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Well-posed boundary value problems for linear evolution equations on a finite interval

BY BEATRICE PELLONI
Department of Mathematics, University of Reading, Reading RG6 6AX.
e-mail: b.pelloni@rdg.ac.uk

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Abstract

We identify the class of smooth boundary conditions that yield an initial-boundary value problem admitting a unique smooth solution for the case of a dispersive linear evolution PDE of arbitrary order, in one spatial dimension, defined on a finite interval.

This result is obtained by an application of a spectral transform method, introduced by Fokas, which allows us to reduce the problem to the study of the singularities of the set of functions arising as the unique solution of a certain linear system.

1. *Introduction*

The Fokas transform method is a method for solving boundary value problems for linear and for integrable nonlinear PDEs in two dimensions, see the review [**3**]. In a recent paper by Fokas and the author [**5**], this method has been applied to solve a particular class of two-point boundary value problems for the general linear dispersive evolution equation of the form

$$\left(\partial_t + i\sum_{j=0}^{n}\alpha_j(-i\partial_x)^j\right)q(x,t) = 0, \qquad 0<x<L, \quad 0<t<T, \tag{1·1}$$

where $q(x,t)$ is a real scalar function, α_j are real constants with $\alpha_n \neq 0$ and L, T are positive constants. In this notation, the dispersion relation of equation (1·1) is given by

$$\omega(k) = \sum_{j=1}^{n}\alpha_j k^j. \tag{1·2}$$

We assume throughout this paper that all given functions are sufficiently smooth (e.g. of $\mathbf{C}^\infty$ class); this is not necessary, but it allows us to simplify the exposition. We also assume that $q(x,t)$ satisfies the initial condition

$$q(x,0) = q_0(x), \qquad 0<x<L, \quad q_0(x)\in\mathbf{C}^\infty[0,L]. \tag{1·3}$$

The aim of the present work is to characterise the class of boundary conditions that, prescribed at $x = 0$ and at $x = L$, give rise to an initial boundary value problem for equations (1·1)–(1·3) which admits a unique smooth solution, and is, in this sense, well-posed. The result we prove is that well-posed problems are precisely those obtained by prescribing N conditions at $x=0$ and $n-N$ at $x=L$; N is equal to $n/2$ when N is even, but can be equal to either $(n+1)/2$ or $(n-1)/2$ when n is odd. Namely,

$$N = \begin{cases} n/2 & n \text{ even}, \\ (n+1)/2 & n \text{ odd}, \quad \alpha_n > 0, \\ (n-1)/2 & n \text{ odd}, \quad \alpha_n < 0, \end{cases} \tag{1·4}$$

where α_n is the highest degree coefficient of the dispersion relation (1·2) of the equation. As a special case, when the prescribed boundary conditions are the values for $q(0,t), \ldots, \partial_x^{N-1}q(0,t)$, $q(L,t), \ldots, \partial_x^{n-N-1}q(L,t)$, we obtain the result of [**5**].

We break the analysis up into three steps:

Step 1: prove that for any well-posed boundary value problem, the unique smooth solution $q(x,t)$ admits an integral representation with explicit exponential x and t dependence; this representation involves certain functions – called the *spectral functions* – defined only in terms of the initial and boundary values of the solution and of its x-derivatives; see Proposition 2·1.

Step 2: prove the existence and uniqueness of the solution under a certain admissibility assumption for the boundary data; we define this notion in Section 3, see Theorem 3·1.

Step 3: characterise the boundary conditions that yield a set of admissible functions, hence a well-posed boundary value problem. This question is the heart of the the present paper, and is addressed in Sections 4–6.

Step 1 and Step 2 can be found in [**5**]; the analysis of Step 3 yields the following theorem.

THEOREM 1·1. *Consider the boundary value problem for equation* (1·1) *obtained when there are prescribed*:

(i) *the initial condition* (1·3);

(ii) *the boundary conditions*

$$(-i\partial_x)^{j_1} q(0,t) = u_{j_1}(t), \quad (-i\partial_x)^{j_2} q(L,t) = v_{j_2}(t),$$

where $(j_1, j_2) \in J_1 \times J_2$, *with* J_1, J_2 *subsets of the index set* $\{0, \ldots, n-1\}$.

Assume that these initial and boundary conditions are of $\mathbf{C}^\infty$ *class and that they are compatible, i.e. that* $(-i\partial_x)^{j_1} q_0(0) = u_{j_1}(0)$, $j_1 \in J_1$ *and* $(-i\partial_x)^{j_2} q_0(L) = v_{j_2}(0)$, $j_2 \in J_2$.

This boundary value problem has a unique solution $q(x,t)$ *such that* $t \to q(\cdot,t)$ *is a* $\mathbf{C}^\infty$ *map from* $[0,T]$ *into* $\mathbf{C}^\infty[0,L]$ *if and only if* $|J_1| = N$ *and* $|J_2| = n-N$, *where* $|\cdot|$ *denotes the cardinality of the index set, and* N *is given by* (1·4).

The paper is organized as follows. In Section 2, we set the notation, give some definitions and summarise the results that yield Step 1. In Section 3, we define the admissibility condition and state the existence theorem, whose proof can be found in [**5**]. In Section 4, we consider some preliminary results in view of the proof of our main theorem. In Section 5 we illustrate the results by discussing two particular examples, and finally in Section 5 we give the proof of Theorem 1·1. We end with Section 6 and some concluding remarks.

2. *Step* 1: *the integral representation of the solution*

We summarise in this and the next section the results of previous work, see [**5**], [**7**]. In order to describe these results, we start by defining certain contours and domains in the complex k-plane.

Definition 2·1. Let D, D_+ and D_- be the domains in the complex k-plane defined by

$$D = \{k \in \mathbb{C} : \ \operatorname{Im}\omega(k) > 0\}, \qquad D_+ = D \cap \mathbb{C}^+, \quad D_- = D \cap \mathbb{C}^-, \tag{2·1}$$

where $\mathbb{C}^+$ and $\mathbb{C}^-$ indicate the upper and lower half plane, respectively.

$$\partial D_+,\ \partial D_- \text{ are the oriented boundaries of } D_+,\ D_-, \tag{2·2}$$

where the orientation is such that D is on the left-hand side of the increasing direction of ∂D.

Remark 2·1. The domain D is precisely the region of $\mathbb{C}$ where the function $\mathrm{e}^{i\omega(k)t}$ is a bounded function of k. Indeed,

$$\operatorname{Re}(i\omega(k)t) = -\operatorname{Im}(\omega(k))t \longrightarrow \left|\mathrm{e}^{i\omega(k)t}\right| = \left|\mathrm{e}^{-\operatorname{Im}(\omega(k))t}\right|$$

and as $t \geqslant 0$, the last term is bounded for any k such that $\operatorname{Im}(\omega(k)) > 0$.

It is easily shown, by induction on the degree of the polynomial $\operatorname{Im}(\omega(k))$, that $\operatorname{Im}\omega(k) = \operatorname{Im}(k)\tilde{p}(k)$, where $\tilde{p}(k)$ is a nonzero polynomial in $\operatorname{Re}(k)$, $\operatorname{Im}(k)$. Hence, for $k \in D$ (when $\operatorname{Im}\omega(k) > 0$),

$$\operatorname{Im}(k) \longrightarrow_{k\to\infty} \infty \Longrightarrow \operatorname{Im}\omega(k) \longrightarrow_{k\to\infty} \infty.$$

We shall use this fact implicitly when considering the behaviour of various functions at infinity.

It is shown in [**7**] that the components of D are simply connected and unbounded, and that there exists an $R > 0$ such that, outside the curve $|\omega(k)| = R$, ∂D is the union of smooth disjoint simple contours that approach asymptotically, as $k \to \infty$, the rays of the variety $\operatorname{Im}(k+\alpha)^n = 0$, where $\alpha = \alpha_{n-1}/(n\alpha_n)$. Moreover, the following lemma holds [**7**].

Lemma 2·1. *Let D_R be defined by*

$$D_R = \{k \in D : \ |\omega(k)| > R\}. \tag{2·3}$$

Let $D_{R,+}$ and $D_{R,-}$ denote the part of D_R in $\mathbb{C}^+$ and $\mathbb{C}^-$ respectively, i.e.

$$D_{R,+} = D_R \cap \mathbb{C}^+, \qquad D_{R,-} = D_R \cap \mathbb{C}^-.$$

If R is sufficiently large, D_R has n components, $D_{R,+}$ has N components, and $D_{R,-}$ has $n-N$ components, where N is given by (1·4).

In what follows, we denote by $D_{R,1}$, $D_{R,2}, \ldots, D_{R,N}$ the N components of $D_{R,+}$, and we denote by $D_{R,N+1}$, $D_{R,N+2}, \ldots, D_{R,n}$ the $n-N$ components of $D_{R,-}$.

The next proposition gives the integral representation of the solution $q(x,t)$; its proof can be found in [**5**].

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PROPOSITION 2·1 (Representation of solutions of equation (1·1)). *Assume that* $q(x,t)$ *is a sufficiently smooth solution of* (1·1). *Then* $q(x,t)$ *is given by*

$$q(x,t) = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} \mathrm{e}^{ikx-i\omega(k)t}\hat{q}_0(k)\,dk + \int_{\partial D_+} \mathrm{e}^{ikx-i\omega(k)t}\hat{Q}(0,k)\,dk \right.$$
$$\left. + \int_{\partial D_-} \mathrm{e}^{ik(x-L)-i\omega(k)t}\hat{Q}(L,k)\,dk\right\}, \tag{2·4}$$

where

$$\hat{q}_0(k) = \int_0^L \mathrm{e}^{-ikx} q_0(x)\,dx, \quad k \in \mathbb{C}, \tag{2·5}$$

the spectral functions $\hat{Q}(0,k)$, $\hat{Q}(L,k)$ *are defined by*

$$\hat{Q}(\alpha,k) = \sum_{j=1}^{n} \alpha_j(\hat{Q}_{j-1}(\alpha,k) + k\hat{Q}_{j-2}(\alpha,k) + \cdots + k^{j-1}\hat{Q}_0(\alpha,k)) \tag{2·6}$$

$$\hat{Q}_j(\alpha,k) = \int_0^T \mathrm{e}^{i\omega(k)t}(-i\partial_x)^j q(\alpha,t)\,dt, \qquad j=0,\ldots,n-1, \quad k \in \mathbb{C},$$

with $\alpha=0$ *or* $\alpha=L$, *and* D_+, D_- *the domains given in Definition* 2·1.

Moreover, the boundary values of $q(x,t)$ *satisfy the global relation*

$$\hat{Q}(0,k) - \mathrm{e}^{-ikL}\hat{Q}(L,k) = -\hat{q}_0(k) + \mathrm{e}^{i\omega(k)T}\hat{q}_T(k), \tag{2·7}$$

where

$$\hat{q}_T(k) = \int_0^L \mathrm{e}^{-ikx} q(x,T)\,dx. \tag{2·8}$$

We stress the fact that to obtain the above representation, which involves the initial condition as well as the values of the solution and of its derivatives at the boundary points $x=0$ and $x=L$, one must *assume* that a solution with sufficient differentiability exists. The question of when this existence can be estabilished is addressed in the next section.

3. *Step 2: admissible functions and the existence theorem*

We start by defining the notion of a set of admissible functions. This notion is important because, given such a set, it is possible to prove the existence theorem 3·1 stated below; the proof is given in [**5**].

Definition 3·1 (Admissible functions). Let $q_0(x) \in \mathbf{C}^\infty[0,L]$, and let $\omega(k)$ be defined by equation (1·2). Let $\{f_0(t),\ldots,f_{n-1}(t),g_0(t),..,g_{n-1}(t)\}$ be a set of $2n$ functions in $\mathbf{C}^\infty[0,T]$ such that $(-i\partial_x)^j q_0(0) = f_j(0)$ and $(-i\partial_x)^j q_0(L) = g_j(0)$, $j=0,\ldots,n-1$.

Let $\hat{q}_0(k)$ be given by equation (2·5), and define two functions $\hat{F}(k)$ and $\hat{G}(k)$ by

$$\hat{F}(k) = \sum_{j=1}^{n} \alpha_j(\hat{f}_{j-1}(k) + k\hat{f}_{j-2}(k) + \cdots + k^{j-1}\hat{f}_0(k)), \tag{3·1}$$

$$\hat{G}(k) = \sum_{j=1}^{n} \alpha_j(\hat{g}_{j-1}(k) + k\hat{g}_{j-2}(k) + \cdots + k^{j-1}\hat{g}_0(k)), \tag{3·2}$$

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where

$$\hat{f}_j(k) = \int_0^T \mathrm{e}^{i\omega(k)t} f_j(t)\, dt, \quad j = 0, \ldots, n-1, \; k \in \mathbb{C}, \tag{3·3}$$

$$\hat{g}_j(k) = \int_0^T \mathrm{e}^{i\omega(k)t} g_j(t)\, dt, \quad j = 0, \ldots, n-1, \; k \in \mathbb{C}. \tag{3·4}$$

The set of smooth functions $\{f_0(t), \ldots, f_{n-1}(t), g_0(t), \ldots g_{n-1}(t)\}$ is called *admissible with respect to* $q_0(x)$ if and only if the functions $\hat{F}(k)$ and $\hat{G}(k)$ satisfy the relation

$$\hat{F}(k) - \mathrm{e}^{-ikL}\hat{G}(k) = -\,\hat{q}_0(k) + \mathrm{e}^{i\omega(k)T}\hat{\gamma}(k), \quad k \in \mathbb{C}, \tag{3·5}$$

where

$$\hat{\gamma}(k) = \int_0^L \mathrm{e}^{-ikx}\gamma(x)\, dx,$$

and $\gamma(x)$ is some function belonging to the space $\mathbf{C}^\infty[0, L]$.

The functions $\hat{F}(k)$ and $\hat{G}(k)$ are called the spectral functions associated with the set $\{f_0, \ldots, f_{n-1}, g_0, \ldots, g_{n-1}\}$.

Remark 3·1. It follows from equation (2·7), setting $f_j(t) = (-i\partial_x)^j q(0,t)$, and $g_j(t) = (-i\partial_x)^j q(L,t)$, that the set of the boundary values of any smooth solution $q(x,t)$ of (1·1)–(1·3) is an admissible set with respect to the initial condition $q_0(x)$. In this case, $\gamma(x) = q(x, T)$.

Remark 3·2. Equation (3·5) can be written in the equivalent form

$$\mathrm{e}^{ikL}\hat{F}(k) - \hat{G}(k) = -\,\mathrm{e}^{ikL}\hat{q}_0(k) + \mathrm{e}^{ikL+i\omega(k)T}\hat{\gamma}(k). \tag{3·6}$$

Although both relations are well defined for all $k \in \mathbb{C}$, as $\mathrm{Im}(k) \to \infty$ the functions appearing in expression (3·5) are bounded only if $k \in D_-$, while those in (3·6) are bounded only if $k \in D_+$. Indeed, the functions $\hat{f}_j(k)$ and $\hat{g}_j(k)$, hence the spectral functions $\hat{F}(k)$ and $\hat{G}(k)$, are holomorphic, and bounded as $\mathrm{Im}(k) \to \infty$ only if $k \in D$ (see Remark 2·1); indeed, these functions vanish as $\mathrm{Im}(k) \to \infty$ for all $k \in D$. The exponential terms $\mathrm{e}^{\pm ikL}$ further restrict the boundedness region to $D \cap \mathbb{C}^\pm = D_\pm$, respectively.

These properties are a *necessary condition* for the functions $\hat{f}_j(k)$ and $\hat{g}_j(k)$ to correspond to smooth functions $f_j(t)$, $g_j(t)$ via equations (3·3)–(3·4).

Remark 3·3. Multiplying the global relations (3·5) and (3·6) by the exponential term $\mathrm{e}^{-i\omega(k)t}$ and integrating the resulting expression along ∂D_- and ∂D_+ respectively, we obtain

$$\int_{\partial D_-} \mathrm{e}^{-i\omega(k)t}[\hat{F}(k) - \mathrm{e}^{-ikL}\hat{G}(k)]\, dk = -\int_{\partial D_-} \mathrm{e}^{-i\omega(k)t}\hat{q}_0(k)\, dk,$$

$$\int_{\partial D_+} \mathrm{e}^{-i\omega(k)t}[\mathrm{e}^{ikL}\hat{F}(k) - \hat{G}(k)]\, dk = -\int_{\partial D_+} \mathrm{e}^{-i\omega(k)t}\mathrm{e}^{ikL}\hat{q}_0(k)\, dk.$$

Indeed, in both cases the integrand involving the term $\hat{\gamma}(k)$ is analytic and bounded in D_-, respectively D_+, hence its integral along the closed contour ∂D_-, respectively D_+, vanishes.

THEOREM 3·1 (Existence of solutions associated with an admissible set). *Assume that $q_0(x) \in \mathbf{C}^{\infty}[0,L]$, and that the set of smooth functions $\{f_j(t), g_j(t)\}$, $0 \leqslant j \leqslant n-1$, is admissible with respect to $q_0(x)$, see Definition* 3·1.

Let $\hat{q}_0(k)$ be defined by equation (2·5), *and let $\hat{F}(k)$ and $\hat{G}(k)$ be defined by equations* (3·1) *and* (3·2) *respectively.*

Define $q(x,t)$ as follows:

$$q(x,t) = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} \mathrm{e}^{ikx-i\omega(k)t}\hat{q}_0(k)\,dk + \int_{\partial D_+} \mathrm{e}^{ikx-i\omega(k)t}\hat{F}(k)\,dk \right.$$
$$\left. + \int_{\partial D_-} \mathrm{e}^{ik(x-L)-i\omega(k)t}\hat{G}(k)\,dk\right\}, \tag{3·7}$$

where $\omega(k)$ is given by (1·2), *and D_+ and D_- are defined by* (2·1).

Then:

(1) *$q(x,t)$ is the unique solution of equation* (1·1) *such that $t \to q(\cdot,t)$ is a $\mathbf{C}^{\infty}$ map from $[0,T]$ into $\mathbf{C}^{\infty}[0,L]$*;

(2) $q(x,0) = q_0(x)$;

(3) $(-i\partial_x)^j q(0,t) = f_j(t), \quad 0 \leqslant j \leqslant n-1$;

(4) $(-i\partial_x)^j q(L,t) = g_j(t), \quad 0 \leqslant j \leqslant n-1$.

The proof of the above theorem, given in [**5**], depends crucially on the relation (3·5), and on Remark 3·3.

Remark 3·4. Although the smooth solution $q(x,t)$ is unique, it is important to note that the spectral functions $\hat{F}(k)$ and $\hat{G}(k)$ appearing in its representation are *not unique*. The function $\hat{F}(k)$ is only defined modulo the class of holomorphic functions $\tilde{f}(k)$ with the property that $\mathrm{e}^{ikx-i\omega(k)t}\tilde{f}(k)$ is bounded for $k \in D_+$; indeed, for any such function, Cauchy's theorem implies that

$$\int_{\partial D_+} \mathrm{e}^{ikx-i\omega(k)t}\tilde{f}(k)\,dk = 0.$$

Similarly, $\hat{G}(k)$ is only defined modulo the class of holomorphic functions $\tilde{g}(k)$ with the property that $\mathrm{e}^{ik(x-L)-i\omega(k)t}\tilde{g}(k)$ is holomorphic and bounded in D_-.

4. *Step* 3: *well-posed boundary value problems*

Theorem 3·1 implies that a boundary value problem is well posed if it is possible to construct a set of admissible functions which includes the given boundary conditions. On the other hand, if a problem *is* well posed, hence if it has a unique smooth solution $q(x,t)$ satisfying the given conditions, it follows from Proposition 2·1 that the boundary values of the solution are a set of admissible functions (see Remark 3·1).

Hence Theorem 1·1 is a consequence of the following result.

PROPOSITION 4·1. *Let the assumptions of Theorem* 1·1 *be given. Then it is possible to define a set of functions $\{f_1,\ldots,f_n,g_1,\ldots,g_n\}$, admissible with respect to $q_0(x)$, and such that*

$$f_{j_1}(t) = u_{j_1}(t), \quad j_1 \in J_1, \qquad g_{j_2}(t) = v_{j_2}(t), \quad j_2 \in J_2,$$

if and only if $|J_1| = N$ and $|J_2| = n-N$.

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In the present section we sketch the line of the proof of Proposition 4·1; the complete proof is given in Section 6.

We start by giving some auxiliary definitions, and proving a preliminary lemma.

Definition 4·1. Let $\omega(k)$ be the dispersion relation of equation (1·1), given by (1·2).

(i) The polynomials $\omega_{n-j}(k)$, $j=1,\ldots,n$, are defined by

$$\omega_{n-j}(k)=\alpha_j+k\alpha_{j+1}+\cdots+\alpha_n k^{n-j}. \tag{4·1}$$

(ii) The map $\lambda_{l,m}:\overline{D_{R,m}}\to\mathbb{C}$ is the biholomorphic map defined by

$$\omega(\lambda_{l,m}(k))=\omega(k),\forall\, k\in\overline{D_{R,m}}. \tag{4·2}$$

It is proved in [**7**] that $\lambda_{l,m}(\overline{D_{R,m}})=\overline{D_{R,l}}$. Since $D_{R,l}\subset D_+$ if $1\leqslant l\leqslant N$, and $D_{R,l}\subset D_-$ if $N+1\leqslant l\leqslant n$, the map $\lambda_{l,m}(k)$ satisfies the following:

$$1\leqslant m\leqslant N:\qquad \begin{cases}\lambda_{l,m}(k)\in D_+ & \text{if}\quad 1\leqslant l\leqslant N,\\ \lambda_{l,m}(k)\in D_- & \text{if}\quad N+1\leqslant l\leqslant n.\end{cases} \tag{4·3}$$

In addition, the definition of $\lambda_{l,m}$ implies

$$\lambda_{l,m}(\lambda_{m,j}(k))=\lambda_{l,j}(k);\qquad \lambda_{l,m}(k)\sim \mathrm{e}^{i(l-m)2\pi/n}k,\quad k\to\infty.$$

In particular, $\lambda_{1,m}(k)+\cdots+\lambda_{n,m}(k)\sim k$, as $\zeta_{1,m}+\cdots+\zeta_{n,m}=1$, $\zeta_{l,m}=\mathrm{e}^{i(l-m)2\pi/n}k$.

Remark 4·1. The maps $\lambda_{l,m}$ are defined as the transformations of the complex k plane into itself that *leave the dispersion relation* $\omega(k)$ *invariant*. In particular, since the spectral functions $\hat{F}$ and $\hat{G}$ depend on k only through the function $\omega(k)$, $\hat{F}(\lambda_{l,m}(k))=\hat{F}(k)$ and $\hat{G}(\lambda_{l,m}(k))=\hat{G}(k)$. See also the proof of Lemma 4·1 below.

Using the definition (3·1)–(3·2) of the spectral functions, and the global relations (3·6) for $k\in D_{R,+}$ and (3·5) for $k\in D_{R,-}$, we can prove the following lemma.

LEMMA 4·1. *Let $q_0(x)\in \mathbf{C}^\infty[0,L]$ be given, and let the set of smooth functions $\{f_j(t),g_j(t)\}$, $0\leqslant j\leqslant n-1$ be admissible with respect to $q_0(x)$.*

For $k\in D_+$, the functions $\{\hat{f}_j(k),\hat{g}_j(k)\}$, $0\leqslant j\leqslant n-1$, defined by (3·3)–(3·4), *satisfy for some m, $1\leqslant m\leqslant N$, the system of equations*

$$\begin{cases}\mathrm{e}^{i\lambda_{l,m}L}\sum_{j=1}^n w_{n-j}(\lambda_{l,m})\hat{f}_{j-1}(k)-\sum_{j=1}^n w_{n-j}(\lambda_{l,m})\hat{g}_{j-1}(k)\\ \quad=-\mathrm{e}^{i\lambda_{l,m}L}\hat{q}_0(\lambda_{l,m})+\mathrm{e}^{i\lambda_{l,m}L+iw(k)T}\hat{\gamma}(\lambda_{l,m}),\qquad 1\leqslant l\leqslant N,\\ \sum_{j=1}^n w_{n-j}(\lambda_{l,m})\hat{f}_{j-1}(k)-\mathrm{e}^{-i\lambda_{l,m}L}\sum_{j=1}^n w_{n-j}(\lambda_{l,m})\hat{g}_{j-1}(k)\\ \quad=-\hat{q}_0(\lambda_{l,m})+\mathrm{e}^{iw(k)T}\hat{\gamma}(\lambda_{l,m}),\qquad (N+1)\leqslant l\leqslant n,\end{cases} \tag{4·4}$$

where the polynomials $\omega_{n-j}(k)$ are defined by (4·1).

Proof. Assume that $k\in D_{R,+}$, with $D_{R,+}$ defined in Lemma 2·1, and R so large, that $D_{R,+}$ has N connected components. Since $D_{R,+}=\bigcup_{m=1}^N D_{R,m}$, there is an m, $1\leqslant m\leqslant N$, such that $k\in D_{R,m}$. We fix this value of m for the rest of the proof.

By definition of the maps $\lambda_{l,m}$, this implies that for $1\leqslant l\leqslant N$, $\lambda_{l,m}(k)\in D_{R,+}$, while for $N+1\leqslant l\leqslant n$, $\lambda_{l,m}(k)\in D_{R,-}$. Hence, when evaluated at $\lambda_{l,m}(k)$, the global relation (3·6) holds and is bounded for $1 \leqslant l \leqslant N$, while the global relation (3·5) holds and is bounded for $N+1 \leqslant l \leqslant n$. Note also that the definition $\lambda_{l,m}$ implies

$$\hat{f}_j(k) = \hat{f}_j(\lambda_{l,m}(k)), \quad \hat{g}_j(k) = \hat{g}_j(\lambda_{l,m}(k)).$$

Using the definition of $\omega_{n-j}(k)$ we can write the spectral functions $\hat{F}(k)$ and $\hat{G}(k)$ given by (3·1)–(3·2) in the form

$$\hat{F}(k) = \sum_{j=1}^{n} w_{n-j}(k)\hat{f}_{j-1}(k); \quad \hat{G}(k) = \sum_{j=1}^{n} w_{n-j}(k)\hat{g}_{j-1}(k).$$

The evaluation of the global conditions (3·6) and (3·5) at the various $\lambda_{l,m}(k)$ then yields the system (4·4).

To end the proof, we only need to show that the restriction to the case that $k \in D_{R,+}$ entails no loss of generality. Indeed, note that both relations (3·5) and (3·6) are bounded in any bounded subset of $\mathbb{C}$. Thus it is sufficient to verify the validity of the statement as $k \to \infty$, or equivalently, for $k \in D_{R,+}$, for any $R > 0$.

In what follows, we always assume $k \in D_{R,m}$ for some fixed value of $m \in [1, N]$; by definition, this implies that $D_{R,m} \subset \mathbb{C}^+$. Without loss of generality, we take $m = 1$, and drop the relevant subscript; henceforth, λ_l will stand for $\lambda_{l,1}$, and D_R for $D_{R,1}$.

Equations (4·4) relate the $2n$ functions $\hat{f}_0(k), \ldots, \hat{f}_{n-1}(k)$, $\hat{g}_0(k), \ldots, \hat{g}_{n-1}(k)$; note that they also contain the unspecified function $\hat{\gamma}(k)$; however, for the moment we consider this function as known.

For $1 \leqslant l \leqslant n$, set

$$E_l = \mathrm{e}^{i\lambda_l(k)L}. \tag{4·5}$$

In the limit as $\mathrm{Im}(k) \to \infty$ in D_+, E_l is bounded if $1 \leqslant l \leqslant N$ while E_l^{-1} is bounded if $N+1 \leqslant l \leqslant n$.

We can write the system (4·4) in the matrix form

$$A(k)(\hat{f}_{j_1}, \hat{g}_{j_2})_{j_1,j_2} = b(k) + \mathrm{e}^{iw(k)T}c(k), \tag{4·6}$$

where

(i) $(\hat{f}_{j_1}, \hat{g}_{j_2})_{j_1,j_2}$ is the $2n$-vector containing the functions $\{\hat{f}_{j_1}\}$ and $\{\hat{g}_{j_2}\}$, $0 \leqslant j_i \leqslant n-1$;

(ii) $b(k)$ is the n-vector containing the values of the function $\hat{q}_0(k)$ given in Definition 3·1, and defined by

$$b(k) = (-E_1\hat{q}_0(\lambda_1, \ldots, -E_N\hat{q}_0(\lambda_N), -\hat{q}_0(\lambda_{N+1}), \ldots, -\hat{q}_0(\lambda_n))^{\tau}, \tag{4·7}$$

where $\lambda_l(k)$ is the map given by (4·2), with $m = 1$;

(iii) $c(k)$ is the n-vector containing the values of the function $\hat{\gamma}(k)$ given in Definition 3·1, and defined by

$$c(k) = (E_1\hat{\gamma}(\lambda_1), \ldots, E_N\hat{\gamma}(\lambda_N), \hat{\gamma}(\lambda_{N+1}), \ldots, \hat{\gamma}(\lambda_n))^{\tau}; \tag{4·8}$$

(iv) A is the $2n \times n$ matrix $A = (A_1|A_2)$, where the $n \times n$ blocks A_1 and A_2 are

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defined by

$$A_1 = \begin{pmatrix} E_1\,\omega_{n-1}(\lambda_1) & \dots & E_1\,\omega_0(\lambda_1) \\ E_2\,\omega_{n-1}(\lambda_2) & \dots & E_2\,\omega_0(\lambda_2) \\ \cdot & \dots & \cdot \\ \cdot & \dots & \cdot \\ E_N\,\omega_{n-1}(\lambda_N) & \dots & E_N\,\omega_0(\lambda_N) \\ \omega_{n-1}(\lambda_{N+1}) & \dots & \omega_0(\lambda_{N+1}) \\ \cdot & \dots & \cdot \\ \cdot & \dots & \cdot \\ \omega_{n-1}(\lambda_n) & \dots & \omega_0(\lambda_n) \end{pmatrix} \tag{4·9}$$

$$A_2 = \begin{pmatrix} -\omega_{n-1}(\lambda_1) & \dots & -\omega_0(\lambda_1) \\ -\omega_{n-1}(\lambda_2) & \dots & -\omega_0(\lambda_2) \\ \cdot & \dots & \cdot \\ \cdot & \dots & \cdot \\ -\omega_{n-1}(\lambda_N) & \dots & -\omega_0(\lambda_N) \\ -E_{N+1}^{-1}\,\omega_{n-1}(\lambda_{N+1}) & \dots & -E_{N+1}^{-1}\,\omega_0(\lambda_{N+1}) \\ \cdot & \dots & \cdot \\ \cdot & \dots & \cdot \\ -E_n^{-1}\,\omega_{n-1}(\lambda_n) & \dots & -E_n^{-1}\,\omega_0(\lambda_n) \end{pmatrix} \tag{4·10}$$

Remark 4·2. In general, if p of the functions $\hat{f}_0, \ldots, \hat{f}_{n-1}, \hat{g}_0, \ldots, \hat{g}_{n-1}$ are known, from equation (4·6) we obtain a $(2n-p)\times n$ system for the remaining functions in the set. Thus to be able to solve for the unknown functions in the generic case, p should be equal to n, i.e. a total of n functions must be prescribed; in this case, we obtain a $n\times n$ square system for the remaining n unknown functions.

In the context of boundary value problems, the set $f_0, \ldots, f_{n-1}, g_0, \ldots, g_{n-1}$ is the set of boundary values of the solution $q(x,t)$ and of its derivatives. As noted previously (see Remark 3·2), the functions $\hat{f}_{j(k)}$, $\hat{g}_j(k)$, $0 \leqslant j \leqslant n-1$, obtained via the transformation (3·3) and (3·4), are holomorphic functions of k which vanish as $\mathrm{Im}(k)\to\infty$ for $k \in D_R$ (indeed, for $k \in D$, hence this is true for every m, $1 \leqslant m \leqslant n$, see Remark 2·1). In view of this fact, when n boundary condition are prescribed, we will only be interested in those solutions of the resulting $n\times n$ system which are a set of holomorphic functions of k, vanishing as $\mathrm{Im}(k)\to\infty$.

The proof of the main result relies on the following lemma.

LEMMA 4·2. *For fixed $k \in D_R$, consider the linear system* (4·12) *obtained by a choice of $n-N_0$ columns of A_1 and of N_0 columns of A_2. If $N_0 = N$, this system has a unique solution $\{h_p(k),\ 0 \leqslant p \leqslant n-1\}$, where each $h_p(k)$ is a meromorphic function in D_R, such that $\lim_{\mathrm{Im}(k)\to\infty} |h_p(k)| = 0$, for k inside D_R. In addition, there exists an appropriate choice of the function $\gamma(x)$ for which the singularity at the poles of the functions $h_p(k)$ is removable; for this choice, it is hence possible to construct a solution of system* (4·12) *which is a set of bounded holomorphic functions, vanishing as* $\mathrm{Im}(k)\to\infty$ *in D_R.*

If $N_0 \neq N$, for any choice of $\gamma(x)$ the unique solution of system (4·12) *is a set of functions of which at least one has an essential singularity as* $\mathrm{Im}(k)\to\infty$, *$k \in D_R$.*

We prove this lemma in Section 5; however we show here how this result yields the proof of Proposition 4·1.

Proof of Proposition 4·1. Consider the boundary value problem obtained by prescribing

$$(-i\partial_x)^{j_1} q(0,t) = u_{j_1}(t), \quad (-i\partial_x)^{j_2} q(L,t) = v_{j_2}(t).$$

We assume that the indices (j_1, j_2) belong to some index set $J_1 \times J_2$, with $J_i \subset \{0, \ldots, n-1\}$; motivated by Remark 4·2, we assume also that the cardinality of $|J_1| + |J_2|$ is equal to n. Let

$$\hat{f}_{j_1}(k) = \int_0^T \mathrm{e}^{i\omega(k)t} u_{j_1}(t)\, dt, \qquad \hat{g}_{j_2}(k) = \int_0^T \mathrm{e}^{i\omega(k)t} v_{j_2}(t)\, dt,$$

$k \in \mathbb{C}$, $(j_1, j_2) \in J_1 \times J_2$.

Moving to the right-hand side of the system (4·4) the terms containing the functions $\hat{f}_{j_1}$ and $\hat{g}_{j_2}$ corresponding to $(j_1, j_2) \in J_1 \times J_2$, for $k \in D_R$ we obtain the $n \times n$ system

$$\begin{cases} \mathrm{e}^{i\lambda_l L} \sum_{j_1' \in J_1'} w_{n-j_1'}(\lambda_l) \hat{f}_{j_1'-1}(k) - \sum_{j_2' \in J_2'} w_{n-j_2'}(\lambda_l) \hat{g}_{j_2'-1}(k) \\ \quad = -\mathrm{e}^{i\lambda_l L} \sum_{j_1 \in J_1} w_{n-j_1}(\lambda_l) \hat{f}_{j_1-1}(k) + \sum_{j_2 \in J_2} w_{n-j_2}(\lambda_l) \hat{g}_{j_2-1}(k) \\ \quad\quad - \mathrm{e}^{i\lambda_l L} \hat{q}_0(\lambda_l) + \mathrm{e}^{i\lambda_l L + iw(k)T} \hat{\gamma}(\lambda_l), \qquad 1 \leqslant l \leqslant N, \\ \sum_{j_1' \in J_1'} w_{n-j_1'}(\lambda_l) \hat{f}_{j_1'-1}(k) - \mathrm{e}^{-i\lambda_l L} \sum_{j_2' \in J_2'} w_{n-j_2'}(\lambda_l) \hat{g}_{j_2'-1}(k) \\ \quad = -\sum_{j_1 \in J_1} w_{n-j_1}(\lambda_l) \hat{f}_{j_1-1}(k) + \mathrm{e}^{-i\lambda_l L} \sum_{j_2 \in J_2} w_{n-j_2}(\lambda_l) \hat{g}_{j_2-1}(k) \\ \quad\quad - \hat{q}_0(\lambda_l) + \mathrm{e}^{iw(k)T} \hat{\gamma}(\lambda_l), \qquad (N+1) \leqslant l \leqslant n, \end{cases} \tag{4·11}$$

where $J_i' = \{0, \ldots, n-1\} \setminus J_i$, $i = 1, 2$. In matrix form, this can be written as

$$\tilde{A}(\hat{f}_{j_1'}, \hat{g}_{j_2'}) = K(\hat{f}_{j_1}, \hat{g}_{j_2}) + b(k) + \mathrm{e}^{i\omega(k)T} c(k), \tag{4·12}$$

where $(\hat{f}_{j_1'}, \hat{g}_{j_2'})$ is the n-vector containing the unknown functions $\hat{f}_{j_1'}$, $\hat{g}_{j_2'}$ corresponding to indices $j_1' \in J_1'$ and $j_2' \in J_2'$, the $n \times n$ matrix $\tilde{A}$ is obtained by choosing the columns of A_1 indexed by the set of j_1's and the columns of A_2 indexed by the set of j_2's, and $K(\hat{f}_{j_1}, \hat{g}_{j_2})$ is an n-vector constructed with the known functions $\hat{f}_{j_1}$, $\hat{g}_{j_2}$.

If we can prove that there exists a unique solution of this system, and that this solution is a set of *holomorphic* functions which vanish as $\mathrm{Im}(k) \to \infty \; \forall\, k \in D_R$, we can use these functions to construct the spectral functions $\hat{F}(k)$ and $\hat{G}(k)$ according to the formulas (3·1)–(3·2). Using $\hat{F}(k)$, and $\hat{G}(k)$, we define the function $q(x,t)$ by equation (3·7). The boundary values of this function and of its derivatives are then by construction admissible, and the functions $\hat{f}_j(k)$, $\hat{g}_j(k)$ constructed from these boundary values via the formulae (3·3) and (3·4) coincide with the solution of system (4·4), up to holomorphic and decaying additive factors. This means that there exist functions $f_{j_1'}$, $g_{j_2'}$ such that the set of functions $f_0, \ldots, f_{n-1}, g_0, \ldots, g_{n-1}$ obtained by considering this solution and its boundary values is by construction admissible![1] Hence, by Theorem 3·1, in the above case the given boundary value problem is well posed. By Remark 3·3, this implies also that the term containing the function $\hat{\gamma}(k)$ does not contribute to the representation (3·7) of the solution.

Conversely, if a problem is well posed, then its boundary values are a set of $2n$ smooth functions; the transforms of these functions given by the formula (3·3) for the

[1] These functions are given by $f_{j_1'} = (-i\partial_x)^{j_1'} q(0,t)$ and $g_{j_2'} = (-i\partial_x)^{j_2'} q(L,t)$.

first n or (3·4) for the last n, are holomorphic functions of k, bounded and decaying for $k \in D_R$, which satisfy a system of the form (4·4) with $\gamma(x) = q(x, T)$.

Hence the identification of the boundary conditions which yield a set of admissible functions, for a given equation (1·1) and initial condition (1·3), is reduced to finding conditions which guarantee that, given a specific set of n holomorphic bounded functions, the system (4·12) has as its unique solution a set of n bounded holomorphic functions, vanishing as $\mathrm{Im}(k) \to \infty$ in D_R.

Assume that the result of Lemma 4·2 holds. Suppose that $\{f_1, \ldots, f_n, g_1, \ldots, g_n\}$ are a set of admissible functions with respect to the given initial condition $q_0(x)$, and that

$$f_{j_1}(t) = u_{j_1}(t), \quad j_1 \in J_1, \qquad g_{j_2}(t) = v_{j_2}(t), \quad j_2 \in J_2.$$

By definition of admissibility, the functions $\hat{f}_j(k)$, $\hat{g}_j(k)$ defined by equations (3·3) and (3·4) respectively, satisfy the global conditions (3·5) and (3·6), hence they satisfy the system (4·6) constructed using these conditions. Moving to the right-hand side of the system the terms corresponding to the given boundary conditions, we obtain a $n \times n$ system of the form (4·12), which is satisfied by the functions $\hat{f}_j$, $\hat{g}_j$. Moreover, by construction these functions have all the required properties. By Lemma 4·2, this can be the case only if the matrix A_n defining the $n \times n$ system (4·6) is obtained by a choice of $n - N$ columns of A_1 and of N columns of A_2, so that the unknown functions in the system are $n - N$ of the $\hat{f}_j$'s and N of the $\hat{g}_j$'s; i.e. only if $|J_1| = N$ and $|J_2| = n - N$.

Conversely, let $|J_1| = N$ and $|J_2| = n - N$. By the result of Lemma 4·2, the system obtained by choosing the N columns of A_1 and $n - N$ columns of A_2 corresponding to the indices in J_1 and J_2, respectively, has as solution a set of n functions $\{\hat{h}_p(k)\}_{p=1}^{n}$, obtained by specifying the function $\gamma(x)$ in a such a way that these functions are bounded and holomorphic in D_R.

We then consider the set $\{\hat{f}_0, \ldots, \hat{f}_{n-1}, \hat{g}_0, \ldots, \hat{g}_{n-1}\}$, where

$$\hat{f}_{j_1}(k) = \int_0^T \mathrm{e}^{i\omega(k)t} u_{j_1}(t)\, dt, \quad \text{if } j_1 \in J_1,$$
$$\hat{g}_{j_2}(k) = \int_0^T \mathrm{e}^{i\omega(k)t} v_{j_2}(t)\, dt, \quad \text{if } j_2 \in J_2,$$

and the remaining n gaps in the sequence of functions are sequentially filled by the functions $\hat{h}_p(k)$, $p = 1, \ldots, n$. By construction, the corresponding set $\{f_0, \ldots, f_{n-1}, g_0, \ldots, g_{n-1}\}$ is admissible with respect to $q_0(x)$. Thus, assuming the validity of Lemma 4·2, the proof is established.

5. *Examples*

Before proving Lemma 4·2, we consider in this section two examples, for a second and a third order equation, which illustrate the ideas involved.

Example 5·1. The equation $iq_t + q_{xx} = 0$.

In this case, $\omega(k) = k^2$; thus $n = 2$, so that two boundary conditions must be prescribed in general for a well-posed boundary value problem, and $N = 1$, and one condition must be prescribed at each end. The polynomials $w_j(k)$ are given by

$$\omega_0(k) = 1; \ \omega_1(k) = k.$$

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The domains D_+ and D_- have only one simply connected component, hence $D_+ = D_1$ and $D_- = D_2$; explicitly, D_1 is the first and D_2 the third quadrant of the complex k-plane. The map $\lambda_{2,1} : D_1 \to D_2$ is given by $\lambda_{2,1}(k) = -k$.

For the rest of this example, we fix $k \in D_1$. The vectors $b(k)$ and $c(k)$ are

$$b(k) = (-\mathrm{e}^{ikL}\hat{q}_0(k), -\hat{q}_0(-k))^\tau, \tag{5·1}$$

$$c(k) = (\mathrm{e}^{ikL}\hat{\gamma}(k), \hat{\gamma}(-k))^\tau, \tag{5·2}$$

and the matrices A_1 and A_2 are given by

$$A_1 = \begin{pmatrix} k\mathrm{e}^{ikL} & \mathrm{e}^{ikL} \\ -k & 1 \end{pmatrix} \tag{5·3}$$

$$A_2 = \begin{pmatrix} -k & -1 \\ k\mathrm{e}^{ikL} & -\mathrm{e}^{ikL} \end{pmatrix}. \tag{5·4}$$

Thus

$$\det(A_1) = \det(A_2) = 2k\mathrm{e}^{ikL}.$$

The four possibilities of 2×2 matrix $\tilde{A}$ obtained by choosing one column of A_1 and one column of A_2 are the following:

$$(i) \qquad \tilde{A}_i = \begin{pmatrix} \mathrm{e}^{ikL} & -1 \\ 1 & -\mathrm{e}^{ikL} \end{pmatrix}; \quad \det(\tilde{A}_i) = 1 - \mathrm{e}^{2ikL};$$

$$(ii) \qquad \tilde{A}_{ii} = \begin{pmatrix} k\mathrm{e}^{ikL} & -k \\ -k & k\mathrm{e}^{ikL} \end{pmatrix}; \quad \det(\tilde{A}_{ii}) = k^2(\mathrm{e}^{2ikL} - 1);$$

$$(iii) \qquad \tilde{A}_{iii} = \begin{pmatrix} \mathrm{e}^{ikL} & -k \\ 1 & k\mathrm{e}^{ikL} \end{pmatrix}; \quad \det(\tilde{A}_{iii}) = k(\mathrm{e}^{2ikL} + 1);$$

$$(iv) \qquad \tilde{A}_{iv} = \begin{pmatrix} k\mathrm{e}^{ikL} & -1 \\ -k & -\mathrm{e}^{ikL} \end{pmatrix}; \quad \det(\tilde{A}_{iv}) = -k(1 + \mathrm{e}^{2ikL}).$$

We now consider all possibilities of prescribing two boundary conditions. Prescribing two conditions at $x = 0$ corresponds to choosing the two columns of A_2, thus in this case $\tilde{A} = A_2$; similarly, if both conditions are prescribed at $x = L$, then the matrix $\tilde{A} = A_1$.

If one condition is prescribed at both ends, then $\tilde{A}$ is one of the four matrices $\tilde{A}_i$, $\tilde{A}_{ii}$, $\tilde{A}_{iii}$ or $\tilde{A}_{iv}$ given above. Explicitly, these matrices correspond to prescribing the following boundary conditions:

$$\begin{array}{llll}
\text{(i)} & \Longrightarrow & -iq_x(0,t) = f_1(t), & -iq_x(L,t) = g_1(t); \\
\text{(ii)} & \Longrightarrow & q(0,t) = f_0(t), & q(L,t) = g_0(t); \\
\text{(iii)} & \Longrightarrow & -iq_x(0,t) = f_1(t), & q(L,t) = g_0(t); \\
\text{(iv)} & \Longrightarrow & q(0,t) = f_0(t), & -iq_x(L,t) = g_1(t).
\end{array}$$

We now analyse the solution of the system arising in correspondence with each one of these possibilities.

(a) $\tilde{\mathbf{A}} = \mathbf{A_i}$: in this case,

$$\hat{f}_1(k) = \int_0^T \mathrm{e}^{ik^2t} f_1(t)\, dt, \quad \hat{g}_1(k) = \int_0^T \mathrm{e}^{ik^2t} g_1(t)\, dt$$

are given, and the unknown functions to be determined through solving the

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system are $\hat{f}_0(k)$ and $\hat{g}_0(k)$. The system they must satisfy is given explicitly by

$$\tilde{A}(k)(\hat{f}_0(k), \hat{g}_0(k))^\tau = \beta(k) + \mathrm{e}^{ik^2T} c(k), \tag{5·5}$$

where the column vector $\beta(k)$ is given by

$$\beta(k) == \begin{pmatrix} -\mathrm{e}^{ikL}(\hat{q}_0(k) + \hat{f}_1(k)) + \hat{g}_1(k) \\ -\hat{q}_0(-k) - \hat{f}_1(k) + \mathrm{e}^{ikL}\hat{g}_1(k) \end{pmatrix},$$

and $c(k)$ is given by equation (5·2). Since $\det(\tilde{A}) = 1 - \mathrm{e}^{2ikL}$, by Cramer's rule the solution is given by

$$\hat{f}_0(k) = \frac{1}{1 - \mathrm{e}^{2ikL}} \big(\det B_1(k) + \mathrm{e}^{ik^2T} \det C_1(k)\big), \tag{5·6}$$

$$\hat{g}_0(k) = \frac{1}{1 - \mathrm{e}^{2ikL}} \big(\det B_2(k) + \mathrm{e}^{ik^2T} \det C_2(k)\big), \tag{5·7}$$

where $B_1(k)$ and $B_2(k)$ are the 2×2 matrices obtained by substituting the first and second column of $\tilde{A}$ respectively by the column vector $\beta(k) = (b_1, b_2)^\tau$, and $C_1(k)$ and $C_2(k)$ are the 2×2 matrices obtained by substituting the first and second column of $\tilde{A}$ respectively by the column vector $c(k) = (c_1, c_2)^\tau$. Namely,

$$\begin{aligned} \det B_1(k) &= \mathrm{e}^{i2kL}(\hat{q}_0(k) - \hat{f}_1(k)) - \hat{q}_0(-k) - \hat{f}_1(k), \\ \det B_2(k) &= (\mathrm{e}^{i2kL} - 1)\hat{g}_1(k)) - \mathrm{e}^{ikL}(\hat{q}_0(-k) - \hat{q}_0(k)) \\ \det C_1(k) &= \hat{\gamma}(-k) - \mathrm{e}^{i2kL}\hat{\gamma}(k), \\ \det C_2(k) &= \mathrm{e}^{ikL}\hat{\gamma}(-k) - \mathrm{e}^{ikL}\hat{\gamma}(k). \end{aligned}$$

For $k \in D_1$, all terms in the expressions above defining $\hat{f}_0$ and $\hat{g}_0$ are bounded except at the zeros $\{k_h = h\pi/L\}_{h \in \mathbf{Z}}$ of $\det(\tilde{A})$, and vanish when $k \to \infty$.

Because of the pole singularities at the points k_h, these functions are not generically holomorphic; however, the terms c_1 and c_2, containing the arbitrary function $\hat{\gamma}(k)$, can be specified in such a way that the contribution of these poles is subtracted, and thus to define holomorphic functions. Indeed, choose c_1, c_2 so that $\mathrm{e}^{ik_h^2T} c_i(k_h) = -b_i(k_h)$, $i = 1, 2$, $h \in \mathbf{Z}$. Then equations (5·6)–(5·7) can be rewritten as

$$\begin{aligned} \hat{f}_0(k) = {} & \frac{-\mathrm{e}^{ikL}\big(b_1(k) + \mathrm{e}^{ik^2T}c_1(k)\big) + \big(b_2(k) + \mathrm{e}^{ik^2T}c_2(k)\big)}{1 - \mathrm{e}^{2ikL}} \\ & - \frac{\sum_h \big(-\mathrm{e}^{ik_hL}\big(b_1(k_h) + \mathrm{e}^{ik_h^2T}c_1(k_h)\big) + \big(b_2(k_h) + \mathrm{e}^{ik_h^2T}c_2(k_h)\big)\big)}{1 - \mathrm{e}^{2ikL}}, \\ \hat{g}_0(k) = {} & \frac{-\mathrm{e}^{ikL}(b_2(k) + \mathrm{e}^{ik^2T}c_2(k)) - (b_1(k) + \mathrm{e}^{ik^2T}c_1(k))}{1 - \mathrm{e}^{2ikL}} \\ & - \frac{\sum_h \big(-\mathrm{e}^{ik_hL}\big(b_2(k_h) + \mathrm{e}^{ik_h^2T}c_2(k_h)\big) - \big(b_1(k_h) + \mathrm{e}^{ik_h^2T}c_1(k_h)\big)\big)}{1 - \mathrm{e}^{2ikL}}. \end{aligned}$$

The expressions above represent two holomorphic functions with the desired asymptotic properties. Moreover, multiplying $\hat{f}_0(k)$ by $\mathrm{e}^{ikx - ik^2t}$, and integrating along ∂D_1, and similarly, multiplying $\hat{g}_0(k)$ by $\mathrm{e}^{ik(L-x) - ik^2t}$, and integrating along ∂D_2, it is easy to show that the terms containing c_1 and c_2 give no contribution, as the overall exponential term appearing in these functions is bounded in the whole of the region enclosed by the relevant integration contour. Hence the solution $q(x,t)$ given in terms of the spectral functions $\hat{F}(k)$, $\hat{G}(k)$ is uniquely defined.

(a') $\tilde{\mathbf{A}} = \mathbf{A_{ii}}$, $\tilde{\mathbf{A}} = \mathbf{A_{iii}}$, or $\tilde{\mathbf{A}} = \mathbf{A_{iv}}$: these cases are all analogous to case (a), and are treated in the same way.

(b) $\tilde{\mathbf{A}} = \mathbf{A_1}$: in this case, the unknown functions to be determined are $\hat{f}_0(k)$ and $\hat{f}_1(k)$. We assume without loss of generality that the given initial condition $q_0(x)$ is equal to zero. Hence $\hat{q}_0(k) = 0$, the vector $\beta(k)$ is given by

$$\beta(k) = \begin{pmatrix} k\hat{g}_0(k) + \hat{g}_1(k) \\ -\mathrm{e}^{ikL}k\hat{g}_0(k) + \mathrm{e}^{ikL}\hat{g}_1(k) \end{pmatrix},$$

and $c(k)$ given by equation (5·2); $\det(\tilde{A}) = 2k\mathrm{e}^{ikL}$. The solution is given by

$$\hat{f}_0(k) = \frac{\mathrm{e}^{-ikL}}{2k}\big(\det B_1(k) + \mathrm{e}^{ik^2T}\det C_1(k)\big),$$
$$\hat{f}_1(k) = \frac{\mathrm{e}^{-ikL}}{2k}\big(\det B_2(k) + \mathrm{e}^{ik^2T}\det C_2(k)\big),$$

where $B_1(k)$, $B_2(k)$, $C_1(k)$ and $C_2(k)$ are obtained as in case (a), and are given by

$$\det B_1(k) = -\mathrm{e}^{ikL}(k\hat{g}_0(k) + \hat{g}_1(k) + \mathrm{e}^{i2kL}\,(k\hat{g}_0(k) - \hat{g}_1(k)))\,,$$
$$\det B_2(k) = k(-\mathrm{e}^{i2kL}(k\hat{g}_0(k) + \hat{g}_1(k)) + k\hat{g}_0(k) + \hat{g}_1(k)),$$
$$\det C_1(k) = \mathrm{e}^{ikL}(\hat{\gamma}(k) - \hat{\gamma}(-k))$$

$$\det C_2(k) = k\mathrm{e}^{ikL}(\hat{\gamma}(-k) + \hat{\gamma}(k)).$$

Consider the term

$$\frac{\mathrm{e}^{ik^2T}\det(C_1)}{\det\tilde{A}} = \frac{\mathrm{e}^{ik^2T}}{2k\mathrm{e}^{ikL}}\mathrm{e}^{ikL}(\hat{\gamma}(k) - \hat{\gamma}(-k)) = \mathrm{e}^{ik^2T}\frac{\hat{\gamma}(k) - \hat{\gamma}(-k)}{2k}.$$

(We can assume that $\hat{\gamma}(k) \neq \hat{\gamma}(-k)$; if not, the argument that follows applies to the term $\mathrm{e}^{ik^2T}\det(C_2)/\det\tilde{A}$.)

By its definition, the function $\mathrm{e}^{ik^2T}\mathrm{e}^{ikL}\hat{\gamma}(k)$ is bounded as $k \to \infty$, $k \in D_+$. However, when the exponential e^{ikL} cancels out with the same term in the denominator, the resulting function $\mathrm{e}^{ik^2T}\hat{\gamma}(k)/2k$ has an essential singularity as $\mathrm{Im}(k) \to \infty$. Hence the function $\hat{f}_0(k)$ is *not* a bounded holomorphic function, and no admissible set can be constructed that contains $f_0(t)$ which transforms to $\hat{f}_0(k)$ by way of the formula (3·3). This implies that no problem obtained by assigning two conditions at $x = L$ and no condition at $x = 0$ can be well posed.

(b') $\tilde{\mathbf{A}} = \mathbf{A_2}$: this case is analogous to the previous one and does not yield a well-posed boundary value problem.

Example 5·2. The equation $q_t + q_{xxx} = 0$.

In this case, $\omega(k) = -k^3$; thus $n = 3$ and three boundary conditions must be prescribed. Also, $a_3 = -1 < 0$, hence $N = 1$, and a boundary value problem is well posed if and only if one condition is prescribed at $x = 0$ and two are prescribed at

$x = L$. The polynomials $w_j(k)$ are given by

$$\omega_0(k) = -1; \quad \omega_1(k) = -k; \quad \omega_2(k) = -k^2.$$

The domain D_+ has only one simply connected component, D_1, while D_- has two connected components, D_2 and D_3; namely,

$$D_1 = \{k : \pi/3 < \arg(k) < 2\pi/3\};$$

$$D_2 = \{k : \pi < \arg(k) < 4\pi/3\}; \quad D_3 = \{k : 5\pi/3 < \arg(k) < 2\pi\}.$$

The maps $\lambda_2 : D_1 \to D_2$ and $\lambda_3 : D_1 \to D_3$ are given by

$$\lambda_2(k) = \zeta k, \;\; \lambda_3 = \zeta^2 k; \qquad \zeta = \mathrm{e}^{2\pi i/3}.$$

For the computations in this example, fix $k \in D_1$. The matrices A_1 and A_2 are given by

$$A_1 = \begin{pmatrix} -k^2\mathrm{e}^{ikL} & -k\mathrm{e}^{ikL} & -\mathrm{e}^{ikL} \\ -\zeta^2 k^2 & -\zeta k & -1 \\ -\zeta k^2 & -\zeta^2 k & -1 \end{pmatrix} \tag{5·8}$$

$$A_2 = \begin{pmatrix} k^2 & k & 1 \\ \zeta^2 k^2 \mathrm{e}^{-i\zeta kL} & \zeta k\mathrm{e}^{-i\zeta kL} & \mathrm{e}^{-i\zeta kL} \\ \zeta k^2 \mathrm{e}^{-i\zeta^2 kL} & \zeta^2 k\mathrm{e}^{-i\zeta^2 kL} & \mathrm{e}^{-i\zeta^2 kL} \end{pmatrix}. \tag{5·9}$$

Thus

$$\det(A_1) = -3k^3\mathrm{e}^{ikL}(\zeta - \zeta^2); \;\; \det(A_2) = k^3\mathrm{e}^{-i(\zeta+\zeta^2)kL}(\zeta - \zeta^2).$$

In view of the relation $1 + \zeta + \zeta^2 = 0$,

$$\det(A_2) = k^3\mathrm{e}^{ikL}(\zeta - \zeta^2).$$

Hence, as in the previous case, the determinants of A_1 and of A_2 contain the same exponential factor. Using the notation (see (4·5))

$$E_1 = \mathrm{e}^{ikL}; \;\; E_2 = \mathrm{e}^{i\zeta kL}; \;\; E_3 = \mathrm{e}^{i\zeta^2 kL},$$

two other possibilities for the determinant of $\tilde{A}$ are:

case 1: $\hat{f}_0, \hat{f}_1, \hat{g}_0$ given
$\implies \det(\tilde{A}) = k\left[E_2^{-1}(\zeta - 1) + E_3^{-1}(\zeta^2 - 1) - E_1^2(\zeta - \zeta^2)\right];$

case 2: $\hat{f}_0, \hat{g}_0, \hat{g}_1$ given
$\implies \det(\tilde{A}) = k\left[(\zeta - \zeta^2) + E_1\left(E_2^{-1} - E_3^{-1}\right) + \zeta E_2^{-1} - \zeta^2 E_3^{-1}\right)\right].$

By Theorem 1·1, the boundary value problem (1) is not well posed, while (2) is well posed. Note that in the latter case, the determinant contains one term $\zeta - \zeta^2$ which is not multiplied by any exponential factor. On the other hand for problem (1) all terms of the determinant contain one of the exponentials E_1, or E_i^{-1}, $i = 2, 3$; these exponentials all have vanishing limit, as $k \to \infty$, $k \in D_R$. Hence the term $1/\det(A)$ has an essential singularity at infinity, for $k \in D_R$. We will show that, as in the previous example, this is the structure of all problems that are *not* well posed.

Case 1. The matrix $\tilde{A}$ is in this case given by

$$\tilde{A} = \begin{pmatrix} -E_1 & k & 1 \\ -1 & \zeta k E_2^{-1} & E_2^{-1} \\ -1 & \zeta^2 k E_3^{-1} & E_3^{-1} \end{pmatrix}.$$

Assume, without loss of generality, that the given initial condition is $q_0(x) = 0$. Then the vector $b(k)$ is given by (4·7), and $c(k)$ is given by

$$c(k) = (E_1 \hat{\gamma}(k), \hat{\gamma}(\zeta k), \hat{\gamma}(\zeta^2 k))^{\tau}.$$

We note that

(i) The exponentials E_1, E_2^{-1} and E_3^{-1} vanish exponentially in the limit when $\mathrm{Im}(k) \to \infty$, $k \in D_1$. Hence the term $1/\det(\tilde{A})$ has an essential singularity at infinity.

(ii) By its definition, $\hat{\gamma}(\zeta k)$ and $\hat{\gamma}(\zeta^2 k)$ are bounded when $k \in D_1$. Indeed if $k \in D_1$, then $\zeta^2 k \in D_2 \subset \mathbb{C}^-$, $\zeta^2 k \in D_3 \subset \mathbb{C}^-$, hence in both cases the exponential term in the definition of $\hat{\gamma}$ vanishes as $\mathrm{Im}(k) \to \infty$, $k \in D_1$.

(iii) The term $\mathrm{e}^{-ik^3 T}$ is bounded, as a function of k for all $k \in D$ (indeed, this is the property defining D), hence in particular it is bounded in D_1.

The matrix C_2 is in the present case given by

$$C_2 = \begin{pmatrix} -E_1 & E_1 \hat{\gamma}(k) & 1 \\ -1 & \hat{\gamma}(\zeta k) & E_2^{-1} \\ -1 & \hat{\gamma}(\zeta^2 k) & E_3^{-1} \end{pmatrix}.$$

Hence the quotient $\mathrm{e}^{-ik^3 T}\det(C_2)/\det(\tilde{A})$ contains the term

$$\frac{\mathrm{e}^{-ik^3 T}\hat{\gamma}(\zeta k)}{\det(\tilde{A})}.$$

As $\mathrm{Im}(k) \to \infty$, the behaviour of this term is $O((\mathrm{e}^{Im(\zeta k)L})$ in the numerator, and $O(\mathrm{e}^{-Im(k)L})^2)$ in the denominator. Since $O((\mathrm{e}^{Im(\zeta k)L}) \sim O(\mathrm{e}^{-Im(k)L})$ in the interior of D_1, overall this ratio is of $O(\mathrm{e}^{Im(k)L})$, hence it has an essential singularity as $\mathrm{Im}(k) \to \infty$, $k \in D_1$. It follows that, in this particular case, the function $\hat{g}_2(k)$ computed by solving this system *cannot* arise as the transform (via the formula (3·4)) of a function $g_2(t) \in \mathbf{C}^\infty[0, T]$.

Case 2. This case is considered in detail in [**8**, section 5·2·1]; see also [**2**]. As before assume for simplicity that $q_0(x) = 0$. The computation of $\hat{f}_1$, $\hat{f}_2$, $\hat{g}_2$ follows the lines of the previous 2×2 example. All terms defining these functions are bounded for $k \in D_1$, except possibly at the zeros of $\det(\tilde{A})$. By specifying the arbitrary function $\gamma(x)$ in such way that the contribution of these poles can be subtracted, the functions can be selected to be holomorphic.

6. *Proof of Lemma* 4·2 *and Theorem* 1·1

We now state and prove a series of lemmas on which the proof of our main Lemma 4·2 relies. We continue to assume, without loss of generality, that $m = 1$ is fixed.

The first result characterises the determinant of the matrix $\tilde{A}$.

LEMMA 6·1. *Let $\tilde{A}$ be an $n \times n$ matrix constructed by choosing $n - N_0$ columns of A_1 and N_0 columns of A_2. Then the determinant of $\tilde{A}$ has the form*

$$P\big(E_1, E_2, \ldots, E_N, E_{N+1}^{-1}, \ldots, E_n^{-1}\big)$$

where $E_l = \mathrm{e}^{i\lambda_l(k)L}$, and P is a polynomial expression in the indicated exponentials, whose coefficients are polynomials in k. Let $r_0(k)$ be the zero degree coefficient of P, i.e. the term that is not multiplied by any of the exponentials $E_i^{\pm 1}$. Then

(a) *if $N_0 = N$, $r_0(k) \neq 0$,*
(b) *if $N_0 \neq N$, $r_0(k) = 0$,*

where N is given by (1·4). *In addition, in case* (*b*), *the term of highest degree of det($\tilde{A}$) is the product of the elements along the diagonal of* ($\tilde{A}$).

Remark 6·1. In example 4, we showed the above lemma by a direct computation, for a case when $n = 2$. Indeed, when $\tilde{A} = A_1$ or $\tilde{A} = A_2$, $\det(\tilde{A}) = e^{2ikL}$, hence $r_0(k) = 0$, while for case (i)–(iv), $r_0(k) \neq 0$. Namely, for case (i), $r_0(k) = 1$; for case (ii), $r_0(k) = -k^2$, and for case (iii) and (iv), $r_0(k) = \pm k$ respectively.

Proof. Suppose that $N_0 = N$. Then $\tilde{A}$ is obtained by $n - N$ columns of A_1, enumerated by $n - j_1, \ldots, n - j_N$ and N of A_2, enumerated by $l_1, \ldots, l_N$; the structure of A_1 and A_2 implies that the $N \times N$ minor at the top right corner of the matrix contains no exponential factors. Expanding the determinant along the first row, we find that $\det(\tilde{A})$ contains the additive factor

$$\pm\omega_{n-j_1}(\lambda_1)\det(\tilde{A}_{n-1}),$$

where $\tilde{A}_{n-1}$ is an $n-1$ matrix, obtained by erasing the first row and the $(n-N+1)$th column of $\tilde{A}$. Repeating the same argument to compute $\det(\tilde{A}_{n-1})$, we find the additive factor

$$\omega_{n-j_1}(\lambda_1)\omega_{n-j_2}(\lambda_2)\det(\tilde{A}_{n-2}),$$

where $\tilde{A}_{n-2}$ is the $n-2$ matrix, obtained by erasing the first row and the $(n-N+1)$th column of $\tilde{A}_{n-1}$. Repeating this process N times, we arrive at the additive factor

$$\omega_{n-j_1}(\lambda_1)\cdots\omega_{n-j_N}(\lambda_N)\,\det\begin{pmatrix}\omega_{n-j_1}(\lambda_{N+1}) & \ldots & \omega_{n-j_N}(\lambda_{N+1})\\ . & \ldots & .\\ . & \ldots & .\\ \omega_{n-j_1}(\lambda_n) & \ldots & \omega_{n-j_N}(\lambda_n)\end{pmatrix},$$

which contains no exponentials. Moreover, it is easy to verify that every other term in this expansion of the determinant contains exponential factors. Hence $r_0(k) \neq 0$.

Conversely, let $\tilde{A}$ be obtained by $n - N_0$ columns of A_1 and N_0 of A_2, with $N_0 \neq N$; assume to fix ideas that $N_0 > N$. Expanding the determinant by the first row, we obtain

$$\begin{aligned}\det(\tilde{A}) = {} & E_1 p_1(k)\det(\tilde{A}_1) + \cdots + E_1 p_{n-N_0}(k)\det(\tilde{A}_{n-N_0})\\ & + p_{n-N_0+1}(k)\det(\tilde{A}_{n-N_0+1}) + \cdots + p_n(k)\det(\tilde{A}_n),\end{aligned} \qquad (6·10)$$

where the $p_j(k)$ are polynomials in k, and the square matrices $\tilde{A}_j$ are of order $(n-1)$. The first $n - N_0$ terms of this expression are multiplied by the exponential term E_1, thus they cannot contribute to the zero degree coefficient $r_0(k)$. The last N_0 terms

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are all multiplied by the determinant of an $(n-1)\times(n-1)$ matrix. The determinant of each of these matrices reduces by successive steps, taking as multipliers always non-exponentials terms, to the computation of the determinant of a matrix of order $n-N$, of the form

$$\begin{pmatrix} * & \dots & * & p_1^{N+1}(k)E_{N+1}^{-1} & \dots & p_{N_0-N}^{N+1}(k)E_{N+1}^{-1} \\ & \dots & * & \dots & \dots & \dots \\ & \dots & * & p_1^{n}(k)E_{n}^{-1} & \dots & p_{N_0-N}^{n}(k)E_{n}^{-1} \end{pmatrix},$$

where in the first $n-N_0$ columns, $*$ denotes an element with no exponential factors. Expanding the determinant by the last column, we see that all its additive factors contain exponentials. Hence $r_0(k)=0$.

To prove that indeed the term of highest degree (in the exponentials) in this determinant is given by the product of the diagonal elements, consider the formula (6·10), but now start with the first term, which multiplies a lower order matrix by the first exponential E_1. By induction, the conclusion easily follows.

Remark 6·2. The fact that $\det(\tilde{A})$ has no zero degree coefficients means that every additive term contains at least one of the exponentials $E_1,\dots, E_N$, $E_{N+1}^{-1},\dots, E_n^{-1}$. All of these exponentials are *exponentially vanishing* when $\mathrm{Im}(k)\to\infty$, $k\in D_R$, of order $\mathrm{e}^{-\mathrm{Im}(k)L}$. Hence the term $1/\det\tilde{A}$ has an *essential singularity* in D_R.

LEMMA 6·2. *Let $\tilde{A}$ be the $n\times n$ matrix constructed by choosing $n-N_0$ columns of A_1 and N_0 of A_2, with $N_0=N\pm 1$. If $N_0=N+1$, let C be the $n\times n$ matrix constructed by substituting the $(n-N)$th column of the matrix $\tilde{A}$ with the column vector $c(k)$ given by* (4·8).

If $N_0=N-1$, let C be the $n\times n$ matrix constructed by substituting the $(n-N+1)$th column of the matrix $\tilde{A}$ with the column vector $c(k)$ given by (4·8).

Then there exists an integer $s\geqslant 1$ such that

$$\frac{\det(C)}{\det(\tilde{A})}=O(\mathrm{e}^{\mathrm{Im}(k)L})^s,\quad \mathrm{Im}(k)\to\infty,\ k\in D_R,$$

hence this function has an essential singularity in the domain $D_R\subset\mathbb{C}^+$.

Proof. We start with the case that $N_0=N+1$. Consider the matrix $C=C_{n-N}$; by definition,

$$C=\begin{pmatrix} p_1^1E_1 & \dots & p_1^{n-N-1}E_1 & E_1\hat{\gamma}_1 & * & \dots & * \\ * & \dots & \dots & \dots & * & \dots & * \\ p_N^1E_N & \dots & p_N^{n-N-1}E_N & E_N\hat{\gamma}_N & * & \dots & * \\ * & \dots & * & \hat{\gamma}_{N+1} & p_{N+1}^{n-N+1}E_{N+1}^{-1} & \dots & p_{N+1}^{n}E_{N+1}^{-1} \\ * & \dots & * & \dots & \dots & \dots & \dots \\ * & \dots & * & \hat{\gamma}_n & p_n^{n-N+1}E_n^{-1} & \dots & p_n^nE_n^{-1} \end{pmatrix}, \qquad (6·11)$$

where $n-N$ columns involve $E_1,\dots, E_N$ and N columns involve $E_{N+1}^{-1},\dots, E_n^{-1}$, $p_j^i(k)$ denote polynomials in k, and $*$ stands for terms containing no exponential, and $\hat{\gamma}_i=\hat{\gamma}(\lambda_i(k))$.

The matrix C has, by construction, the exponentials $E_1, E_2,\dots, E_{N+1}^{-1},\dots, E_n^{-1}$ in the same positions as the matrix $\tilde{A}$ obtained from $n-N$ columns of A_1 and N of A_2 (i.e. the "good" case). As in the proof of Lemma 6·1, one can show that $\det(C)$

contains an additive term of the form

$$* \begin{pmatrix} * & \dots & * & \hat{\gamma}_{N+1} \\ * & \dots & * & \dots \\ * & \dots & * & \hat{\gamma}_n \end{pmatrix} = p_0(k) \sum_{j=N+1}^{n} p_j(k)\hat{\gamma}_j,$$

where $\hat{\gamma}_j = \hat{\gamma}(\lambda_j(k))$, and the $p_j(k)$'s are polynomials in k.

On the other hand, the product of the diagonal elements of $\tilde{A}$ is equal to

$$\tilde{a}_{11} \cdots \tilde{a}_{nn} = \begin{cases} E_1 \cdots E_{N-1} * E_{N+1}^{-1} \cdots E_n^{-1} & N \geqslant n - N, \\ E_1 \cdots E_N * E_{N+2}^{-1} \cdots E_n^{-1} & N < n - N. \end{cases}$$

The behaviour as $\mathrm{Im}(k) \to \infty$ of any addictive factors in the ratio

$$\frac{\det(C)}{\det(\tilde{A})} \tag{6·12}$$

depends on the highest power term in the denominator. This term, by Lemma 6·1, is given by the product of the diagonal elements of $\tilde{A}$ (recall that, by the same lemma, all terms in the denominator decay to zero at infinity). The ratio (6·12) contains either the term

$$\frac{p_0(k) \sum_{j=N+1}^{n} p_j(k)\hat{\gamma}_j}{E_1 \cdots E_{N-1} * E_{N+1}^{-1} \cdots E_n^{-1} + l.o.t.}$$

or the term

$$\frac{p_0(k) \sum_{j=N+1}^{n} p_j(k)\hat{\gamma}_j}{E_1 \cdots E_N * E_{N+2}^{-1} \cdots E_n^{-1} + l.o.t.}$$

where *l.o.t.* denotes terms of lower degree in the exponentials.

Now consider the case that $N_0 = N - 1$, and $C = C_{n-N+1}$; by definition, the matrix C has the same form as given in (6·11), but now $n - N + 1$ columns involve $E_1, \dots, E_N$ and $N - 1$ columns involve $E_{N+1}^{-1}, \dots, E_n^{-1}$. Hence by a computation similar to the one above, it is not difficult to show that $\det(C)$ always contains a term of the form

$$\begin{cases} \tilde{p}(k)\hat{\gamma}_1 E_1 \dots E_N E_{N+1}^{-1} \dots E_n^{-1} & N > n - N, \\ \tilde{p}(k)\hat{\gamma}_1 E_1 \dots E_N * E_{N+2}^{-1} \dots E_n^{-1} & N = n - N, \\ \tilde{p}(k)\hat{\gamma}_1 E_1 \dots E_N * E_{N+3}^{-1} \dots E_n^{-1} & N < n - N. \end{cases}$$

where the gap indicated by $*$ may contain both or just one of the missing exponentials, depending on the parity of n and the value of N, and $\tilde{p}(k)$ is a generic polynomial in k. We note that in this case, the product of the diagonal elements of $\tilde{A}$ is equal to

$$\tilde{a}_{11} \cdots \tilde{a}_{nn} = \begin{cases} E_1 \cdots E_N E_{N+1}^{-1} \cdots E_n^{-1} & N > n - N, \\ E_1 \cdots E_N * E_{N+2}^{-1} \cdots E_n^{-1} & N = n - N, \\ E_1 \cdots E_N * E_{N+3}^{-1} \cdots E_n^{-1} & N < n - N. \end{cases}$$

Hence, except for the multiplicative term $\tilde{p}(k)\hat{\gamma}_1$, these two expression contain, in every case, the same product of exponentials. Hence the ratio (6·12) contains in this

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case the term

$$\begin{cases} \dfrac{\hat{\gamma}_1 E_1 \cdots E_N\, E_{N+1}^{-1} \cdots E_n^{-1}}{E_1 \cdots E_N\, E_{N+1}^{-1} \cdots E_n^{-1} + l.o.t.} & N > n - N, \\ \dfrac{\hat{\gamma}_1 E_1 \cdots E_N * E_{N+2}^{-1} \cdots E_n^{-1}}{E_1 \cdots E_N * E_{N+2}^{-1} \cdots E_n^{-1} + l.o.t.} & N = n - N, \\ \dfrac{\hat{\gamma}_1 E_1 \cdots E_N * E_{N+3}^{-1} \cdots E_n^{-1}}{E_1 \cdots E_N * E_{N+3}^{-1} \cdots E_n^{-1} + l.o.t.} & N < n - N. \end{cases}$$

In both cases, as $\mathrm{Im}(k) \to \infty$, the ration $\det(C)/\det(\tilde{A})$ has an essential singularity of order at least $O(\mathrm{e}^{Im(k)L})$.

Proof of Lemma 4·2. We assume that N_0 boundary conditions are given at $x=0$ and $n - N_0$ at $x=L$. Suppose first that $N_0 = N$. It is proved in [**5**] that when the first N boundary conditions are given for $q(0,t), \ldots, \partial_x^{N-1} q(0,t)$, and the remaining $n-N$ for $q(L,t), \ldots, \partial_x^{n-N-1} q(L,t)$, then it is possible to construct a set of admissible functions with respect to the given initial condition $q_0(x)$; hence the corresponding boundary value problem is well posed. The proof relies only on the structure of the determinant of the relevant matrix $\tilde{A}$; namely, on the fact that this matrix has an additive factor containing no exponentials. Moreover, it exploits the possibility of specifying the function $\gamma(x)$ appearing in the matrices C_j in order to remove the singularity at the poles of $\det(\tilde{A})$. Hence, by the result of Lemma 6·1, the same proof goes through in exactly the same way for the case that any N boundary conditions are given at $x=0$, and any $n-N$ are given at $x=L$; hence when $\tilde{A}$ is constructed by chosing $n-N$ columns of A_1 and N of A_2, with N given by (1·4).

Conversely, let $N_0 \neq N$. It is enough to prove the statement for $N_0 = N \pm 1$; in this case, by the result of Lemma 6·2, we have that $\det(C)/\det(\tilde{A})$ has an essential singularity in D_R, where $C = C_{n-N}$ or $C = C_{n-N+1}$. Since the function $\hat{h}_p$ is given by Cramer's rule by

$$\hat{h}_p(k) = \frac{\det(C_p)}{\det(\tilde{A})},$$

in either case there exist an index ($p = n - N$ or $p = n - N + 1$) for which this function has an essential singularity, and hence it cannot arise from any $f(t)$ which is an element of a set of admissible functions according to Definition 3·1.

We remarked in Section 4 how the result of Theorem 1·1 is a consequence of Proposition 4·1; but the proof of the latter had been given modulo the result of Lemma 4·2, which we have just proved. Hence the proof of Theorem 1·1 is complete.

7. *Conclusions*

(1) The result proved here does not depend on the assumption that the functions involved are smooth. For a more detailed regularity study, see [**7**].

(2) We stress that the unknown function $\gamma(x)$ does not play a role in the final representation of the solution; however it does play a role when the problem has a discrete spectrum, or equivalently when the functions obtained as the solution of the linear system (4·4) are meromorphic. It is then crucial to be able to choose $\gamma(x)$ in such a way that the residues at the poles can be computed giving rise explicitly to the discrete spectrum contribution in the representation (3·7). See [**5**] for a careful analysis of this phenomenon.

(3) An example of a well-posed problem of the kind covered by this analysis has been studied by Cattabriga [**1**], who proved existence and uniqueness of classical solutions for a particular boundary value problem for a third order equation of the type (1·1), with both $\alpha_3 > 0$ and $\alpha_3 < 0$, but with the boundary of the domain *depending on t*. His results are consistent with the results of the present study and of the study of time-dependent domain, using the Fokas transform method, to be found in [**6**].

(4) The analysis of linear problems carried out here generalises to the case of nonlinear *integrable* evolution dispersive equations in one spatial dimension, such as the nonlinear Schrödinger and Koteweg–deVries equations, on a finite interval. Indeed if the equation is integrable, a similar analysis to the one yielding the result in the linear case can be performed, based on expressing these equations as the closure condition of a certain differential form, see [**3**].

For the nonlinear Schrödinger equation, this approach reproduces the results obtained for the initial value problems with periodic boundary conditions [**4**]. In addition, some results on finite intervals for the KdV equation have recently been established by Colin and Ghidaglia, see [**2**]; the problems considered correspond to problems that have been proved here to be (linearly) well posed.

If the nonlinear equation is not integrable, it should still be possible to use the results established here to yield at least *local* well-posedness of specific boundary value problems, by treating the nonlinear term as a forcing and considering only small times. The extent of validity of this statement will be investigated in future work.

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Errata Corrige. This work depends on the work presented in [**5**]. Unfortunately, the latter has two errors:

(1) In the statement of Lemma 3.1, conclusion (b) should read:

(b) For $k \in D_{R,m}$, $1 \leqslant l \leqslant n$, and $1 \leqslant m \leqslant N$,

$$\int_{\partial D_{R,m}} \mathrm{e}^{-i\omega(k)t} \left[\sum_{j=1}^{n} \omega_{n-j}(\lambda_{l,m}) \hat{f}_{j-1}(k) - \mathrm{e}^{-i\lambda_{l,m}L} \sum_{j=1}^{n} \omega_{n-j}(\lambda_{l,m}) \hat{g}_{j-1}(k) \right] dk = 0,$$

while for $N+1 \leqslant m \leqslant n$,

$$\int_{\partial D_{R,m}} \mathrm{e}^{-i\omega(k)t} \left[\mathrm{e}^{i\lambda_{l,m}L} \sum_{j=1}^{n} \omega_{n-j}(\lambda_{l,m}) \hat{f}_{j-1}(k) - \sum_{j=1}^{n} \omega_{n-j}(\lambda_{l,m}) \hat{g}_{j-1}(k) \right] dk = 0.$$

(2) On page 530, the two lines following equation (3·3) should read:

Using the property (b) of Lemma (3·1) we can show that

$$\int_{\partial D_{R,-}} \mathrm{e}^{-i\omega(k)t}\left[\mathrm{e}^{-ikL}\sum_{j=1}^{n}\omega_{n-j}(k)\hat{g}_{j-1}(k)-\sum_{j=1}^{n}\omega_{n-j}(k)\hat{f}_{j-1}(k)\right]dk=0, \quad k\in D_{R,-}.$$

REFERENCES

[**1**] L. CATTABRIGA. Un problema al contorno per una equazione parabolica di ordine dispari. *Ann. Scuola Norm. Sup. Pisa* **13** (1959), 163–203.

[**2**] J. L. GHIDAGLIA and T. COLIN. An initial-boundary value problems for the Korteweg-deVries equation posed on a finite interval. *Adv. Differential Equations* **6**(12) (2001), 1463–1492.

[**3**] A. S. FOKAS. On the integrability of linear and nonlinear PDEs. *J. Math. Phys.* **41** (2000), 4188.

[**4**] A. S. FOKAS and A. R. ITS. The nonlinear Schrödinger equation on a finite interval, preprint, 2000.

[**5**] A. S. FOKAS and B. PELLONI. Two-point boundary value problems for linear evolution equations. *Math. Proc. Camb. Phil. Soc.* **131** (2001), 521.

[**6**] A. S. FOKAS and B. PELLONI. Boundary value problems for linear evolution PDEs in time-dependent domains. *Inverse Problems* **17** (2001), 919–935.

[**7**] A. S. FOKAS and L. Y. SUNG. Initial boundary value problems for linear evolution equations on the half line, preprint, 2000.

[**8**] M. D. RAMAZANOV. A boundary-value problem for a class of differential equations (in Russian). *Mat. Sbornik* **64**(106), (1964), 234–261.