsilon).$$

This follows from (ii) of Corollary 4.81.

For (ii), we need only prove that $l(F_k\,.\,D)-\alpha\geqslant s+\mathfrak{c}_k(\varepsilon)$. The left-hand side is not less than $(s+\mu+\tau+1)-(\tau+1+\alpha_{kl})$, which, by (4.35), is not less than $s+\lambda_k-\alpha_k-\alpha_{kl}$, and the result follows easily.

Proof of Lemma 4.1. In view of the preceding lemmas, we need only consider the case in which (C′) holds and the assumptions of Lemma 4.9 do not apply. Thus we assume that

$$\alpha>\mu+\tau+1-\lambda_k+\alpha_k-\phi_k(\varepsilon) \qquad \text{(4.91)}$$

and

$$\alpha>p+\theta_l+\alpha_l-\phi_l(\varepsilon). \qquad \text{(4.92)}$$

We shall obtain a contradiction; it will be shown that

$$l(F_k\,.\,D)-\alpha\geqslant s+\lambda_k-2\alpha_k+\phi_k(\varepsilon) \qquad \text{(4.93)}$$

and

$$l(F_l)-\alpha\geqslant p+\lambda_l-2\alpha_l+\phi_l(\varepsilon). \qquad \text{(4.94)}$$

From these, *making our only use of the fact that*

$$\lambda_i>6\alpha_i \quad (i = 1, 2, \ldots, h),$$

it follows that (*cf.* Lemma 2.2 (i))

$$l(F_k\,.\,D)-\alpha\geqslant s+\mathfrak{b}_k^*(\varepsilon) \text{ and } l(F_l)-\alpha\geqslant p+\mathfrak{b}_l^*(\varepsilon).$$

Since $\tilde{C}$ is simple (Corollary 4.81), we have that $\tilde{C}$ has double barriers, in contradiction to the hypothesis that C is open.

First, we note that $p\leqslant\tau$ and equality implies that $\alpha=\tau+1+\alpha_{kl}$ and $\theta_l\varepsilon=0$. This follows by combining (4.92) with $\alpha\leqslant\tau+1+\alpha_{kl}$ and $\alpha_{kl}\leqslant\alpha_l$.

Next, we prove (4.93). Since $\alpha\leqslant\tau+1+\alpha_{kl}$ and $\mu\geqslant\lambda_k-\alpha_k+\phi_k(\varepsilon_1)$, we have

$$l(F_k\,.\,D)-\alpha\geqslant s+\lambda_k-2\alpha_k+\phi_k(\varepsilon_1),$$

where equality implies that $\alpha=\tau+1+\alpha_{kl}$. If there is strict inequality, then (4.93) follows at once. If not, the result is trivial when $\phi_k(\varepsilon_1)\geqslant\phi_k(\varepsilon)$; but when $\phi_k(\varepsilon_1)<\phi_k(\varepsilon)$, we have $\theta_k=1$, $\varepsilon=0$ and $\varepsilon_1=1$, in contradiction to Lemma 4.7.

Finally, we prove (4.94), *i.e.*, we prove that $\alpha\leqslant p+\theta_l+2\alpha_l-\phi_l(\varepsilon)$. By Corollary 4.81, $\alpha\leqslant p+\theta_l+2\alpha_l-\phi_l(\varepsilon_2)$; we need only consider the case of equality, with $\theta_l=1$, $\varepsilon=0$ and

$\varepsilon_2 = 1$. But equality together with (iv) of the corollary implies that $\alpha \geqslant \tau + 1 + \alpha_l$. Hence $\alpha = \tau + 1 + \alpha_{kl}$; so Lemma 4.7 is applicable and we obtain a contradiction.

This completes the proof of Lemma 4.1.

5. Normal form of an open chain. The remainder of the proof of the theorem is straightforward; it consists merely of a number of applications of Lemma 1A. This lemma gives us information about the normal form of a sum of chains in terms of the normal forms of the summands. In this section we show that any open chain can be built up from left-closed or right-closed chains in such a way that the lemma can be applied, and in § 6 we prove that an arbitrary chain can be built up in a similar way from open chains and chains with double barriers. We shall then, in particular, possess information about the normal form of the chain $\tilde{U}_0$, that is, about the normal form of the element U_0.

Let $\tilde{S} = \langle F_i, F_{i+1}, \ldots, F_j \rangle$ be an open chain. It follows at once from Definition 2.5 that any subchain of $\tilde{S}$ is open. Thus there exists a decomposition of $\tilde{S}$ into a sum of subchains, where each subchain is either left-closed or right-closed, namely, that in which each subchain has two terms (*cf.* § 3). Of the decompositions of this type we choose one in which the number of subchains is minimal. Let this be

$$\tilde{S} = \tilde{S}_1 + \tilde{S}_2 + \ldots + \tilde{S}_n, \qquad (5.01)$$

where $\tilde{S}_\nu = \langle F_{k(\nu-1)}, F_{k(\nu-1)+1}, \ldots, F_{k(\nu)} \rangle \quad (\nu = 1, 2, \ldots, n)$.

Thus $k(0) = i$ and $k(n) = j$. Let

$$S_\nu = F^l_{k(\nu-1)} c_\nu^{\varepsilon_\nu} F^r_{k(\nu)},$$

in the usual notation for simple chains.

If $n > 1$, we can apply Lemma 1A, provided that

$$l(F^r_{k(\nu)}) + l(F^l_{k(\nu)}) \geqslant l(F_{k(\nu)}) \quad (\nu = 1, 2, \ldots, n-1),$$

where equality implies that $\varepsilon_\nu = 1$ or $\varepsilon_{\nu+1} = 1$.

Now, by Lemma 4.1, $\tilde{S}_\nu$ and $\tilde{S}_{\nu+1}$ cannot be left-closed and right-closed respectively, since n was chosen minimally. Assume, as we may do without loss of generality, that $\tilde{S}_\nu$ is right-closed. Then, whether $S_{\nu+1}$ is left-closed or right-closed, we have (if $k = k(\nu)$)

$$l(F^r_k) + l(F^l_k) \geqslant (\delta_k + \lambda_k - \alpha_k + \phi_k(\varepsilon_\nu)) + (\delta_k + c_k(\varepsilon_{\nu+1})).$$

By Lemma 2.2 (vii), the right-hand side is not less than $2\delta_k + b^*_k(\varepsilon_\nu) + c_k(\varepsilon_{\nu+1})$ which, by (2.12), is not less than $2\delta_k + \sigma_k$, which is equal to $l(F_k)$, and equality implies that $\varepsilon_\nu = 1$ or $\varepsilon_{\nu+1} = 1$.

Applying Lemma 1A, we have

$$\left.\begin{aligned} S &= F^l_i c_1^{\varepsilon_1} J_1 \ldots c_{n-1}^{\varepsilon_{n-1}} J_{n-1} c_n^{\varepsilon_n} F^r_j \\ &= F^l_i B' F^r_j, \text{ say.} \end{aligned}\right\} \qquad (5.02)$$

Note that (5.02) is also an expression for the normal form of S when $n = 1$.

LEMMA 5.1. *An open chain $\tilde{S} = \langle F_i, F_{i+1}, \ldots, F_j \rangle$ has normal form given by* (5.02), *where*

(i) *if $B' = I$, then $\tilde{S}$ is simple,*

(ii) $l(F^l_i) \geqslant \delta_i + c_i(\varepsilon_1)$ *and* $l(F^r_j) \geqslant \delta_j + c_j(\varepsilon_n)$,

(iii) $\varepsilon_1 = 1$ *if and only if* $F^l_i \,\hat{\epsilon}\, \mathscr{L}(F_i, S)$,

(iv) $\varepsilon_n = 1$ *if and only if* $F^r_j \,\hat{\epsilon}\, \mathscr{R}(F_j, S)$,

(v) *at least one of the following conditions is satisfied:*

(5.1A) $l(F_i^l) \geqslant \delta_i + \lambda_i - \alpha_i + \phi_i(\varepsilon_1)$.

(5.1B) $l(F_j^r) \geqslant \delta_j + \lambda_j - \alpha_j + \phi_j(\varepsilon_n)$.

(5.1C) *$n \geqslant 2$ and there exists an integer p such that $1 \leqslant p \leqslant n-1$ and J_p contains at least*

$$\lambda_k - 2\alpha_k - \theta_k + \phi_k(\varepsilon_p) + \phi_k(\varepsilon_{p+1})$$

elements of the kernel of some F_k, where $i < k < j$.

Proof. (i), (ii), (iii) and (iv) are trivial consequences of Lemma 1A if $n \geqslant 2$, or of the fact that $\tilde{S}$ is either left-closed or right-closed if $n = 1$. (Note that $n \geqslant 2$ implies that $B' \neq I$.)

To prove (v), we assume that neither (5.1A) nor (5.1B) holds. Then $\tilde{S}_1$ is right-closed and $\tilde{S}_n$ is left-closed. But, for $\nu = 1, 2, \ldots, n-1$, we have seen that the chains $\tilde{S}_\nu$ and $\tilde{S}_{\nu+1}$ cannot be left-closed and right-closed, respectively. Therefore there is an integer p in the range $1 \leqslant p \leqslant n-1$, such that $\tilde{S}_p$ is right-closed and $\tilde{S}_{p+1}$ is left-closed.

Write $k = k(p)$; then

$$l(F_k^r) \geqslant \delta_k + \lambda_k - \alpha_k + \phi_k(\varepsilon_p)$$

and

$$l(F_k^l) \geqslant \delta_k + \lambda_k - \alpha_k + \phi_k(\varepsilon_{p+1}).$$

Therefore

$$\begin{aligned} l(J_p) &= l(F_k^r) + l(F_k^l) - l(F_k) \\ &\geqslant \lambda_k - 2\alpha_k - \theta_k + \phi_k(\varepsilon_p) + \phi_k(\varepsilon_{p+1}), \end{aligned}$$

and it is not difficult to see that J_p (and hence B') contains at least this last number of elements of the kernel of F_k.

This proves the lemma.

An alternative expression for S. Since $S = F_i^l B' F_j^r$, we may write

$$S = A_i B C_j, \qquad \text{(5.11)}$$

where

$$l(A_i) = \begin{cases} \delta_i + c_i(0) & \text{if} \quad l(F_i^l) \geqslant \delta_i + c_i(0), \\ \delta_i + c_i(1) & \text{if} \quad l(F_i^l) < \delta_i + c_i(0), \end{cases}$$

and C_j is defined similarly. Note that if $l(F_i^l) < \delta_i + c_i(0)$, then, by Lemma 5.1 (ii), $\varepsilon_1 = 1$ and $l(F_i^l) = \delta_i + c_i(1)$. Also, $F_i^l \,\hat{\epsilon}\, \mathscr{L}(F_i, S)$. Thus $A_i \in \mathscr{L}(F_i)$ and $C_j \in \mathscr{R}(F_j)$, and it follows that $B = XB'Y$. (We may have $X = I$ or $Y = I$.)

Thus if $B = I$, then $\tilde{S}$ is simple, by Lemma 5.1 (i).

6. Normal form of an arbitrary chain. In the fixed minimal representation

$$U_0 = F_1 \,.\, F_2 \,.\, \ldots \,.\, F_h, \qquad \text{(6.01)}$$

let us insert brackets between each pair of factors F_i, F_{i+1} for which $\alpha(F_i, F_{i+1}) = 0$. Thus

$$U_0 = U_1 \,.\, U_2 \,.\, \ldots \,.\, U_m, \qquad \text{(6.02)}$$

where $1 \leqslant m \leqslant h$ and U_q is a product of, say, m_q factors F_ν such that if $m_q > 1$, then there is at least an amalgamation between each adjacent pair of factors.

To discuss the normal form of any U_q with $m_q > 1$, we use Lemma 2.6, which shows that

$$\tilde{U}_q = \tilde{S}_1 + \tilde{S}_2 + \ldots + \tilde{S}_f, \qquad \text{(6.03)}$$

where the subchains $\tilde{S}_\nu$ are open or have double barriers. Consider a typical subchain $\tilde{S}_\nu = \langle F_i, F_{i+1}, \ldots, F_j \rangle$. We write $S_\nu = A_i B C_j$, where if $\tilde{S}_\nu$ has double barriers the factorization is taken as in Definition 2.3, but if $\tilde{S}_\nu$ is open we take the factorization (5.11). The following remarks show that Lemma 1A is applicable.

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Assume, as we may do without loss of generality, that $\tilde{S}_\nu$ has double barriers and $\tilde{S}_{\nu+1}$ is open. Suppose that $S_{\nu+1} = \langle F_j, F_{j+1}, \ldots, F_k \rangle$ and $S_{\nu+1} = A_j B_1 C_k$. Now C_j has length $\delta_j + \mathfrak{b}_j^*(0)$ or $\delta_j + \mathfrak{b}_j^*(1)$, and A_j has length $\delta_j + \mathfrak{c}_j(0)$ or $\delta_j + \mathfrak{c}_j(1)$. By (2.12), the sum of the lengths of C_j and A_j is either greater than $l(F_j)$ or else it equals $l(F_j)$ and then $C_j \,\hat{\epsilon}\, \mathscr{R}(F_j, S_\nu)$ or $A_j \,\hat{\epsilon}\, \mathscr{L}(F_j, S_{\nu+1})$.

LEMMA 6.1. *Let U_q be a product of $m_q \geqslant 2$ consecutive factors of* (6.01), *say*

$$U_q = F_i \,.\, F_{i+1} \,.\, \ldots \,.\, F_j,$$

such that $\alpha(F_\nu, F_{\nu+1}) \neq 0$ $(\nu = i, i+1, \ldots, j-1)$. Then

(A) *$l > \delta_i + \delta_j + \lambda_0$, where $l = l(U_q)$ and $\lambda_0 = \mathrm{Min}\ \lambda_\nu$ $(\nu = 1, 2, \ldots, h)$,*

(B) *there exists a factorization $U_q = XKZ$, where K is part of the kernel of some factor F_ν $(i \leqslant \nu \leqslant j)$ and*

either $l(K) > \frac{1}{2}\lambda_\nu$,

or $l(K) = \frac{1}{2}\lambda_\nu$ and at least one of the following conditions (i) *and* (ii) *is satisfied,*

or $l(K) = \frac{1}{2}(\lambda_\nu - 1)$ and both (i) *and* (ii) *are satisfied.*

The conditions (i) *and* (ii) *are :*

(i) *The kernel of F_ν does not start with the component* In (K), *$X \neq I$ and the component of F_ν next to the left of* In (K) *is in the same constituent group as* Fin (X).

(ii) *The kernel of F_ν does not end with* Fin (K), *$Z \neq I$ and the component of F_ν next to the right of* Fin (K) *is in the same constituent group as* In (Z).

Further,

(C) *$l(K) \geqslant \lambda_\nu - 3\alpha_\nu - 1$ and equality implies that both* (i) *and* (ii) *hold.*

Note 6.2. If $\lambda_\nu \geqslant 6\alpha_\nu + 1$, then (C) implies (B) ; if $\lambda_\nu \leqslant 6\alpha_\nu + 1$, then (B) implies (C).

Proof of the lemma. The following inequalities are trivial ; II, III and IV are consequences of the inequality $\lambda_i \geqslant 4\alpha_i + 1$ $(i = 1, 2, \ldots, h)$.

I. $\mathfrak{b}_i(\varepsilon) > \frac{1}{2}\lambda_i + \theta_i$.
II. $\lambda_i - \alpha_i \geqslant \frac{1}{4}(3\lambda_i + 1)$.
III. $\lambda_i - 2\alpha_i \geqslant \frac{1}{2}(\lambda_i + 1)$.
IV. $\mathfrak{c}_i(\varepsilon) \geqslant [\frac{1}{4}(\lambda_i + 2\theta_i - 2\varepsilon + 3)]$.
V. If N is an integer, then $[\frac{1}{4}N] \geqslant \frac{1}{4}(N - 3)$.

1°. We first consider the case in which $f = 1$ in (6.03). Thus $U_q = S_1 = A_i B C_j$.

If $\tilde{S}_1$ has double barriers, then, by I,

$$l(U_q) \geqslant l(A_i) + l(C_j) \geqslant \delta_i + \mathfrak{b}_i(1) + \delta_j + \mathfrak{b}_j(1) > \delta_i + \delta_j + \lambda_0.$$

Now the part of the kernel of F_i appearing in A_i has length $l(A_i) - \delta_i - \theta_i \geqslant \mathfrak{b}_i^*(1) - \theta_i > \frac{1}{2}\lambda_i$. Finally, if $\lambda_\nu > 6\alpha_\nu + 1$, we have

$$\mathfrak{b}_i^*(1) - \theta_i = \lambda_i - 2\alpha_i - \theta_i > \lambda_i - 3\alpha_i - 1,$$

by Lemma 2.2 (i). By Note 6.2, we have proved (A), (B) and (C).

Now assume that $\tilde{S}_1$ is open. Then S_1 can be expressed in the form (5.02), so that

$$\left.\begin{array}{ll} \text{if } n = 1, & l = l(F_i^l) + \varepsilon_1 + l(F_j^r), \\ \text{if } n = 2, & l \geqslant l(F_i^l) + 1 + l(F_j^r), \\ \text{if } n \geqslant 3, & l \geqslant l(F_i^l) + l(J_t) + 1 + l(F_j^r) \quad (t = 1, 2, \ldots, n-1). \end{array}\right\}$$

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We first prove (A). If (5.1A) holds, it is sufficient to prove that

$$(\lambda_i - \alpha_i) + \varepsilon_1 + c_j(\varepsilon_1) > \lambda_0 \quad \text{if } n = 1$$

and

$$(\lambda_i - \alpha_i) + 1 + c_j(\varepsilon_n) > \lambda_0 \quad \text{if } n > 1.$$

These are consequences of II, IV and V. There is a similar argument when (5.1B) holds. Now assume that (5.1C) holds. Here $n \geqslant 2$; hence, by (5.02),

$$\begin{aligned} l &\geqslant l(F_i^l) + \varepsilon_p + l(J_p) + \varepsilon_{p+1} + l(F_j^r) \qquad (6.21) \\ &\geqslant (\delta_i + c_i(\varepsilon_1) + \varepsilon_p + (\lambda_k - 2\alpha_k - \theta_k + \phi_k(\varepsilon_p) + \phi_k(\varepsilon_{p+1})) + \varepsilon_{p+1} + (\delta_j + c_j(\varepsilon_n)). \end{aligned}$$

But $c_i(\varepsilon_1) + (\lambda_k - 2\alpha_k) + c_j(\varepsilon_n) \geqslant \lambda_0 - \frac{1}{2}$, by III, IV and V, so that (A) will follow if

$$\varepsilon_p - \theta_k + \phi_k(\varepsilon_p) + \phi_k(\varepsilon_{p+1}) + \varepsilon_{p+1} \geqslant 1,$$

i.e., if

$$\varepsilon_p(1 - \theta_k) + \varepsilon_{p+1}(1 - \theta_k) + \theta_k \geqslant 1.$$

If this is not the case, then $\varepsilon_p = \varepsilon_{p+1} = \theta_k = 0$. We have seen that, if $n \geqslant 3$, then (6.21) remains true if we replace $\varepsilon_p + \varepsilon_{p+1}$ by 1, so that we need only consider $n = 2$ in the case we are discussing. Since $1 \leqslant p \leqslant n - 1$, we have $p = 1$, $\varepsilon_1 = \varepsilon_2 = 0$ and hence $\varepsilon_n = 0$. Therefore

$$\begin{aligned} l &\geqslant (\delta_i + c_i(0)) + (\lambda_k - 2\alpha_k) + (\delta_j + c_j(0)) \\ &\geqslant \delta_i + \delta_j + \lambda_0 + \tfrac{1}{2}, \quad \text{by III, IV and V.} \end{aligned}$$

Finally, we prove (B) and (C). If (5.1A) holds, there appear in the normal form of U_q at least $(\delta_i + \lambda_i - \alpha_i + \phi_i(\varepsilon)) - (\delta_i + \theta_i)$ components of the kernel of F_i. This number is strictly greater than $\lambda_i - 3\alpha_i - 1$ and is also, by II, strictly greater than $\frac{1}{2}\lambda_i$. Case (5.1B) is similar. If (5.1C) holds, then the normal form of U_q contains at least

$$\lambda_k - 2\alpha_k - \theta_k + \phi_k(\varepsilon_p) + \phi_k(\varepsilon_{p+1})$$

components of the kernel of F_k. This number is not less than $\frac{1}{2}(\lambda_k - 1)$, by III, and if it equals $\frac{1}{2}(\lambda_k - 1)$, then $\varepsilon_p = \varepsilon_{p+1} = 1$. If it is greater than $\frac{1}{2}(\lambda_k - 1)$, it is, obviously, not less than $\frac{1}{2}\lambda_k$ and if it equals $\frac{1}{2}\lambda_k$, then again $\varepsilon_p = \varepsilon_{p+1} = 1$. This proves (B). The proof of (C) is trivial.

2°. Now we consider the case in which $f \geqslant 2$ in (6.03). We have seen that Lemma 1A is applicable; so writing

$$S_1 = A_i B_1 C_m, \quad S_f = A_n B_f C_j,$$

we obtain $U_q = A_i B_1 E B_f C_j$, say.

3°. To prove (A), it is sufficient to show that $A_i B_1$ and $B_f C_j$ have lengths strictly greater than $\delta_i + \frac{1}{2}\lambda_0$ and $\delta_j + \frac{1}{2}\lambda_0$, respectively. We shall only prove that $l(A_i B_1) > \delta_i + \frac{1}{2}\lambda_0$; the other case is similar.

If $\tilde{S}_1$ has double barriers, then

$$l(A_i B_1) \geqslant l(A_i) \geqslant \delta_i + b_i^*(1) > \delta_i + \tfrac{1}{2}\lambda_0, \quad \text{by I.}$$

If $\tilde{S}_1$ is open, we have, by (5.02), $A_i B_1 = F_i^l c_1^{\varepsilon_1} Y$, say. If (5.1A) holds, there is no difficulty, since $l(A_i B_1) \geqslant l(F_i^l) > \delta_i + \frac{1}{2}\lambda_0$.

Now suppose that (5.1B) is satisfied. Then

$$l(F_m^r) \geqslant \delta_m + \lambda_m - \alpha_m + \phi_m(\varepsilon_n) \geqslant \delta_m + c_m(0),$$

so that, by definition, $l(C_m) = \delta_m + c_m(0)$. Further, from (5.02) (with an appropriate change of notation) it follows that $l(S_1) \geqslant l(F_i^l) + \varepsilon_n + l(F_m^r)$. Therefore

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$$\begin{aligned} l(B_1) &= l(S_1) - l(A_i) - l(C_m) \\ &\geqslant (l(F_i^l) - l(A_i)) + \varepsilon_n + (l(F_m^r) - l(C_m)) \\ &\geqslant 0 + \varepsilon_n + (\lambda_m - \alpha_m + \phi_m(\varepsilon_n) - \mathfrak{c}_m(0)) \\ &\geqslant \varepsilon_n + \lambda_m - \alpha_m + \phi_m(\varepsilon_n) - (\tfrac{1}{2}(\sigma_m + 1) - \alpha_m) \\ &= \tfrac{1}{2}\lambda_m + \varepsilon_n(1 - \theta_m) + \tfrac{1}{2}(\theta_m - 1) \\ &\geqslant \tfrac{1}{2}(\lambda_m - 1). \end{aligned}$$

But $l(A_i) \geqslant \delta_i + \mathfrak{c}_i(1) \geqslant \delta_i + 1$, so that $l(A_iB_1) > \delta_i + \frac{1}{2}\lambda_0$.

Finally, let (5.1C) hold. Then there exists an integer k such that $l(B_1) \geqslant \lambda_k - 2\alpha_k - \theta_k$ and, by III, we obtain $l(B_1) \geqslant \frac{1}{2}(\lambda_0 - 1)$, and hence $l(A_iB_1) > \delta_i + \frac{1}{2}\lambda_0$.

4°. We now prove (B) and (C). If $\tilde{S}_1$ has double barriers, we use the same argument as in 1°. We assume, then, that $\tilde{S}_1$ is open. Again the case (5.1A) presents no difficulties.

Now let (5.1B) hold. We write

$$S_1 = A_iB_1C_m, \quad S_2 = A_mB_2C_r,$$

so that, by Lemma 1A,

$$U_q = A_iB_1XB_2T,$$

say, where $X_m = C_m \,.\, F_m^{-1} \,.\, A_m$. Since $\tilde{S}_1$ is open, $\tilde{S}_2$ has double barriers and hence A_m has length $\delta_m + \mathfrak{b}_m^*(0)$ or $\delta_m + \mathfrak{b}_m^*(1)$. Define the symbol ε by

$$\varepsilon = \begin{cases} 0 & \text{if } l(A_m) = \delta_m + \mathfrak{b}_m^*(0), \\ 1 & \text{otherwise.} \end{cases}$$

Then $l(A_m) = \delta_m + \mathfrak{b}_m^*(\varepsilon)$. Also, $\varepsilon = 1$ implies that $A_m \hat{\epsilon} \mathscr{L}(F_m, S_2)$. By (5.02), we may write $S_1 = A_iB_1C_m = A_iYc_n^{\varepsilon_n}F_m^r$, say. By (5.1B), $l(F_m^r) \geqslant \delta_m + \lambda_m - \alpha_m + \phi_m(\varepsilon_n)$. Now

$$\begin{aligned} U_q &= A_iB_1C_m \,.\, F_m^{-1} \,.\, A_mB_2T \\ &= A_iYc_n^{\varepsilon_n} \,.\, Z_m \,.\, B_2T, \qquad (6.22) \end{aligned}$$

where $Z_m = F_m^r \,.\, F_m^{-1} \,.\, A_m$. We proceed to show that the number $l(F_m^r) + l(A_m) - l(F_m)$ is strictly positive, which will imply both that this number equals $l(Z_m)$ and that in (6.22) the dots can be removed. The number in question is not less than N, where

$$\begin{aligned} N &= (\delta_m + \lambda_m - \alpha_m + \phi_m(\varepsilon_n)) + (\delta_m + \mathfrak{b}_m^*(\varepsilon)) - (2\delta_m + \sigma_m) \\ &= \mathfrak{b}_m^*(\varepsilon) - \alpha_m + \phi_m(\varepsilon_n) - \theta_m. \end{aligned}$$

First, if $\lambda_m \geqslant 6\alpha_m + 1$, then, by Lemma 2.2 (i),

$$N = (\lambda_m - 3\alpha_m - 1) + (1 + \phi_m(\varepsilon) + \phi_m(\varepsilon_n) - \theta_m) \geqslant \lambda_m - 3\alpha_m - 1,$$

where equality implies that $\varepsilon = \varepsilon_n = 1$. Thus certainly $N > 0$. Second, if $\lambda_m < 6\alpha_m + 1$, then $\mathfrak{b}_m^*(\varepsilon) = \mathfrak{b}_m(\varepsilon)$. Hence $N = [\frac{1}{2}\lambda_m + x]$, where $x = \theta_m(\frac{1}{2} - \varepsilon_n) + 1 - \frac{1}{2}\varepsilon$. Clearly $x \geqslant 0$ and $x = 0$ implies that $\varepsilon = \varepsilon_n = 1$. If $x > 0$, then $x \geqslant \frac{1}{2}$ and moreover $x = \frac{1}{2}$ implies that $\varepsilon = 1$ or $\varepsilon_n = 1$. In any case, we have $N > 0$, as required.

Now write $F_m = MLKJ$, where $MLK = A_m$, $LKJ = F_m^r$ and $l(KJ) = \delta_m + \lambda_m - \alpha_m + \phi_m(\varepsilon_n)$. Then $l(K) = N$ and K is part of the kernel of F_m. Further, $Z_m = LK$ and, since the dots can be removed from (6.22),

$$U_q = XKZ,$$

where $X = A_iYc_n^{\varepsilon_n}L$ and $Z = B_2T$. We wish to prove (B) and (C) for this factorization of U_q. Let us first show that conditions (i) and (ii) of the lemma are implied by $\varepsilon_n = 1$ and $\varepsilon = 1$, respectively.

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If $\varepsilon_n = 1$, we need only prove that Fin $(ML) \sim$ Fin (X), *i.e.*, that Fin $(ML) \sim$ Fin $(c_n L)$. This is easily verified; if $L = I$ we use the fact that $F_m^r \hat{\epsilon} \mathscr{R}(F_m, S_1)$, *i.e.*, that Fin $(M) \sim c_n$.

If $\varepsilon = 1$, it is sufficient to prove that $B_2 \neq I$ and In $(J) \sim$ In (Z). But if B_2 were equal to I, then S_2 would be simple and S_2 would be equal to $A_m C_r$, in contradiction to

$$A_m \hat{\epsilon} \mathscr{L}(F_m, S_2).$$

Also, $A_m \hat{\epsilon} \mathscr{L}(F_m, S_2)$ implies that In $(J) \sim$ In $(B_2 C_r)$ and hence that In $(J) \sim$ In (Z).

The above discussion of the number N now shows that (C) is true when $\lambda_m \geqslant 6\alpha_m + 1$, and that (B) is true when $\lambda_m < 6\alpha_m + 1$. By Note 6.2, this completes the proof of (B) and (C) for the case under discussion.

The remaining case is that in which $\tilde{S}_1$ is open and satisfies (5.1C). Thus, since $S_1 = A_i Y c_n^{\varepsilon_n} F_m^r = V F_m^r$, say, we have that V contains a factor $c_p^{\varepsilon_p} J_p c_{p+1}^{\varepsilon_{p+1}}$. But the normal form of U_q starts with V, since the dots can be removed from (6.22). (B) and (C) now follow as in 1°.

This completes the proof of Lemma 6.1.

7. Proof of the theorem. In the decomposition (6.02) of U_0, consider any particular U_q. If $m_q \neq 1$, we have seen that U_q satisfies the conditions (A), (B) and (C) of Lemma 6.1. But if $m_q = 1$, these conditions are trivially satisfied, except that equality may occur in (A). Further, if $U_q = F_i . F_{i+1} . \dots . F_j$, then In $(U_q) =$ In (F_i) and Fin $(U_q) =$ Fin (F_j), so that

$$l(U_0) = \sum_{q=1}^{m} l(U_q).$$

Thus $l(U_0) \geqslant m\lambda_0 \geqslant \lambda_0 \geqslant l_0$, in the notation of the theorem.

Now suppose that $l(U_0) = l_0$. Then $m = 1$, *i.e.*, $U = U_1$, and $m_1 = 1$, for $m_1 > 1$ implies that $l(U_1) > \lambda_0$, by Lemma 6.1. Therefore $U_0 = F_1$. Comparing lengths, we have

$$l_0 = 2\delta_1 + \theta_1 + \lambda_1 \geqslant \lambda_1 \geqslant \lambda_0 \geqslant l_0,$$

so that $\delta_1 = \theta_1 = 0$ and $F_1 \epsilon \Omega^*$.

This proves (i) and (ii). But (iii) follows from (C) of Lemma 6.1.

This completes the proof of the theorem.

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