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An integral representation for a generalised variation of a function

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In this note we present sufficient conditions for the continuity of the total kth variation of a function defined on a closed interval $[a, b]$. We also give an integral representation for total kth variation, thus obtaining an extension of the classical result

$$V(f; a, x) = \int_a^x |f'(t)|dt , \quad a \le x \le b .$$

The results presented in this note will be a continuation of results obtained in [2], and unless otherwise stated, all definitions and notation will be taken from [2].

THEOREM 1. *If* $f \in BV_k[a, b]$, $k \ge 2$, *and* f *has a* $(k-1)th$ *Riemann* derivative throughout* $[a, b]$ *(one-sided, of course, at* a *and* b *), then* $V_k(f; a, x)$ *is a continuous function of* x .

Proof. Let $\varepsilon > 0$ be arbitrary, and let $V_k(x)$ denote $V_k(f; a, x)$. Then there exists a π subdivision $a = y_0, y_1, \ldots, y_n = x$ of $[a, x]$ with $y_{n-1} = x'$ arbitrarily close to x , such that

$$\sum_{i=0}^{n-k} |Q_{k-1}(f; y_i, \ldots, y_{i+k-1})-Q_{k-1}(f; y_{i+1}, \ldots, y_{i+k})| > V_k(x) - \frac{\varepsilon}{2} ,$$

and

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$$|Q_{k-1}(f;\, y_{n-k},\, \dots,\, y_{n-1}) - Q_{k-1}(f;\, y_{n-k+1},\, \dots,\, y_n)| < \frac{\varepsilon}{2} .$$

Therefore,

$$\begin{aligned} V_k(x') &\geq \sum_{i=0}^{n-k-1} |Q_{k-1}(f;\, y_i,\, \dots,\, y_{i+k-1}) - Q_{k-1}(f;\, y_{i+1},\, \dots,\, y_{i+k})| \\ &= \sum_{i=0}^{n-k} |Q_{k-1}(f;\, y_i,\, \dots,\, y_{i+k-1}) - Q_{k-1}(f;\, y_{i+1},\, \dots,\, y_{i+k})| \\ &\qquad - |Q_{k-1}(f;\, y_{n-k},\, \dots,\, y_{n-1}) - Q_{k-1}(f;\, y_{n-k+1},\, \dots,\, y_n)| \\ &> V_k(x) - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = V_k(x) - \varepsilon . \end{aligned}$$

Consequently we have $V_k(x) \geq V_k(x') > V_k(x) - \varepsilon$, and so it follows that $V_k(x') \to V_k(x)$ as $x' \to x - 0$. Similarly it can be shown that $V_k(x') \to V_k(x)$ as $x' \to x + 0$, and this now completes the proof.

We make two observations.

The result of Theorem 1 is an extension of the classical case in which the total variation $V(f;\, a,\, x)$ of a continuous function f is continuous.

The hypothesis of existence of the $(k-1)$th Riemann* derivative is necessary, as shown by the following example. Let $f(x) = |x|$, $-1 \leq x \leq +1$. If $k = 2$, then $V_2(f;\, -1,\, 0) = 0$, whereas $V_2(f;\, -1,\, x) = 2$ for all $x > 0$.

We now obtain an integral representation for the total kth variation $V_k(f;\, a,\, x)$. This will be a generalisation of the classical result

$$V(f;\, a,\, x) = \int_a^x |f'(t)|dt ,$$

when f has an integrable derivative.

We make use of the following

THEOREM 2. *If f is a real-valued function whose kth derivative exists and is bounded on $[a,\, b]$, then $f \in BV_k[a,\, b]$.*

The proof follows readily from [1, Section 1.2].

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THEOREM 3. *Let* f *be a function whose* kth *derivative is continuous in* $[a, b]$. *Then* $f \in BV_k[a, b]$, *and*

$$(k-1)!V_k(f; a, x) = \int_a^x |f^{(k)}(t)|dt , \quad a \le x \le b .$$

Proof. It follows immediately from Theorem 2 that $f \in BV_k[a, b]$. Let us denote $V_k(f; a, x)$ by $V_k(a, x)$. Then, using [2, Theorem 7], and noting that f has a $(k-1)$th Riemann* derivative in $[a, b]$, we obtain

$$\frac{1}{h}\left[V_k(a, x+h)-V_k(a, x)\right] = \frac{1}{h} V_k(x, x+h) \quad \text{when} \quad h > 0 .$$

Let $y_0, y_1, \ldots, y_n$ be a π subdivision of $[x, x+h]$ such that $x = y_0 < y_1 < \ldots < y_n = x + h$ and all sub-intervals $\left[y_i, y_{i+1}\right]$ are of equal length l , so that $nl = h$. Then, using [1, Section 1.2], we obtain

$$\begin{aligned} V_k(x, x+h) &= \sup_{\pi} \sum_{i=0}^{n-k} \left(y_{i+k}-y_i\right)\left|Q_k\left(f; y_i, \ldots, y_{i+k}\right)\right| \\ &\ge \frac{1}{k!} \sum_{i=0}^{n-k} \left(y_{i+k}-y_i\right)\left|f^{(k)}\left(\xi_i\right)\right| , \\ &\qquad\qquad \text{where } y_i < \xi_i < y_{i+k} , \quad i = 0, 1, \ldots, k , \\ &\ge \inf_{x\le t\le x+h} \left|f^{(k)}(t)\right| \frac{1}{k!} \sum_{i=0}^{n-k} (kl) \\ &= \frac{h}{(k-1)!} \inf_{x\le t\le x+h} \left|f^{(k)}(t)\right| \cdot \left\{1 - \frac{k-1}{n}\right\} . \end{aligned}$$

Letting n tend to infinity gives us the result

$$V_k(x, x+h) \ge \frac{h}{(k-1)!} \inf_{x\le t\le x+h} \left|f^{(k)}(t)\right| \quad \text{when} \quad h > 0 . \tag{1}$$

We now consider any $\pi\left(x_0, \ldots, x_n\right)$ subdivision of $[x, x+h]$ and use [1, Section 1.2] again to show that

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$$\sum_{i=0}^{n-k} \left(x_{i+k}-x_i\right)\left|Q_k\left(f;\ x_i,\ \ldots,\ x_{i+k}\right)\right| \le \sup_{x\le t\le x+h} \left|f^{(k)}(t)\right| \frac{1}{k!} \sum_{i=0}^{n-k} \left(x_{i+k}-x_i\right)$$

$$\le \frac{h}{(k-1)!} \sup_{x\le t\le x+h} \left|f^{(k)}(t)\right| .$$

Consequently,

$$V_k(x,\ x+h) \le \frac{h}{(k-1)!} \sup_{x\le t\le x+h} \left|f^{(k)}(t)\right| . \tag{2}$$

Combining inequalities (1) and (2) gives

$$\inf_{x\le t\le x+h} \left|f^{(k)}(t)\right| \le (k-1)!\, \frac{V_k(x,x+h)}{h} \le \sup_{x\le t\le x+h} \left|f^{(k)}(t)\right| . \tag{3}$$

Similar inequalities would be obtained when $h < 0$, so we conclude that $\frac{d}{dx} V_k(a,\ x)$ exists, and equals $\frac{1}{(k-1)!} \left|f^{(k)}(x)\right|$. The required result now follows.

REMARK. Continuity of $f^{(k)}$ in the preceding theorem is not necessary. If $f^{(k)}$ is bounded and continuous almost everywhere, then $V_k(f;\ a,\ x)$ is absolutely continuous by (3), and at a point x of continuity of $f^{(k)}$ we can show, as before, that

$$(k-1)!\, \frac{d}{dx} V_k(f;\ a,\ x) = \left|f^{(k)}(x)\right| .$$

Consequently, the integral representation still holds, the integral of course being Lebesgue.

We conclude with the following

THEOREM 4. *Let f be a function whose kth derivative is continuous. Then $f^{(k-s)} \in BV_s[a,\ b]$, $s = 1,\ 2,\ \ldots,\ k$, and*

$$(k-1)!\,V_k(f;\ a,\ x) = (s-1)!\,V_s\left(f^{(k-s)};\ a,\ x\right), \quad a \le x \le b ,$$

$$s = 1,\ 2,\ \ldots,\ k .$$

Proof. It follows immediately from Theorem 3 that $f \in BV_k[a,\ b]$. That $f^{(k-s)} \in BV_s[a,\ b]$, $s = 1,\ 2,\ \ldots,\ k$ follows from [2, Theorem 12].

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The required result now follows readily by an application of Theorem 3.

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