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ABSOLUTE RETRACTS AND AMALGAMATION IN CERTAIN CONGRUENCE DISTRIBUTIVE VARIETIES

BY

PETER JIPSEN AND HENRY ROSE

ABSTRACT. It is shown that if $\mathcal{V}$ is a congruence distributive variety whose members have one element subalgebras, then the class of absolute retracts of $\mathcal{V}$ is closed under direct products. If $\mathcal{V}$ is residually small, then a characterisation of the amalgamation class of $\mathcal{V}$ is given.

0. **Introduction.** We consider congruence distributive varieties whose members have one element subalgebras. The varieties of lattices, weakly associative lattices, lattice ordered monoids and Browerian algebras are examples of such varieties.

Theorem 5.1 in Jónsson [2] characterises the amalgamation class of finitely generated lattice varieties. Such varieties, or in fact any finitely generated congruence distributive variety has up to isomorphism only finitely many subdirectly irreducible members, and is therefore residually small. Our Theorem 2.1 is an improvement of the result of Jónsson: we show that Jónsson characterisation is valid for any residually small congruence distributive variety whose members have one element subalgebras.

Every maximal subdirectly irreducible member of a residually small variety $\mathcal{V}$ is an absolute retract in $\mathcal{V}$ (see Corollary 1.3). It is shown in Jónsson [2] Theorem 5.4, that if $\mathcal{V}$ is a finitely generated lattice variety then any product of maximal subdirectly irreducible members of $\mathcal{V}$ is an absolute retract in $\mathcal{V}$. Our Theorem 3.1 generalises this result: if $\mathcal{V}$ is a congruence distributive variety whose members have one element subalgebras, then the class of all absolute retracts of $\mathcal{V}$ is closed under direct products.

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1. **Preliminaries. The Amalgamation Class of a Variety.** By a *diagram* in a variety $\mathcal{V}$ we mean a quintuple (A, f, B, g, C) with $A, B, C \in \mathcal{V}$ and $f\colon A \hookrightarrow B, g\colon A \hookrightarrow C$ embeddings. By an *amalgam* in $\mathcal{V}$ of this diagram we mean a triple (f_1, g_1, D)

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with $D \in \mathcal{V}$ and with $f_1 : B \hookrightarrow D, g_1 : C \hookrightarrow D$ embeddings such that $ff_1 = gg_1$. If such an amalgam exists we say that the diagram can be *amalgamated* in $\mathcal{V}$. An algebra $A \in \mathcal{V}$ is called an *amalgamation base* for $\mathcal{V}$ if every diagram (A, f, B, g, C) can be amalgamated in $\mathcal{V}$. The class of all amalgamation bases for $\mathcal{V}$ is called the *amalgamation class* and is denoted by *Amal*($\mathcal{V}$). A variety $\mathcal{V}$ is said to have *the amalgamation property* if *Amal*($\mathcal{V}$) $= \mathcal{V}$.

Absolute Retracts and Essential Extensions. The symbolism $A \leqq B(f : A \hookrightarrow B)$ indicates that an algebra B is an extension of A (f is an embedding of A into B). An extension B of an algebra A is said to be *essential* if each non-0 congruence of +B restricts to a non-0 congruence of A. An algebra $A \in \mathcal{K}$ is said to be an *absolute retract* in $\mathcal{K}$ if for any embedding $f : A \hookrightarrow B \in \mathcal{K}$ there is an epimorphism $g : B \longrightarrow A$ such that gf is an identity map on A.

The next result can be found in [3].

PROPOSITION 1.1. (i) *An essential extension of a subdirectly irreducible algebra is subdirectly irreducible.*

(ii) *If B is an essential extension of A, then among congruences θ on B with $\theta|A = 0$ there is a maximal one θ_2 and the extension $A \leqq B/\theta_0$ is essential.*

(iii) *An algebra has a proper essential extension iff it is not an absolute retract.*

Residually Small Varieties. A variety $\mathcal{V}$ is said to be *residually small* if $\mathcal{V}$ satisfies the two equivalent conditions of the following theorem:

THEOREM 1.2. (Taylor [5]). (i) *There exists a cardinal α such that every subdirectly irreducible member of $\mathcal{V}$ has cardinality $\leqq \alpha$.*

(ii) *Every member of $\mathcal{V}$ has a maximal essential extension in $\mathcal{V}$.*

Combining Proposition 1.1 and Theorem 1.2 we have the following corollary:

COROLLARY 1.3. *Let $\mathcal{V}$ be a residually small variety. Then every subdirectly irreducible member of $\mathcal{V}$ has a maximal essential extension which is subdirectly irreducible. Moreover, every such maximal essential extension is an absolute retract in $\mathcal{V}$ and therefore does not have proper essential extensions.*

We define a maximal irreducible algebra in a variety $\mathcal{V}$ to be a subdirectly irreducible algebra of $\mathcal{V}$ with no essential extensions in $\mathcal{V}$. The class of all maximal irreducibles of $\mathcal{V}$ is denoted $\mathcal{V}_{MI}$. It follows from Corollary 1.3 that if $\mathcal{V}$ is residually small then every subdirectly irreducible member of $\mathcal{V}$ has a maximal essential extension in $\mathcal{V}_{MI}$.

LEMMA 1.4. (Bergman [1] Lemma 3.8). *Let $\mathcal{V}$ be a residually small variety, $A \in \mathcal{V}$. Suppose that for every $B \in \mathcal{V}$ extending A and every $M \in \mathcal{V}_{MI}$, any homomorphism from A to M can be extended to a homomorphism from B M. Then $A \in$ Amal($\mathcal{V}$).*

REMARK 1.5. It follows from Proposition 1.1 (ii) that it suffices to consider all essential extensions of A in the preceeding lemma. We shall see later (Theorem 2.1) that the converse of Lemma 1.4 also holds for congruence distributive varieties whose members have one-element subalgebras.

Congruence Distributive Algebras. Call an algebra A congruence distributive if the lattice Con(A) of all congruences on A is distributive. It is assumed throughout this section that every algebra is congruence distributive.

LEMMA 1.6. (cf[2] Lemma 4.1). *Let A and B be algebras and let g be a homomorphism on $A \times B$. Then there are congruences $\theta \in \mathrm{Con}(A)$ and $\phi \in \mathrm{Con}(B)$ such that for $(x, y), (x', y') \in A \times B$*

$$(x, y) \ker g (x', y') \text{ if and only if } x\theta x' \text{ and } y\phi y'.$$

LEMMA 1.7. *Let A and B be algebras, $a \in A$ and let a be a subalgebra of A. Let h_a: $B \hookrightarrow A \times B$ be the embedding given by $h_a(b) =(a, b)$ for all $b \in B$. Then the projection π_B: $A \times B \longrightarrow B$ is the only retraction onto B.*

PROOF. Suppose g: $A \times B$ is a retraction, that is gh_a is an identity map on B. Let $\theta \in \mathrm{Con}(A)$ and $\phi \in \mathrm{Con}(B)$ be as in Lemma 1.6. Since gh_a is an identity map on B, ϕ must be a trivial congruence on B. To prove that $g = \pi_B$ it suffices to show that for any $a' \in A$ we have $a\theta a'$. Suppose the contrary. First observe that for $b, b' \in B$ with $b \neq b'$ we always have

$$g(a, b) = b \neq b' = g(a, b').$$

Now if $(a, a') \notin \theta$ for some $a' \in A$, then there exist $b, b' \in B$ such that

$$g(a, b) = b \neq b' = g(a', b).$$

Thus $g(a, b') = g(a', b)$ and so Lemma 1.6 implies $a\theta a'$ and $b\phi b'$, a contradiction. □

COUNTEREXAMPLE: The assumption that $h = h_a$ is a one-element subalgebra embedding cannot be dropped. Indeed, let $\mathbf{2} = \{0, 1\}$ be a two-element chain and consider a lattice embedding $h : \mathbf{2} \hookrightarrow \mathbf{2} \times \mathbf{2}$ given by $h(0) = (0, 0)$ and $h(1) = (1, 1)$. Then both projections on $\mathbf{2} \times \mathbf{2}$ are retractions onto $\mathbf{2}$.

COROLLARY 1.8. *Let A, B and a be as in Lemma 1.7. Suppose further that $A, B \in \mathcal{V}$ and B is an absolute retract in $\mathcal{V}$. If $k : A \times B \hookrightarrow C \in \mathcal{V}$ is an embedding, then the projection π_B: $A \times B \longrightarrow B$ can be extended to an epimorphism of C onto B.*

PROOF. If h_a: $B \hookrightarrow A \times B$ is an embedding as in Lemma 1.7, then kh_a is an embedding of B and C. Since B is an absolute retract in $\mathcal{V}$ there is a retraction p of C onto B. It follows from Lemma 1.7 that $p|A \times B = \pi_B$.

2. **The Amalgamation Class.** Theorem 2.1 below is a generalisation of Theorem 5.1 in [**2**]. The latter was proved for finitely generated lattice varieties.

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THEOREM 2.1. *Let $\mathcal{V}$ be a residually small congruence distributive variety, and assume that every member of $\mathcal{V}$ has a one- element subalgebra. Then the following conditions are equivalent:*

(i) $A \in Amal(\mathcal{V})$

(ii) *For any embedding* $f : A \hookrightarrow B \in \mathcal{V}$ *and any homomorphism* $g : A \longrightarrow M \in \mathcal{V}_{MI}$ *there is a homomorphism* $h : B \longrightarrow M$ *such that* $g = hf$.

(iii) *Let* $g : A \longrightarrow M \in \mathcal{V}_{MI}$ *be a homomorphism and* $k : A \hookrightarrow A \times M$ *be the embedding given by* $k(a) = (a, g(a))$ *for all* $a \in A$. *If* $f : A \hookrightarrow B \in \mathcal{V}$ *is an essential embedding then the diagram* $(A, f, B, k, A \times M)$ *can be amalgamated in* $\mathcal{V}$.

PROOF. (ii) implies (i) by Lemma 1.4, and trivially (i) implies (iii). Suppose (iii) holds. It follows from Proposition 1.1 (ii) that in order to prove (ii), we may assume that the embedding $f : A \hookrightarrow B$ is essential. Let $k : A \hookrightarrow A \times M$ be an embedding given by $k(a) = (a, g(a))$ where $g : A \longrightarrow M$ is a homomorphism as in (ii). Notice that $g = \pi_M k$ where π_M is the projection from $A \times M$ onto M. By (iii) the diagram $(A, f, B, k, A \times M)$ has an amalgam (C, f_1, k_1) in $\mathcal{V}$. It follows from Corollary 1.8 that there is a retraction $p : C \longrightarrow M$ such that $g = pk_1k = pf_1f$. Letting $h = pf_1$ we have $g = hf$. □

The following corollary generalises the well-known result of Pierce [**4**] that the variety of distributive lattices has the amalgamation property.

COROLLARY 2.2. *Let $\mathcal{V}$ be a congruence distributive variety whose members have one element subalgebras, and assume that $\mathcal{V}$ is generated by a finite simple algebra S that has no nontrivial subalgebras. Then $\mathcal{V}$ has the amalgamation property.*

PROOF. We show that $\mathcal{V}$ satisfies part (ii) of Theorem 2.1. Observe that since $\mathcal{V}$ is congruence distributive it follows from Jónsson's Lemma that S is, up to isomorphism, the only subdirectly irreducible member of $\mathcal{V}$. So let $A, B \in \mathcal{V}$, $f : A \hookrightarrow B$ and $g : A \longrightarrow S$. Since S has no nontrivial subalgebra, either $g(A)$ is a one-element subalgebra of S or $g(A) = S$. The first case is trivial, so we may assume that g is an epimorphism. Let $k : B \hookrightarrow S^I$ be a subdirect decomposition of B. Since $\ker g$ is meet irreducible in Con(A), another application of Jónsson's Lemma implies that there is an ultrafilter U on I such that

$$\psi_U | kf(A) \subseteq \ker g$$

where ψ_U is the congruence on S^I induced by U. Since S is finite, the above inclusion is in fact an equality. Denoting the canonical epimorphism from S^I onto $S^I/\psi_U \cong S$ by γ and defining $h = \gamma k$, we see that $hf = g$. □

3. **Products of Absolute Retracts.** It is shown in Taylor [**6**] that, in general, the product of absolute retracts is not an absolute retract, even if $\mathcal{V}$ is a congruence distributive variety. Theorem 3.1 however shows that absolute retracts are preserved

under arbitrary products in a congruence distributive variety, provided that every member of this variety has a one-element subalgebra. The theorem is a generalisation of Theorem 5.4 in [**2**], which states that if $\mathcal{V}$ is a finitely generated lattice variety then the product of members of $\mathcal{V}_{MI}$ is an absolute retract in $\mathcal{V}$. (Note that by Proposition 1.1 every member of $\mathcal{V}_{MI}$ is an absolute retract in $\mathcal{V}$).

THEOREM 3.1. *Let $\mathcal{V}$ be a congruence distributive variety and assume that every member of $\mathcal{V}$ has a one-element subalgebra. Then every product of absolute retracts in $\mathcal{V}$ is an absolute retract in $\mathcal{V}$.*

PROOF. Suppose $A = \prod_{i\in I} A_i$ is a direct product of absolute retracts in $\mathcal{V}$, and consider an embedding $f : A \hookrightarrow B \in \mathcal{V}$. For $i \in I$, let $\pi_i : A \longrightarrow A_i$ be the ith projection. By Corollary 1.8 there is a homomorphism $h_i : B \longrightarrow A_i$ such that $\pi_i = h_i f$. Let $h : B \longrightarrow A$ be the homomorphism that satisfies $\pi_i h = h_i$ for each $i \in I$. Then $\pi_i f = h_i f = \pi_i$ and so $hf : A \longrightarrow A$ is an identity map. □

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Department of Mathematics
University of Cape Town
Rondebosch 7700
Cape Town
Republic of South Africa

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