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# The link of $\{f(x,y)+z^n=0\}$ and Zariski's conjecture

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Abstract

We consider suspension hypersurface singularities of type $g=f(x,y)+z^n$, where $f$ is an irreducible plane curve singularity. For such germs, we prove that the link of $g$ determines completely the Newton pairs of $f$ and the integer $n$ except for two pathological cases, which can be completely described. Even in the pathological cases, the link and the Milnor number of $g$ determine uniquely the Newton pairs of $f$ and $n$. In particular, for such $g$, we verify Zariski's conjecture about the multiplicity. The result also supports the following conjecture formulated in this paper: if the link of an isolated hypersurface singularity is a rational homology 3-sphere, then it determines the equisingularity type, the embedded topological type, the equivariant Hodge numbers and the multiplicity of the singularity. The conjecture is verified for weighted homogeneous singularities too.

## 1. Introduction

In the past few decades intense research effort has been concentrated on the following problem: what kind of analytic invariants or smoothing invariants (if they exist) can be determined from the topology of a normal surface singularity $(X,x)$.

Some of the results have already become classical: e.g. Mumford's result, which states that $(X,x)$ is smooth if and only if the fundamental group of the link $L_X$ is trivial [Mum61]; or its generalization by Neumann [Neu81], which claims that the oriented homeomorphism type of the link contains the same information as the resolution graph of $(X,x)$; or Artin's computations of the multiplicity and the embedded dimension of rational singularities [Art62, Art66]; and their generalizations by Laufer [Lau77] for minimally elliptic singularities, and by Yau [Yau80] for some elliptic singularities.

In general, these questions are very difficult, even if we restrict ourselves to some special families, e.g. to complete intersections or hypersurface singularities; and even if we permit ourselves to use, instead of the topology of $(X,x)$, richer topological information, e.g. in the case of hypersurfaces the *embedded topological type*. For example, Zariski conjectured three decades ago that the embedded topological type of an isolated hypersurface singularity determines its multiplicity [Zar71]. This has been verified up to now only for quasi-homogeneous singularities [Gre86, OSh87, XY89, Yau89] (and some other sporadic cases).

In a different direction, the conjecture of Neumann and Wahl [NW90] about a possible connection between the Casson invariant of the link (provided that it is an integral homology sphere) and the signature of the Milnor fiber opened new windows for the theory.

Recently, the subject has been revived with an even larger intensity; see e.g. [Nem99b, NW02, NW03, NN02a, NN04, NN02b], their introductions and listed references. Basically, these articles

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claim that if the link of a $\mathbb{Q}$-Gorenstein singularity is a rational homology sphere, then it codifies extremely rich analytic information about the singularity.

The present paper is in the spirit of the above efforts. We will consider the family of suspension hypersurface singularities of type $f(x,y)+z^n$, where $f$ is an irreducible plane curve singularity.

For this family, we not only answer positively both main conjectures (namely Zariski's conjecture, and the possibility to recover the main analytic and smoothing invariants from the link), but also we succeed in obtaining much sharper statements.

The main result of the paper is the following (cf. Theorem 5.4).

Theorem 1. *Let $f:(\mathbb{C}^2,0)\to(\mathbb{C},0)$ be an irreducible plane curve singularity with Newton pairs $\{(p_i,q_i)\}_{i=1}^s$ and let $n\geqslant 2$ be an integer. Let $L_X$ be the link of the hypersurface singularity $(X,0)=(\{f(x,y)+z^n=0\},0)$. Then, except for two pathological cases S1 and S2 (which are described completely in §§ 5.1 and 5.3, and can be characterized perfectly in terms of $L_X$), from the link $L_X$ one can recover completely the Newton pairs of $f$ and the integer $n$ (provided that we disregard the 'z-axis ambiguity', cf. Remark 5.2.) In both exceptional cases the links have nontrivial first Betti numbers. In particular, the above statement holds without any exception provided that the link is a rational homology sphere.*

*On the other hand, in the cases S1 and S2, the link together with the Milnor number of the hypersurface singularity $f+z^n$ determine completely the Newton pairs of $f$ and the integer $n$ (cf. the two paragraphs at the ends of §§ 5.1 and 5.3).*

Here some remarks are in order.

1) $L_X$ determines the number $s$ of Newton pairs of $f$ in all cases.

The exceptional case S1 appears when $s=1$, and the corresponding singularities have the equisingular type of some special Brieskorn singularities. This case can be easily classified.

The exceptional case S2 appears for $s=2$ with some other strong additional restrictions. In this case any link $L_X$ can be realized by at most two possible pairs $(f,n)$. This case again is completely clarified.

In all other cases, e.g. when $s\geqslant 3$, the theorem assures uniqueness. This is slightly surprising. At the beginning of our study, here we expected more and more complicated special families providing interesting coincidences for their links. But, it turns out that this is not the case: if the plumbing graph of the link (or equivalently the resolution graph of $(X,x)$) has more and more complicated structure, then it becomes more 'over-determined', and it leaves no room for any ambiguity for $f$ and $n$.

In fact, in order to reach our goal, it was sufficient to consider rather limited information about this graph: the determinants of its maximal strings and the determinants of some subgraphs with only one rupture vertex. Except for the two pathological cases, in all other cases these determinants already determine all the Newton pairs and $n$.

2) In general, it is very difficult to characterize those resolution graphs (or links) which can be realized by, say, hypersurface singularities, or complete intersections, or by any family of germs defined by some analytic property.

Our proof gives a complete characterization of those graphs which can be realized as resolution graphs of some $\{f+z^n=0\}$ for some irreducible $f$. Indeed, the proof is a precise recipe for how one can recover the Newton pairs of $f$ and $n$. If one runs this algorithm (the steps of the proof of Theorem 5.4) for an arbitrary minimal resolution graph, and at some point it fails, then the graph definitely is not of this type. If the algorithm goes through and provides some candidates for the Newton pairs of $f$ and for $n$, then one has to compute the minimal resolution graph of $f+z^n=0$

(using e.g. the algorithm of [Nem99d, Appendix 1], see also below) and compare it with the initial graph. If they are the same, then the answer is 'yes'. But it could happen that these two graphs are not the same (since our algorithm is based on a very limited number of determinants: these determinants could be the same even if the graphs are not). In this case again the answer is 'no'.

3) For the case when $f$ is arbitrary (i.e. reduced), but $n = 2$, and the link is a rational homology sphere, Laufer established uniqueness [Lau78].

Pichon [Pic97, Pic99] proved by a different method that, for any *fixed* $n$, any link can be realized in *finitely many* ways as the link of $f + z^n = 0$ (in her case $f$ is reduced too). Some of our partial results (after some – sometimes nontrivial – identifications) can be compared with some of her results. For example, our last formula from Proposition 3.5 can be compared with Proposition 3 of [Pic97], and our Proposition 4.1 with Theorem (6.4) of [Pic99] (formulated in terms of Waldhausen decomposition). Nevertheless, we decided to present our arguments as well, in order to provide a self-contained proof and uniform picture.

4) We expect that a possible generalization of the above result for reduced $f$ inevitably will contain a much larger list of pathological cases, with a lot of numerical, case-by-case verifications in the proof. For some more concrete 'warnings' in this case, see § 5.7.

Now we return to our main theorem and its corollaries.

Corollary 1. *Assume that $g(x, y, z) = f(x, y) + z^n$ is a suspension hypersurface singularity with $f$ irreducible, and not of type described in the pathological cases S1 and S2 (cf. §§ 5.1 and 5.3). Then the link $L_X$ of $(X, 0) = (\{g = 0\}, 0)$ determines completely the following data:*

1) *$(X, 0)$ up to equisingular deformation (within the class of suspension of irreducible plane curve singularities);*

2) *the embedded topological type of $(X, 0)$ (i.e. the embedding $L_X \subset S^5$), in particular, the Milnor fibration and all the homological package derived from it;*

3) *all the equivariant Hodge numbers associated with the vanishing cohomology of $g$, in particular, the geometric genus of $(X, 0)$;*

4) *the multiplicity of $g$.*

*In particular, if $L_X$ is a rational homology sphere, then $L_X$ determines all the four data 1–4. If $g$ is a pathological case listed in S1 or S2, then $L_X$ together with the Milnor number of $g$ determines completely data 1–4.*

Indeed, the Newton pairs of $f$ determine the equisingularity type of $f$. Data 2–4 follow from the description of the corresponding invariants for plane curve singularities and different 'Sebastiani–Thom type' formulae; see e.g. [AGV88, SSS91, Nem98, Nem99c]. In fact, recently in [NN02b], the geometric genus (together with the Milnor number and the signature of the Milnor fiber) was computed in terms of the Seiberg–Witten invariant of the link, provided that the link is a rational homology sphere.

It is well known that the Milnor number of $g$ can be determined from the embedded topological type of $g$ (a fact first noticed by Teissier see also [Yau89]). Therefore, we get the following:

Corollary 2 (Zariski's conjecture for this family). *The multiplicity of $g = f(x, y) + z^n$ ($f$ irreducible) is determined by the embedded topological type of $g$.*

Corollary 1 (see also Theorem 2 below) motivates the following conjecture.

CONJECTURE. *Let $g:(\mathbb{C}^3,0)\to(\mathbb{C},0)$ be an isolated hypersurface singularity whose link $L_X$ is a rational homology sphere. Then the fundamental group of the link characterizes completely the equisingularity type, the embedded topological type, the equivariant Hodge numbers and the multiplicity of $g$.*

This can be verified in the following cases. If $g=f+z^n$ with $f$ irreducible, then the Conjecture is true by Theorem 1 above (see also [Neu81] for the relation between $L_X$ and its fundamental group). If $f$ is arbitrary but $n=2$, then it is true by [Lau78].

The next theorem provides a positive answer for weighted homogeneous singularities. Since its proof is short and involves a different terminology than the body of the paper, we decided to separate the whole statement at the end of this Introduction.

THEOREM 2. *The above conjecture is true for any weighted homogeneous hypersurface singularity.*

*Proof.* The Poincaré polynomial $p_A(t)$ of the singularity can be determined from the link by [Dol75] or [Pin77] (see also [Neu83]). Then, by a recent result of Ebeling [Ebe02, Theorem 1] it follows that the characteristic polynomial $\Delta(t)$ of the algebraic monodromy can be recovered from the link $L_X$. Indeed, notice that $\psi_A(t)$ used by Ebeling is link-invariant as well. By [Ebe02, Theorem 1], $\Delta(t)=[\psi_A(t)p_A(t)]^*$, where $*$ stands for K. Saito's duality. In general, $*$ is defined in terms of some integer $h$ (cf. [Ebe02, p. 3]), but here one can verify (using e.g. [MO70]) that if $\psi_A(t)p_A(t)=\prod_m(1-t^m)^{\chi_m}$, then one can take $h:=\operatorname{lcm}\{m:\chi_m\neq 0\}$; hence $\Delta(t)=\prod_{k|h}(1-t^k)^{-\chi_{h/k}}$ for this $h$.

Then, by [XY89] (see also [OW71]), we obtain from $L_X$ the weights, multiplicity and the embedded topological type. The statement about the Hodge data follows from [Ste77]. □

## 2. Preliminaries about resolution graphs

### 2.1 Definitions and notation

Assume that $(X,x)$ is a normal surface singularity, $f:(X,x)\to(\mathbb{C},0)$ is a germ of an analytic function, and $\phi$ is an embedded resolution of the pair $(f^{-1}(0),x)\subset(X,x)$. We denote by $E$ the exceptional divisor of $\phi$ and by $S$ the strict transform of $f^{-1}(0)$. Let $\bigcup_{w\in\mathcal{W}}E_w$ (respectively $\bigcup_{a\in\mathcal{A}}S_a$) be the irreducible decomposition of $E$ (respectively $S$). We will assume that $\mathcal{W}\neq\varnothing$, that any two irreducible components of $E$ have at most one intersection point, and that no irreducible exceptional divisor has a self-intersection point. The (good) dual embedded resolution graph associated with $\phi$ will be denoted by $\Gamma(X,f)$. Its vertices $\mathcal{V}=\mathcal{W}\amalg\mathcal{A}$ consist of the nonarrowhead vertices $\mathcal{W}$ and arrowhead vertices $\mathcal{A}$. Any $w\in\mathcal{W}$ is decorated by the self-intersection $e_w$ and genus $g_w$ of $E_w$; and any $v\in\mathcal{V}$ by the multiplicity $m_v$ of $f$. In all our graph diagrams, we put the multiplicities in parentheses (e.g. (3)) and the genera in brackets (e.g. [3]), with the convention that we omit [0].

If we delete the arrows and multiplicities of the graph $\Gamma(X,f)$, we get a possible (good) resolution graph of $(X,x)$.

If $\Gamma$ is a decorated graph (with or without arrowheads), then for $w\in\mathcal{W}$ we denote by $\delta_w$ the number of vertices $v\in\mathcal{V}$ adjacent to $w$. A vertex $w\in\mathcal{W}$ is called a *rupture* (respectively *leaf*) vertex if either $g_w>0$ or $\delta_w\geqslant 3$ (respectively if $\delta_w=1$).

There are many (embedded) resolutions, but they are all connected by quadratic modifications. By the above convention we can blow down a $(-1)$-curve $E_w$ with $g_w=0$ if and only if $\delta_w\leqslant 2$. If the (embedded) resolution graph has no such curve then we say that it is *minimal*. There is a unique minimal (embedded) resolution graph denoted by $\Gamma^{\min}(X)$ (respectively $\Gamma^{\min}(X,f)$).

Let $I:=(E_w\cdot E_v)_{(w,v)\in\mathcal{W}\times\mathcal{W}}$ be the (negative definite) intersection matrix, and write $\det(\Gamma):=\det(-I)>0$. By convention, the determinant of the empty graph is 1.

We will use the following relations connecting the ingerers $\{m_v\}_{v\in\mathcal{V}}$ and $\{e_w\}_{w\in\mathcal{W}}$. Fix a total ordering of the set $\mathcal{W}$. Let $\mathbf{m}_{\mathcal{W}}$ be the column vector with $|\mathcal{W}|$ entries $\{m_w\}_{w\in\mathcal{W}}$. Similarly, define the column vector $\mathbf{m}_{\mathcal{A}}$ with $|\mathcal{W}|$ entries whose $w$th entry is $\sum_{a\in\mathcal{A}\cap\mathcal{V}_w} m_a$. Then

$$I \cdot \mathbf{m}_{\mathcal{W}} + \mathbf{m}_{\mathcal{A}} = 0. \tag{2.1.1}$$

Moreover, if $\Gamma = \Gamma(X)$ *is a tree*, then the inverse matrix $I^{-1}$ can be computed in terms of determinants of some subgraphs as follows. Consider two vertices $w_1, w_2 \in \mathcal{W}$ and the shortest path which connects them. Let $\Gamma_{w_1w_2}$ be the maximal (in general nonconnected) subgraph of $\Gamma$ which has no vertices on this path. Then the $(w_1, w_2)$th entry of $I^{-1}$ is given by

$$I^{-1}_{w_1w_2} = -\det(\Gamma_{w_1w_2})/\det(\Gamma).$$

Let $L_X$ be the (oriented) link of $(X, x)$. It is known that $L_X$ and $\Gamma(X)$ codify the same amount of information. For example, $L_X$ is a rational homology sphere if and only if $g_w = 0$ for any $w$ and $\Gamma(X)$ is a tree.

For more details about the notions of this paragraph, see [Nem00] or [Nem99d].

### 2.2 Irreducible plane curve singularities [BK86, EN85]

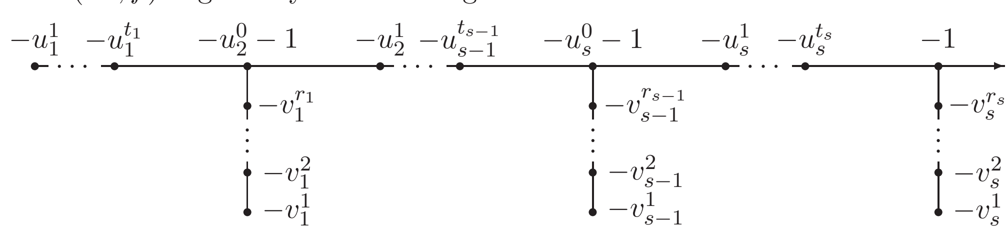
In this case $(X, x) \approx (\mathbb{C}^2, 0)$, $\Gamma(\mathbb{C}^2, f)$ is a tree, and $g_w = 0$ for any $w \in \mathcal{W}$.

In this paper we are mainly interested in *irreducible* plane curve singularities (i.e. when $|\mathcal{A}| = 1$). Their equisingular type and link are completely characterized by the set of *Newton pairs* $\{(p_k, q_k)\}_{k=1}^s$ (see e.g. [EN85, p. 49]). Here $(p_k, q_k) = 1$, $p_k \geqslant 2$, $q_k \geqslant 1$ and $q_1 > p_1$.

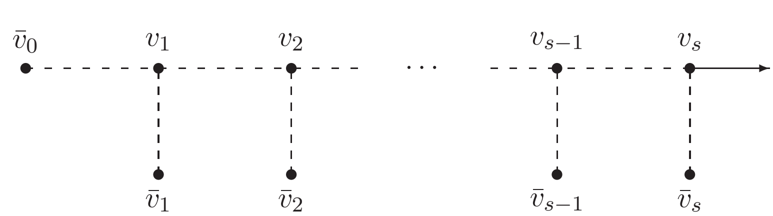
The *minimal* embedded resolution graph can be reconstructed from the Newton pairs as follows (see e.g. [BK86, EN85, Nem99a]). First determine $u_i^l$ and $v_i^l$ ($u_i^0, v_i^0 \geqslant 1$, and $u_i^l, v_i^l \geqslant 2$ for $l > 0$) from the continued fractions:

$$\frac{p_i}{q_i} = u_i^0 - \cfrac{1}{u_i^1 - \cfrac{1}{\ddots - \cfrac{1}{u_i^{t_i}}}}; \qquad \frac{q_i}{p_i} = v_i^0 - \cfrac{1}{v_i^1 - \cfrac{1}{\ddots - \cfrac{1}{v_i^{r_i}}}}.$$

Then $\Gamma^{\min}(\mathbb{C}^2, f)$ is given by the following:




This has the following schematic form:




Here we have emphasized only those vertices $\{\bar{v}_k\}_{k=0}^s$ and $\{v_k\}_{k=1}^s$ which have degree $\delta \neq 2$. We denote the set of these vertices by $\mathcal{W}^*$. The dashed line between two such vertices replaces a string . In our discussions below, the corresponding self-intersections will be less important, but the multiplicities of the vertices $v \in \mathcal{W}^*$ will be crucial. They can be easily described in terms of the integers $\{a_k\}_{k=1}^s$:

$$a_1 = q_1 \quad \text{and} \quad a_{k+1} = q_{k+1} + p_{k+1}p_k a_k \quad \text{if } s-1 \geqslant k \geqslant 1. \tag{2.2.1}$$

Then again, $(p_k, a_k) = 1$ for any $k$. Clearly, the two sets of pairs $\{(p_k, q_k)\}_{k=1}^s$ and $\{(p_k, a_k)\}_{k=1}^s$ determine each other completely. In fact, the set of pairs $\{(p_k, a_k)\}_{k=1}^s$ constitutes the set of decoration of the so-called *splice* (or *Eisenbud–Neumann*) diagram of $f$; cf. [EN85, p. 51]. Then by [EN85, § 10], one has:

$$\begin{aligned} m_{v_k} &= a_k p_k p_{k+1} \cdots p_s, \quad \text{for } 1 \leqslant k \leqslant s; \\ m_{\bar{v}_0} &= p_1 p_2 \cdots p_s; \\ m_{\bar{v}_k} &= a_k p_{k+1} \cdots p_s, \quad \text{for } 1 \leqslant k \leqslant s-1; \\ m_{\bar{v}_s} &= a_s. \end{aligned} \tag{2.2.2}$$

### 2.3 Cyclic coverings

For any isolated plane curve singularity $f$ we write

$$X_{f,n} := \{z^n = f(x,y)\} \subset (\mathbb{C}^3, 0).$$

Then $(x,y,z) \mapsto (x,y)$ induces a $\mathbb{Z}_n$-Galois covering of $(X_{f,n}, 0)$ over $(\mathbb{C}^2, 0)$, and the $z$-projection induces a germ $(X_{f,n}, 0) \to (\mathbb{C}, 0)$, still denoted by $z$.

From any fixed (good) embedded resolution graph $\Gamma(\mathbb{C}^2, f)$ of $f$ and the integer $n$, one can construct a possible dual resolution graph $\Gamma(X_{f,n}, z)$, respectively $\Gamma(X_{f,n})$. A precise algorithm is given in [Nem99c] (or [Nem99d]; for a more general situation, see [Nem00]).

Assume that in this algorithm, we start with the *minimal* (good) embedded resolution graph $\Gamma^{\min}(\mathbb{C}^2, f)$ of $f$. Then the output graph provided by the algorithm (without any modification by any blow-up or blow-down) will be called the *canonical* embedded resolution graph of $(X_{f,n}, z)$, and will be denoted by $\Gamma^{\mathrm{can}}(X_{f,n}, z)$. (The name is motivated by [Lau78], where Laufer proved that the above algorithm for a plane curve singularity $f$ provides exactly the canonical resolution of $X_{f,n}$ in the sense of Zariski, provided that $n = 2$.)

## 3. The embedded resolution graph $\Gamma^{\mathrm{can}}(X_{f,n}, z)$

In this section we make the above-mentioned algorithm for $\Gamma^{\mathrm{can}}(X_{f,n}, z)$ explicit. We assume that $n \geqslant 2$ and $f$ is irreducible.

First, we fix some notation. Recall that $\{(p_k, q_k)\}_{k=1}^s$ denotes the set of Newton pairs of $f$, and the integers $\{a_k\}_{k=1}^s$ are defined in (2.2.1). Additionally, we define

- $d_k := (n, p_{k+1}p_{k+2} \cdots p_s)$ for $0 \leqslant k \leqslant s-1$, and $d_s := 1$;
- $h_k := d_{k-1}/d_k = (p_k, n/d_k)$ and $p'_k := p_k/h_k$ for $1 \leqslant k \leqslant s$;
- $\tilde{h}_k := (a_k, n/d_k)$ and $a'_k := a_k/\tilde{h}_k$ for $1 \leqslant k \leqslant s$.

In order to run the algorithm, besides (2.2.2), one also needs some additional data about $\Gamma(\mathbb{C}^2, f)$. For the convenience of the reader we collect them in the next lemma. For the proof, see e.g. [Neu87] or the proof of (3.2) in [Nem98]. Set $M_w := \gcd\{m_v \mid v = w \text{ or } v \text{ is adjacent to } w\}$.

Lemma 3.1. *In $\Gamma^{\min}(\mathbb{C}^2, f)$ one has: $M_{\bar{v}_k} = m_{\bar{v}_k}$ $(0 \leqslant k \leqslant s)$; $M_{v_k} = p_{k+1} \cdots p_s$ $(1 \leqslant k \leqslant s-1)$; and $M_{v_s} = 1$.*

*Moreover, for any $1 \leqslant k \leqslant s$, fix two integers $i_k$ and $j_k$ with $a_k i_k + p_k j_k = 1$. Then the multiplicities of the three vertices adjacent to $v_k$, modulo $m_{v_k} = a_k p_k p_{k+1} \cdots p_s$, are: $(-i_k a_k p_{k+1} \cdots p_s;\ -j_k p_k p_{k+1} \cdots p_s;\ p_{k+1} \cdots p_s)$ if $k \leqslant s-1$, and $(-i_s a_s;\ -j_s p_s;\ 1)$ if $k = s$.*

Now we start to discuss $\Gamma^{\mathrm{can}}(X_{f,n}, z)$. By the algorithm, this graph can be considered as a 'covering' $q : \Gamma^{\mathrm{can}}(X_{f,n}, z) \to \Gamma^{\min}(X, f)$. Above $w \in \mathcal{W}(\Gamma^{\min}(\mathbb{C}^2, f))$ there are $(M_w, n)$ vertices $v' \in q^{-1}(w)$ of $\Gamma^{\mathrm{can}}(X_{f,n}, z)$, each with some multiplicity $\mathsf{m}_{v'}$ and genus $\mathsf{g}_{v'}$. For a fixed $w \in \mathcal{W}^*(\Gamma^{\min}(\mathbb{C}^2, f))$, the Galois action guarantees that the integers $\mathsf{m}_{v'}$ and $\mathsf{g}_{v'}$ do not depend on the choice of $v' \in q^{-1}(w)$, but only on $w$. Sometimes, we denote them by $\mathsf{m}_w$ and $\mathsf{g}_w$. The reader can easily verify that (2.2.2), Lemma 3.1 and the algorithm gives the following corollary:

Corollary 3.2. a) *For any irreducible $f$ and $n$, $\Gamma^{\mathrm{can}}(X_{f,n}, z)$ is a tree with*

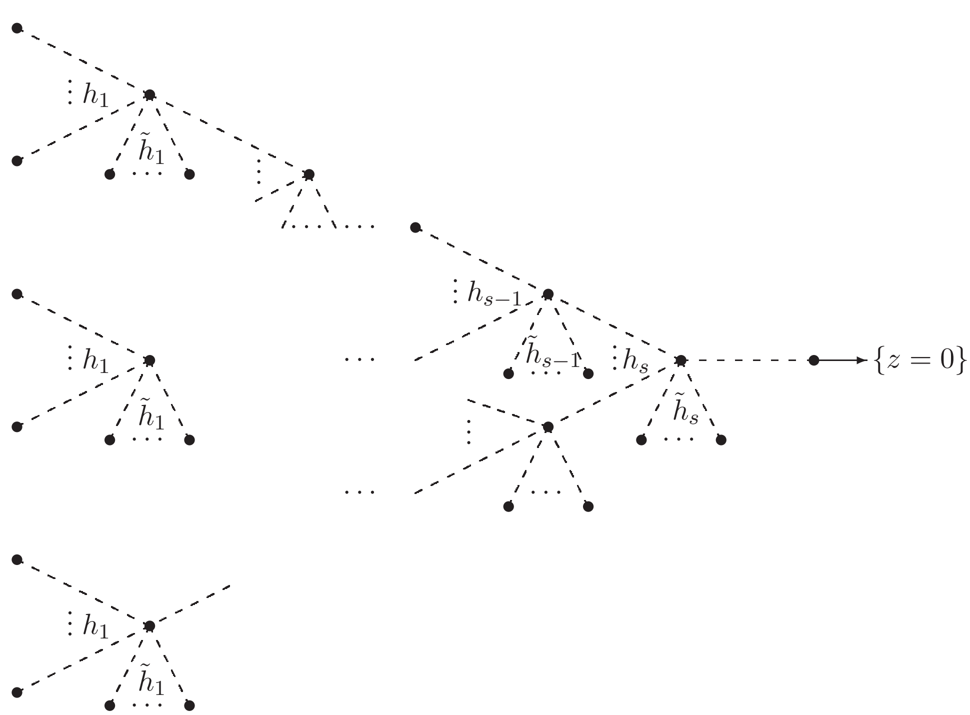
$$\#q^{-1}(v_s) = 1,\ \#q^{-1}(v_k) = h_{k+1}\cdots h_s \quad (1 \leqslant k \leqslant s-1),$$
$$\#q^{-1}(\bar{v}_s) = \tilde{h}_s,\ \#q^{-1}(\bar{v}_k) = \tilde{h}_k h_{k+1}\cdots h_s \quad (1 \leqslant k \leqslant s-1),$$
$$\#q^{-1}(\bar{v}_0) = h_1\cdots h_s.$$

b)
$$\mathsf{m}_{\bar{v}_0} = p'_1 p'_2 \cdots p'_s,$$
$$\mathsf{m}_{\bar{v}_k} = a'_k p'_{k+1}\cdots p'_s \quad (1 \leqslant k \leqslant s-1),$$
$$\mathsf{m}_{\bar{v}_s} = a'_s,$$
$$\mathsf{m}_{v_k} = a'_k p'_k p'_{k+1}\cdots p'_s \quad (1 \leqslant k \leqslant s).$$

c)
$$\mathsf{g}_{\bar{v}_k} = 0 \quad (0 \leqslant k \leqslant s),$$
$$\mathsf{g}_{v_k} = (h_k - 1)(\tilde{h}_k - 1)/2 \quad (1 \leqslant k \leqslant s).$$

*In particular, the link of $X_{f,n}$ is a rational homology sphere if and only if $(h_k - 1)(\tilde{h}_k - 1) = 0$ for any $1 \leqslant k \leqslant s$.*

The graph $\Gamma^{\mathrm{can}}(X_{f,n}, z)$ has the following schematic form (where the dashed lines replace strings as above, and we omit the genera and the self-intersections):



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*Example* 3.3. Assume that $s=1$ and write $p=p_1$ and $a=a_1$. Take $n$ such that $h=(p,n)$ and $\tilde{h}=(a,n)=1$. Then $X_{f,n}$ can be identified with the Brieskorn hypersurface singularity $\{(x,y,z)\in\mathbb{C}^3 : x^a+y^p+z^n=0\}$. Then the link is a Seifert 3-manifold with Seifert invariants: $a,a,\dots,a,p/h,n/h$ ($a$ appearing $h$ times, hence all together there are $h+2$ special fibers corresponding to the $h+2$ arms cf. the above graph diagram). These numbers also give (up to a sign) the determinants of the corresponding arms of the graph $\Gamma(X_{f,n})$. For more details about Seifert manifolds and their plumbings, see e.g. [JN83, NR78].

### 3.4 The maximal strings of $\Gamma^{\mathrm{can}}(X_{f,n},z)$

The next goal is to compute the determinants of the maximal strings of $\Gamma^{\mathrm{can}}(X_{f,n},z)$. For this, fix a vertex $w\in\mathcal{W}^*(\Gamma^{\min}(\mathbb{C}^2,f))$ and $v'\in q^{-1}(w)$. Consider the shortest path in $\Gamma^{\mathrm{can}}(X_{f,n},z)$ which connects $v'$ and the arrowhead.

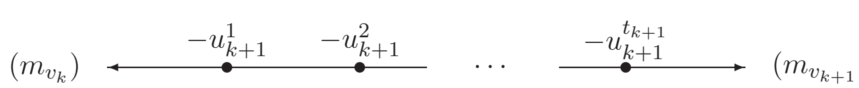
If $w\neq v_s$, then on this path there is at least one rupture vertex of $\Gamma^{\mathrm{can}}(X_{f,n},z)$. Let $v''$ ($v''\neq v'$) be the closest one to $v'$. If $w=v_k$ ($1\leqslant k\leqslant s-1$), then let $\Gamma(v')$ be the string which contains all the vertices between $v'$ and $v''$ (excluding $v'$ and $v''$), and all the edges connecting them. If $w=\bar{v}_k$ ($1\leqslant k\leqslant s$), then $\Gamma(v')$ is the string constructed similarly, but this time we include $v'$ and its connecting edge as well. If $w=v_s$, then the above path is already a string. Let $\Gamma(v')$ be the string which contains all the vertices between $v'$ and the arrowhead (excluding $v'$), and all the edges connecting them.

In this way we have a codification of all the maximal strings of $\Gamma^{\mathrm{can}}(X_{f,n},z)$. Notice also that the isomorphism type of the string $\Gamma(v')$ does not depend on the choice of $v'\in q^{-1}(w)$, but only on $w$. Therefore, sometimes it is preferable to denote this type by $\Gamma(w)$. Denote by $D(v')$ (or by $D(w)$) the determinant $\det(\Gamma(v'))$. If $\Gamma(w)=\varnothing$, then by definition $D(w)=1$.

PROPOSITION 3.5. *The possible values of $D(w)$ are the following:*

$$\begin{aligned} &D(\bar{v}_0)=a'_1,\\ &D(\bar{v}_k)=p'_k \quad (1\leqslant k\leqslant s),\\ &D(v_s)=n/(h_s\tilde{h}_s),\\ &D(v_k)=n\cdot q_{k+1}/(d_{k-1}\tilde{h}_k\tilde{h}_{k+1}) \quad (1\leqslant k\leqslant s-1).\end{aligned}$$

*Proof.* We start with the ('difficult') case $D(v_k)$ ($1\leqslant k\leqslant s-1$). Using the notation of § 2.2, the maximal string in $\Gamma^{\min}(\mathbb{C}^2,f)$ between $v_k$ and $v_{k+1}$ has the following form:




Here $m_{v_k}=a_kp_k\cdots p_s$ and $m_{v_{k+1}}=a_{k+1}p_{k+1}\cdots p_s$; cf. (2.2.2). Moreover, $p_{k+1}/q_{k+1}=u^0_{k+1}-\lambda/q_{k+1}$, and the quotient $q_{k+1}/\lambda$ gives the continued fraction $[u^1_{k+1},\cdots,u^{t_{k+1}}_{k+1}]$. This graph can be identified with the embedded resolution graph of a germ defined on a Hirzebruch–Jung singularity. Indeed, using e.g. [Nem00, p. 102], one can verify that one can take the germ $(z^{p_ka_k}y)^{p_{k+1}\cdots p_s}$ defined on the normalization of $\{z^{q_{k+1}}=xy^{p_{k+1}}\}$. In particular, the collection of graphs $\{\Gamma(v')\}_{v'\in q^{-1}(v_k)}$ is exactly the (nonconnected) graph of the normalization of

$$X=\{(x,y,z,w): z^{q_{k+1}}=xy^{p_{k+1}};\ w^n=(z^{p_ka_k}y)^{p_{k+1}\cdots p_s}\}.$$

This $X$ has $d_k=(n,p_{k+1}\cdots p_s)$ (isomorphic) irreducible components, a number that agrees exactly with $\#q^{-1}(v_k)$. Hence, $D(v')$ is the graph of the normalization of

$$X_1=\{z^{q_{k+1}}=xy^{p_{k+1}};\ w^{n/d_k}=(z^{p_ka_k}y)^{(p_{k+1}\cdots p_s)/d_k}\}.$$



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Let us write $q = q_{k+1}$, $p = p_{k+1}$, $N = n/d_k$, $r = p_k a_k$, $P = p_{k+1} \cdots p_s/d_k$ and $a = a_{k+1}$. Then $(q,p) = 1$, $(N,P) = 1$, $a = rp + q$, and $X_1 = \{z^q = xy^p;\ w^N = (z^r y)^P\}$ in $(\mathbb{C}^4, 0)$. We will verify that the determinant of the graph of $X_1$ is

$$n = \frac{Nq}{(N,r)(N,a)}.$$

In order to prove this last statement, first we show that we can assume $P = 1$. Indeed, consider the space $Y_1 := \{z^q = xy^p,\ w = t^P,\ t^N = z^r y\}$. Then $X_1$ and $Y_1$ are birationally equivalent (eliminate $t$ from the equation for $Y_1$); hence their normalizations are the same. On the other hand, $Y_1$ is isomorphic with $X_1' := \{z^q = xy^p,\ t^N = z^r y\}$ (eliminate $w$).

Then notice that $X_1$ is irreducible and its normalization is a Hirzebruch–Jung singularity. Indeed, the discriminant of the finite map $\rho : (X_1, 0) \to (\mathbb{C}^2, 0)$ induced by the $(x,y)$-projection is the union of the coordinate lines. Let $\pi_1 = \mathbb{Z}^2$ be the fundamental group of $\{xy \neq 0\}$ generated by $e_1$ and $e_2$ representing two elementary loops around the axes $x$ and $y$. Let $\rho_* : \pi_1 \to G$ be the monodromy representation of the restriction of $\rho$ above $\{xy \neq 0\}$, and $\rho_*|\mathbb{Z}(e_i)$ be its restriction to $\mathbb{Z}(e_i)$, the subgroup generated by $e_i$ $(i = 1,2)$. Then one can verify (using e.g. [BPV84, III.5]) that $\mathbb{Z}_n \approx \ker(\rho_*)/\ker(\rho_*|\mathbb{Z}(e_1)) \times \ker(\rho_*|\mathbb{Z}(e_2))$.

In our case the Galois group $G$ of the induced regular covering can be identified with $G = \{(\xi,\eta) \in \mathbb{C}^* \times \mathbb{C}^* \ :\ \xi^q = 1, \eta^N = \xi^r\}$. Since $\rho_*$ is onto, $\ker(\rho_*)$ has index $Nq$ in $\mathbb{Z}^2$. Moreover, $\rho(e_1) = (\exp(2\pi i/q), \exp(2\pi i r/(Nq)))$, and hence $\ker(\rho_*|\mathbb{Z}(e_1)) = k_1\mathbb{Z}$ for $k_1 = Nq/(N,r)$. Similarly, $\rho(e_2) = (\exp(2\pi i p/q), \exp(2\pi i a/(Nq)))$, and hence $\ker(\rho_*|\mathbb{Z}(e_1)) = k_2\mathbb{Z}$ for $k_2 = Nq/(N,a)$. Hence the claim follows.

The other identities can be computed by a similar argument. But also notice that in all other cases the corresponding maximal string contains a leaf vertex of $\Gamma^{\text{can}}(X_{f,n})$. Therefore, $D(v')$ can be identified with the corresponding Seifert invariant, similarly as in Example 3.3. Hence these identities also follow from Example 3.3. □

*Remark* 3.6. In $\Gamma^{\text{can}}(X_{f,n}, z)$ the following hold:

1) If $w = \bar{v}_k$ $(0 \leqslant k \leqslant s)$ then $\Gamma(w) \neq \varnothing$. Indeed, $\Gamma(w)$ contains at least as many vertices as the corresponding arm in $\Gamma^{\min}(\mathbb{C}^2, f)$, which is clearly not empty.

2) The same argument is valid for any $\Gamma(v_k)$ $(1 \leqslant k \leqslant s-1)$ provided that $q_{k+1} > 1$. In fact, for such $w = v_k$, $\Gamma(v_k) = \varnothing$ if and only if $q_{k+1} = 1$ and $n = d_{k-1}\tilde{h}_k\tilde{h}_{k+1}$.

3) $\Gamma(v_s) = \varnothing$ if and only if $n = h_s\tilde{h}_s$.

Here a natural question appears: Is it possible to distinguish the arms $\Gamma(v')$ $(v' \in q^{-1}(\bar{v}_0))$ from the arms of type $\Gamma(v')$ $(v' \in q^{-1}(\bar{v}_1))$? The next corollary says that if $\mathsf{g}_{v_1} = 0$ then already their determinants are different.

Corollary 3.7. a) *If $D(\bar{v}_0) = D(\bar{v}_1)$ then $D(\bar{v}_0) = D(\bar{v}_1) = 1$ and $\mathsf{g}_{v_1} \neq 0$.*

b) *If $D(\bar{v}_s) = D(v_s)$ then $D(\bar{v}_s) = D(v_s) = 1$.*

*Proof.* a) If $D(\bar{v}_0) = D(\bar{v}_1)$, then $a_1/\tilde{h}_1 = p_1/h_1$ by Proposition 3.5. Since $(a_1, p_1) = 1$, one gets $a_1/\tilde{h}_1 = p_1/h_1 = 1$. But then $h_1 \geqslant 2$ and $\tilde{h}_1 \geqslant 2$ since $a_1 = q_1 > p_1 \geqslant 2$.

b) Similarly by Proposition 3.5 one has $p_s/h_s = n/h_s\tilde{h}_s$. But these two numbers are also relatively prime. □

### 3.8 The subgraphs $\Gamma_{\pm}(v_k)$

Above we discussed the case of maximal strings of

$$\Gamma^{\text{can}}(X_{f,n}, z).$$

Obviously, one can consider the determinants of much bigger subgraphs delimited by different rupture vertices. In this way one obtains a large number of rather subtle invariants of this graph. Nevertheless, in order to recover the Newton pairs of $f$ and the integer $n$ from this graph, it is enough to consider only a restrictive subfamily of them.

Let us fix an integer $k$ ($1 \leqslant k \leqslant s$). Consider the maximal subgraph of $\Gamma^{\mathrm{can}}(X_{f,n}, z)$ which does not contain any vertex from the set $q^{-1}(v_k)$. It has many connected components. The component which supports the arrowhead of $\Gamma^{\mathrm{can}}(X_{f,n}, z)$ is denoted by $\Gamma_+(v_k)$. There are $\tilde{h}_k h_{k+1} \cdots h_s$ more components (isomorphic to each other), which contain vertices above $\bar{v}_k$. They are strings of type $\Gamma(\bar{v}_k)$ (cf. Corollary 3.2, part a and § 3.4). Finally, there are $h_k \cdots h_s$ isomorphic components containing vertices above $\bar{v}_0$. We denote such a component by $\Gamma_-(v_k)$, and $D_\pm(v_k)$ denotes $\det(\Gamma_\pm(v_k))$.

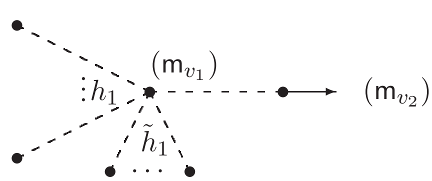
Obviously, $\Gamma_-(v_1) = \Gamma(\bar{v}_0)$, and $\Gamma_+(v_s) = \Gamma(v_s)$ whose determinants are computed in Proposition 3.5.

Proposition 3.9. *Assume that $s \geqslant 2$.*

a) $D_-(v_2) = (a'_1)^{h_1-1} \cdot (p'_1)^{\tilde{h}_1-1} \cdot a'_2$.

b) $D_+(v_{s-1}) = n \cdot D(v_{s-1})^{h_s-1} \cdot D(\bar{v}_s)^{\tilde{h}_s-1}/(h_s h_{s-1} \tilde{h}_{s-1})$.

*Proof.* a) Fix a vertex $v' \in q^{-1}(v_2)$ and one of the graphs $\Gamma_-(v')$. Let $w_1$ be its unique rupture point, and let $w_2$ denote that vertex which was connected by an edge with $v'$ in $\Gamma^{\mathrm{can}}(X_{f,n}, z)$. (If $\Gamma(v_1) = \varnothing$ then $w_1 = w_2$, but the proof is valid in this case as well.) We put back on the vertices of $\Gamma_-(v')$ the multiplicities of $\Gamma^{\mathrm{can}}(X_{f,n}, z)$. They will form a compatible set (i.e. will satisfy (2.1.1)) provided that we put on $w_2$ an arrow with multiplicity $\mathsf{m}_{v'} = \mathsf{m}_{v_2}$. This graph with arrowhead and multiplicities has the following schematic form:

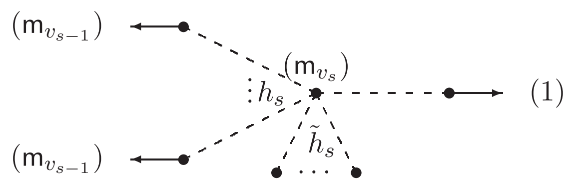



Notice that $\Gamma_-(v') \setminus \{w_1\}$ has $h_1 + \tilde{h}_1 + 1$ connected components, $h_1$ of type $\Gamma(\bar{v}_0)$, $\tilde{h}_1$ of type $\Gamma(\bar{v}_1)$, and one of type $\Gamma(v_1)$. Therefore, by § 2.1 one gets the following:

$$\frac{\mathsf{m}_{v_1}}{\mathsf{m}_{v_2}} = -I^{-1}_{w_1 w_2} = \frac{D(\bar{v}_0)^{h_1} \cdot D(\bar{v}_1)^{\tilde{h}_1}}{D_-(v_2)}.$$

Now, use Corollary 3.2, part b and Proposition 3.5.

b) We proceed similarly, but now with the graph $\Gamma_+(v_{s-1})$. Its schematic form, together with the multiplicities of $\Gamma^{\mathrm{can}}(X_{f,n}, z)$, is as follows:




If from this graph we delete its rupture point (and the arrows and multiplicities) then we get the following connected components: $h_s$ of type $\Gamma(v_{s-1})$, one of type $\Gamma(v_s)$, and $\tilde{h}_s$ of type $\Gamma(\bar{v}_s)$.



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Therefore, from § 2.1, similarly as above, one gets the following:

$$\mathsf{m}_{v_s} = \frac{D(v_{s-1})^{h_s} \cdot D(\bar{v}_s)^{\tilde{h}_s}}{D_+(v_{s-1})} + h_s \cdot \frac{D(v_{s-1})^{h_s-1} \cdot D(\bar{v}_s)^{\tilde{h}_s} \cdot D(v_s)}{D_+(v_{s-1})} \cdot \mathsf{m}_{v_{s-1}}.$$

Then use again Corollary 3.2, part b and Proposition 3.5 (and $a_s = q_s + p_s p_{s-1} a_{s-1}$). □

*Remark* 3.10. By a similar argument one can prove the next identity for $k \geqslant 2$ (which is not needed later):

$$\frac{D_-(v_k)}{a'_k} = \left[\frac{D_-(v_{k-1})}{a'_{k-1}}\right]^{h_{k-1}} \cdot (a'_{k-1})^{h_{k-1}-1} \cdot (p'_{k-1})^{\tilde{h}_{k-1}-1}.$$

## 4. From $\Gamma^{\mathrm{can}}(X_{f,n}, z)$ to $\Gamma^{\mathrm{min}}(X_{f,n})$

Let $\Gamma^{\min}(X_{f,n}, z)$ be the *minimal* embedded resolution graph of $(X_{f,n}, z)$. This can be obtained from $\Gamma^{\mathrm{can}}(X_{f,n}, z)$ by a sequence of blow-downs (and without any blow-up).

PROPOSITION 4.1. *All the rupture vertices of $\Gamma^{\mathrm{can}}(X_{f,n}, z)$ survive in $\Gamma^{\min}(X_{f,n}, z)$ as rupture vertices (i.e. they are not blown down in the minimalization procedure, and in $\Gamma^{\min}(X_{f,n}, z)$ they still live as rupture vertices).*

*Proof.* From the theory of Hirzebruch–Jung singularities it follows that a string of type $\Gamma(\bar{v}_k)$ $(0 \leqslant k \leqslant s)$ is completely collapsed in the minimalization procedure if and only if its determinant $D(\bar{v}_k)$ equals 1. First we verify that all the rupture vertices above $v_1$ will survive (as rupture vertices). Let $v'$ be one of them considered in $\Gamma^{\mathrm{can}}$. It supports $h_1$ strings of type $\Gamma(\bar{v}_0)$, $\tilde{h}_1$ strings of type $\Gamma(\bar{v}_1)$ and another edge, denoted by $e$. Recall that $D(\bar{v}_0) = a'_1$ and $D(\bar{v}_k) = p'_k$; cf. Proposition 3.5. By Corollary 3.7, if both $D(\bar{v}_0)$ and $D(\bar{v}_1)$ equal 1, then $\mathsf{g}_{v'} \neq 0$. Hence $v'$ will be a rupture vertex in $\Gamma^{\min}(X_{f,n}, z)$.

If $D(\bar{v}_0) \neq 1$ but $D(\bar{v}_1) = p_1/h_1 = 1$ then the strings of type $\Gamma(\bar{v}_0)$ will survive. Their number is $h_1 = p_1 \geqslant 2$. Symmetrically, if $D(\bar{v}_1) \neq 1$ but $D(\bar{v}_0) = a_1/\tilde{h}_1 = 1$ then $\tilde{h}_1 = a_1 \geqslant 2$ strings of type $\Gamma(\bar{v}_1)$ will survive. If both determinants are greater than 1, then all the strings will survive with total number $h_1 + \tilde{h}_1 \geqslant 2$. Since the arrowhead survives, and $\Gamma^{\min}(X_{f,n}, z)$ is connected, the edge $e$ will survive as well. Hence $v'$ has degree at least 3 in $\Gamma^{\min}(X_{f,n}, z)$.

By induction, we assume that for a fixed $k$, all the rupture vertices above any $v_i \in \mathcal{W}^*$ survive for any $i \leqslant k-1$. We show that this is the case for $v_k$ as well. For this, fix an arbitrary $v' \in q^{-1}(v_k)$. First notice that by the inductive step, the $h_k$ subgraphs $\Gamma_-(v')$ will survive (they cannot be completely contracted since they contain rupture points that survive). Similarly as above, since the arrowhead survives, the edge connecting $v'$ with $\Gamma_+(v_k)$ will also survive. If $D(\bar{v}_k) = 1$, then $h_k = p_k \geqslant 2$. If $D(\bar{v}_k) \neq 1$, then all the graphs $\Gamma(\bar{v}_k)$ will survive. Hence, in any case $\delta_{v'} \geqslant 3$ in $\Gamma^{\min}(X_{f,n}, z)$. □

Now, recall that $\Gamma^{\min}(X_{f,n})$ denotes the minimal (good) resolution graph of $(X_{f,n}, 0)$. It can be obtained from $\Gamma^{\min}(X_{f,n}, z)$ by deleting its arrowhead (and all the multiplicities) and blowing down successively all the $(-1)$-curves with genus 0 and *new* degree $\leqslant 2$. In fact, there is exactly one case when after deleting the arrowhead of $\Gamma^{\min}(X_{f,n}, z)$ we do not obtain a minimal graph, and this is described completely in the next proposition. In the sequel we refer to this 'pathological' situation as the 'P-case'.

PROPOSITION 4.2. *Assume that by deleting the arrowhead of $\Gamma^{\min}(X_{f,n}, z)$ we obtain a nonminimal graph. Then $\Gamma^{\min}(X_{f,n}, z)$ has the following schematic form with the two left branches isomorphic*



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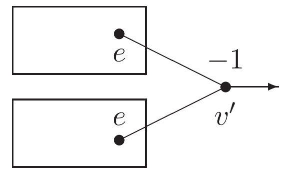
*and with $e \leqslant -3$ (we omit the multiplicities). The rational $(-1)$-curve is the unique vertex $v' = q^{-1}(v_s)$ (which survives in $\Gamma^{\min}(X_{f,n}, z)$; cf. Proposition 4.1).*




*This situation can happen if and only if $n = p_s = 2$.*

*In this case, $\Gamma^{\min}(X_{f,n})$ is obtained from $\Gamma^{\min}(X_{f,n}, z)$ by deleting its arrowhead and blowing-down $v'$. No other blow-downs are necessary.*

*Moreover, in this case, all the vertices of $\Gamma^{\min}(X_{f,n}, z)$ have genus zero.*

*Proof.* If the graph obtained from $\Gamma^{\min}(X_{f,n}, z)$ by deleting its arrowhead is not minimal, then the vertex $v'$ in $\Gamma^{\min}(X_{f,n}, z)$ which supports the arrowhead should be a $(-1)$ rational curve of degree 3 in $\Gamma^{\min}(X_{f,n}, z)$. This can happen only if this vertex $v'$ is exactly the unique vertex $q^{-1}(v_s)$ (and $\mathsf{g}_{v_s} = 0$). This also shows that $\Gamma(v_s)$ was collapsed in $\Gamma^{\min}(X_{f,n}, z)$; hence $D(v_s) = n/h_s\tilde{h}_s = 1$. Hence we obtain that

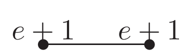
$$h_s\tilde{h}_s = n \geqslant 2 \quad \text{and} \quad (h_s-1)(\tilde{h}_s-1) = \mathsf{g}_{v_s} = 0.$$

Assume first that $h_s = 1$ and $\tilde{h}_s > 1$. Since $D(\bar{v}_s) = p_s > 1$, the $\tilde{h}_s$ strings $\Gamma(\bar{v}_s)$ are present in $\Gamma^{\min}(X_{f,n}, z)$. This can happen if and only if $\tilde{h}_s = 2$ and $\Gamma_-(v_s)$ is collapsed completely in $\Gamma^{\min}(X_{f,n}, z)$. Since for $s \geqslant 2$ the rupture points $q^{-1}(v_1)$ survive in $\Gamma^{\min}(X_{f,n}, z)$, this can happen only if $s = 1$ and $D(\bar{v}_0) = a_1/\tilde{h}_1 = 1$. This shows that $a_1 = \tilde{h}_1 = 2$, which contradicts the inequality $a_1 = q_1 > p_1 \geqslant 2$.

Therefore $\tilde{h}_s = 1$ and $h_s > 1$. Since the degree of $v'$ (in $\Gamma^{\min}(X_{f,n}, z)$) is at least $1 + h_s$ (hence $1 + h_s \leqslant 3$), one gets $h_s = 2$, and also the fact that the graphs of type $\Gamma(\bar{v}_s)$ are collapsed in $\Gamma^{\min}(X_{f,n}, z)$; hence $p_s/h_s = 1$. Therefore, $n = h_s = p_s = 2$.

Then $\tilde{h}_k = 1$, and hence $\mathsf{g}_{v_k} = 0$ for any $k$.

Finally, notice that $e \leqslant -3$ (cf. the diagram) since after we blow down $v'$ we get a subgraph of type




which must be negative definite.

On the other hand, one can verify easily that, if $n = p_s = 2$, then the above situation always occurs. □

*Remark* 4.3. Assume that $\mathsf{g}_{v_k} = 0$. If a family of strings supported by any fixed $v' \in q^{-1}(v_k)$ is collapsed completely during the minimalization procedure, then the cardinality of this family (in spite of the fact that it is missing in $\Gamma^{\min}(X_{f,n})$) can be determined, and it is 1. More precisely, if for $k \geqslant 1$, the $\tilde{h}_k$ graphs of type $\Gamma(\bar{v}_k)$ are completely collapsed, then $h_k = p_k \geqslant 2$. Then by the genus formula of Corollary 3.2, part c, one gets $\tilde{h}_k = 1$. Similarly, if $k = 1$ and the $h_1$ graphs of type $\Gamma(\bar{v}_0)$ are collapsed, then $h_1 = 1$.

In order to recover the Newton pairs of $f$ and the integer $n$ from the graph $\Gamma^{\min}(X_{f,n})$, we need some information about some subgraphs $\mathcal{G}$ of $\Gamma^{\min}(X_{f,n})$ of the following type. Each $\mathcal{G}$ is a connected component of $\Gamma^{\min}(X_{f,n}) \setminus \{v\}$ for some rupture vertex $v$ of $\Gamma^{\min}(X_{f,n})$, and it contains exactly one rupture vertex of $\Gamma^{\min}(X_{f,n})$. In a general setting their precise definition is the following.



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### 4.4 Definitions

Let $\Gamma$ be a decorated tree (with self-intersections and genera $\{g_v\}_v$, without arrowheads and multiplicities). Assume that it has at least two rupture vertices.

1) Let $\Gamma(\mathcal{R})$ be the minimal connected subgraph of $\Gamma$ which contains all the rupture vertices $\mathcal{R}$ of $\Gamma$. Let $\mathcal{L}(\Gamma(\mathcal{R}))$ be the set of leaf vertices of $\Gamma(\mathcal{R})$. For any $v \in \mathcal{L}(\Gamma(\mathcal{R}))$ let $\mathcal{G}(v)$ be the maximal connected subgraph of $\Gamma$ which contains $v$ but contains no other rupture vertex of $\Gamma$. The determinant $\det(\mathcal{G}(v))$ is denoted by $\mathcal{D}(v)$.

2) For any $v \in \mathcal{L}(\Gamma(\mathcal{R}))$, let $v_{\mathrm{root}}$ be the unique rupture vertex of $\Gamma$ with the property that on the shortest path in $\Gamma$ connecting $v$ and $v_{\mathrm{root}}$ there are no other rupture vertices of $\Gamma$. Then clearly $v_{\mathrm{root}}$ is adjacent with a certain vertex of $\mathcal{G}(v)$ (in fact $\mathcal{G}(v)$ is one of the connected components of $\Gamma \setminus \{v_{\mathrm{root}}\}$).

3) For each rupture vertex $v \in \mathcal{R}$, denote by $\mathcal{S}t(v)$ the set of maximal strings of $\Gamma$ which are supported by $v$ (on one end) and contain a leaf vertex of $\Gamma$ (on the other end). More precisely, these strings are those connected components of $\Gamma \setminus \Gamma(\mathcal{R})$ which have an adjacent vertex with $v$ (in $\Gamma$). We write $\mathcal{S}t(v)$ as a disjoint union of its subsets $\{\mathcal{S}t_i(v)\}_{i\in I(v)}$ which are the level sets of $\det : \mathcal{S}t(v) \to \mathbb{Z}$. We set $D_i := \det(St)$ for $St \in \mathcal{S}t_i(v)$ and $\#_i := \#\mathcal{S}t_i(v)$. Then we define

$$\mathcal{D}_{St}(v) := \begin{cases} \prod_{i\in I(v)} D_i^{\#_i} & \text{if } \mathcal{S}t(v) \neq \varnothing, \\ 1 & \text{if } \mathcal{S}t(v) = \varnothing, \end{cases}$$

$$\mathcal{D}_{St}^{\mathrm{red}}(v) := \begin{cases} \prod_{i\in I(v)} D_i^{\#_i - 1} & \text{if } \mathcal{S}t(v) \neq \varnothing, \\ 1 & \text{if } \mathcal{S}t(v) = \varnothing, \end{cases}$$

and $\alpha(v) \in \mathbb{Q} \cup \{\infty\}$ by

$$\alpha(v) = \begin{cases} \prod_{i\in I(v)} \#_i & \text{if } \mathcal{S}t(v) \neq \varnothing, \\ 1 & \text{if } \mathcal{S}t(v) = \varnothing \text{ and } g_v = 0, \\ \dfrac{2g_v}{\delta_v - 2} + 1 & \text{if } \mathcal{S}t(v) = \varnothing \text{ and } g_v \neq 0. \end{cases}$$

[$\alpha(v) = \infty$ if and only if the degree $\delta_v$ of $v$ in $\Gamma$ is 2, $\mathcal{S}t(v) = \varnothing$ and $g_v \neq 0$.]

4) For each $v \in \mathcal{L}(\Gamma(\mathcal{R}))$ we define the $\beta$-invariant by

$$\beta(v) := \frac{\mathcal{D}(v)}{\mathcal{D}_{St}(v)} \cdot \frac{\alpha(v_{\mathrm{root}})}{\alpha(v)}.$$

### 4.5

In the next paragraphs we apply these definitions for $\Gamma = \Gamma^{\min}(X_{f,n})$. Here, we prefer to regard $\Gamma^{\min}(X_{f,n})$ together with $\Gamma^{\mathrm{can}}(X_{f,n}, z)$, as a minimalization of $\Gamma^{\mathrm{can}}(X_{f,n}, z)$. In particular, we will define subsets, subgraphs, etc. in $\Gamma^{\min}(X_{f,n})$ as the images of well-defined subsets, subgraphs, etc. of $\Gamma^{\mathrm{can}}(X_{f,n}, z)$ by the minimalization procedure. (Of course, in the next section it will be a crucial task to recover some of these sets only from the abstract graph $\Gamma^{\min}(X_{f,n})$. The key result for this is Proposition 4.7.)

In order to avoid any confusion, for any subset of vertices of $\Gamma^{\mathrm{can}}(X_{f,n}, z)$, we will denote by $\pi(A)$ the image of $A$ by the minimalization procedure. Hence, $\pi(A)$ denotes those vertices of $\Gamma^{\min}(X_{f,n})$ which have ancestors in $A$, and survive in $\Gamma^{\min}(X_{f,n})$; in some cases this set can be empty.

The following facts follow easily from the structure results proved in § 4 and Propositions 4.1 and 4.2.



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**4.6 Facts**

Assume that $\Gamma = \Gamma^{\min}(X_{f,n})$ with $s \geqslant 2$. Then the following hold:

a) The set $\mathcal{L}(\Gamma(\mathcal{R}))$ is the disjoint union of two sets $\mathcal{R}_1$ and $\mathcal{LR}_s$, where

i) $\mathcal{R}_1 := \pi(q^{-1}(v_1))$;

ii) $\mathcal{LR}_s := \varnothing$ if $h_s > 1$; otherwise $\mathcal{LR}_s := \pi(q^{-1}(v_s))$, the image by the minimalization procedure of the unique rupture vertex of $\Gamma^{\mathrm{can}}(X_{f,n}, z)$ sitting above $v_s$.

(In both cases, by Propositions 4.1 and 4.2, these sets are subsets of the rupture vertices of $\Gamma^{\min}(X_{f,n})$.)

b) The subgraphs $\mathcal{G}(v)$ for $v \in \mathcal{L}(\Gamma(\mathcal{R})) = \mathcal{R}_1 \cup \mathcal{LR}_s$ (cf. part a) can be identified as follows:

i) Assume that we are not in the 'P-case' with $s = 2$. For each $v \in q^{-1}(v_1)$ consider the unique subgraph of type $\Gamma_-(v_{\mathrm{root}})$ in $\Gamma^{\mathrm{can}}(X_{f,n}, z)$, for some $v_{\mathrm{root}} \in q^{-1}(v_2)$, which contains $v$. Then its image in $\Gamma^{\min}(X_{f,n})$ by the minimalization procedure is $\mathcal{G}(v)$.

ii) Assume that $h_s = 1$. For $v = q^{-1}(v_s)$ consider $\Gamma_+(v_{\mathrm{root}})$ in $\Gamma^{\mathrm{can}}(X_{f,n}, z)$ with $v_{\mathrm{root}} := q^{-1}(v_{s-1})$. Then its image in $\Gamma^{\min}(X_{f,n})$ by the minimalization procedure is $\mathcal{G}(v)$.

c) The $\alpha$ and $\beta$ defined in § 4.4 are constant on $\mathcal{R}_1$.

(The motivation for the notation $\mathcal{LR}_s$ is the following: Later we will use the symbol $\mathcal{R}_s$ for $\pi(q^{-1}(v_s))$; hence $\mathcal{LR}_s = \mathcal{R}_s$ if $\pi(q^{-1}(v_s))$ is a 'leaf rupture vertex', otherwise it is empty.)

The main point is that in part a, the cases i and ii can be distinguished by the genus and $\beta$-invariant.

PROPOSITION 4.7. *Assume that $\Gamma = \Gamma^{\min}(X_{f,n})$ with $s \geqslant 2$.*

a) *If there exists at least one $v \in \mathcal{L}(\Gamma(\mathcal{R}))$ with $\mathsf{g}_v \neq 0$, then $\mathcal{R}_1 = \{v \in \mathcal{L}(\Gamma(\mathcal{R})) : \mathsf{g}_v \neq 0\}$ and $\mathcal{LR}_s = \{v \in \mathcal{L}(\Gamma(\mathcal{R})) : \mathsf{g}_v = 0\}$ ($\mathcal{LR}_s$ can be empty).*

b) *If $\mathsf{g}_v = 0$ for any $v \in \mathcal{L}(\Gamma(\mathcal{R}))$ and $\mathcal{LR}_s \neq \varnothing$, then $\beta(v) \in (0,\infty)$ and*

$$\begin{aligned} &\beta(v) > 2 && \text{if } v \in \mathcal{R}_1,\\ &\beta(v) \leqslant 1/2 && \text{if } v \in \mathcal{LR}_s. \end{aligned}$$

*Proof.* a) Genus $\mathsf{g}_v$ is constant on $\mathcal{R}_1$ and $\mathsf{g}_v = 0$ for (the unique) $v \in \mathcal{LR}_s$ provided that $\mathcal{LR}_s \neq \varnothing$, since in this case $h_s = 1$ (cf. Corollary 3.2).

b) Since $h_s = 1$ we can exclude the 'P-case'. First assume that $s \geqslant 3$.

If $v \in \mathcal{R}_1$ then by Corollary 3.7 and Remark 4.3 one has $\alpha(v) = h_1\tilde{h}_1$. For $v_{\mathrm{root}}$, analyzing the three different cases from the definition of $\alpha(v_{\mathrm{root}})$, and using Remark 4.3 and the genus formula, we get $\alpha(v_{\mathrm{root}}) = \tilde{h}_2$. On the other hand, $\mathcal{D}(v) = (a_1')^{h_1-1}(p_1')^{\tilde{h}_1-1}a_2'$ (cf. Proposition 3.9, part a) and $\mathcal{D}_{St}(v) = (a_1')^{h_1}(p_1')^{\tilde{h}_1}$ (use Proposition 3.5 and notice that if a string is collapsed completely then its determinant is 1). Therefore, $\beta(v) = a_2/(a_1p_1) > p_2 \geqslant 2$; cf. (2.2.1).

If $v = \pi(q^{-1}(v_s))$ then $\alpha(v) = \tilde{h}_s$ (use $\mathsf{g}_s = 0$, Remark 4.3 and Corollary 3.7). By similar argument as above, $\alpha(v_{\mathrm{root}}) = \tilde{h}_{s-1}$. By Proposition 3.9, part b and $h_s = 1$ one has $\mathcal{D}(v) = D_+(v_{s-1}) = n(p_s')^{\tilde{h}_s-1}/(h_{s-1}\tilde{h}_{s-1})$. By Proposition 3.5, $\mathcal{D}_{St}(v) = (p_s')^{\tilde{h}_s}\cdot n/(h_s\tilde{h}_s)$. Therefore, using again $h_s = 1$, one gets $\beta(v) = 1/(h_{s-1}p_s) \leqslant 1/2$.

Assume that $s = 2$ and let $v_i' = q^{-1}(v_i)$ $(i = 1,2)$. Then $\alpha(v_i') = h_i\tilde{h}_i$ $(i = 1,2)$. Hence the computation of $\beta(v_1')$ is the same as above, and it gives $a_2/(a_1p_1) > 2$. For $v_2'$ we have an additional $h_1$ and we get $\beta(v_2') = 1/p_2 \leqslant 1/2$. □

## 5. From $\Gamma^{\min}(X_{f,n})$ back to $f$ and $n$

Our final goal is to recover the Newton pairs of $f$ and the integer $n$ from the graph $\Gamma^{\min}(X_{f,n})$. In general, this is not possible. Nevertheless, by our main theorem, there are only two cases when such an ambiguity appears (the S1- and S2-coincidences). They are presented in the following subsections.

### 5.1 S1-coincidence

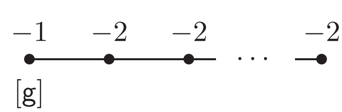
Assume that $(X,0) = (x^{q_1} + y^{p_1} + z^n = 0, 0)$ is a Brieskorn singularity with $(q_1, p_1) = 1$. Let us first analyze how one can recover the set of integers $\{q_1, p_1, n\}$ from the minimal resolution graph $\Gamma$ of $(X,0)$. In this case, the computation of the graph $\Gamma$ from the integers $\{q_1, p_1, n\}$ is a classical, well-known fact (cf. also our algorithm). The graph is either a string (with all genera zero) or a star-shaped graph (where only the central vertex might have a nonzero genus). If $\Gamma$ is a string, then $(X,0)$ is a Hirzebruch–Jung *hypersurface* singularity. But there is only one family of such singularities, namely the $A_{q_1-1}$ singularities provided by the integers $\{q_1, 2, 2\}$. In this case, $q_1$ is just the determinant of the string.

There is a rich literature of star-shaped graphs and Seifert 3-manifolds, and also of their subclass given by Brieskorn hypersurface singularities. The reader is invited to consult [OW71, § 3, case (I)], (cf. also [JN83, NR78]).

If one wants to recover the integers $\{q_1, p_1, n\}$, then one considers the set of strings $\mathcal{S}t(v)$ of the central vertex $v$. Recall definition 3 in § 4.4 for the notation. Then $\#I(v) \leqslant 3$. If $\#I(v) = 3$ then $\{q_1, p_1, n\} = \{D_1\#_2\#_3, D_2\#_1\#_3, D_3\#_1\#_2\}$. If one $\mathcal{S}t_{i_0}$ is missing (empty) then $D_{i_0} = 1$ and $\#_{i_0}$ can determined from the genus of the central vertex (see e.g. our genus formula of Corollary 3.2, part c or [OW71, (3.5)]). Hence the previous procedure still works.

Similarly, in our special situation $(q_1, p_1) = 1$, one can show that, if two subsets $\mathcal{S}t_i$ are empty, then one can still recover $\{q_1, p_1, n\}$ excepting only one case, namely when $\mathcal{S}t(v)$ consists of only one string. In our terminology, this can happen only when the string which supports the arrowhead survives and all the others are contracted (i.e. $p'_1 = a'_1 = 1$). Similar ambiguity appears when $\mathcal{S}t(v) = \varnothing$.

But all these ambiguity cases can be classified very precisely. Consider an identity of type $(h_1 - 1)(\tilde{h}_1 - 1) = 2\mathsf{g} > 0$ and an arbitrary positive integer $l$. Then the triplet $\{q_1, p_1, n\} = \{\tilde{h}_1, h_1, \tilde{h}_1 h_1 l\}$ provides the following graph $\Gamma$ (with $l-1$ $(-2)$-vertices):




Now fix $l > 0$ and $\mathsf{g} > 0$. Then, different triplets $\{\tilde{h}_1, h_1, \tilde{h}_1 h_1 l\}$ with $(h_1 - 1)(\tilde{h}_1 - 1) = 2\mathsf{g}$, $h_1 > 1$ and $\tilde{h}_1 > 1$ provide the same graph. [For example, $(3, 7, 21)$ and $(4, 5, 20)$ provide the same graph consisting of a vertex with $\mathsf{g} = 6$ and self-intersection $-1$.]

This is the only coincidence in the case of Brieskorn singularities with $(q_1, p_1) = 1$. Obviously, this cannot happen if $\mathsf{g} = 0$.

*Relation with the Milnor number.* Notice that in those cases when $\Gamma$ fails to determine the integers $\{q_1, p_1, n\}$, $\Gamma$ together with the Milnor number $\mu$ of the Brieskorn singularity do determine $\{q_1, p_1, n\}$. Indeed, in the 'ambiguity cases' one has $(q_1, p_1, n) = (\tilde{h}_1, h_1, \tilde{h}_1 h_1 l)$, where $2\mathsf{g} = (h_1 - 1)(\tilde{h}_1 - 1) > 0$ and $l$ are readable from the graph. But $\mu = (h_1 - 1)(\tilde{h}_1 - 1)(\tilde{h}_1 h_1 l - 1) = 2\mathsf{g}(\tilde{h}_1 h_1 l - 1)$. This determines $\tilde{h}_1 h_1$, and finally $h_1$ and $\tilde{h}_1$ (using the genus formula).

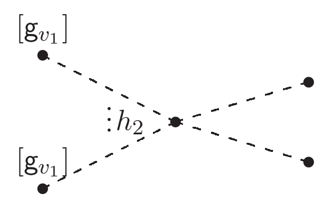
*Remark* 5.2 ('$z$-axis ambiguity'). Recall that by our general aim, we have to recover the Newton pairs of $f$ and the integer $n$. In the Brieskorn case, after we recover the set $(q_1,p_1,n)$ we have to make a choice for the $z$-axis. Recall that $(p_1,q_1)=1$. If $k$ integers among of $(p_1,q_1),(p_1,n)$ and $(q_1,n)$ equal 1, then there are $k$ possibilities for the choice of the $z$-axis.

### 5.3 S2-coincidence

The next coincidence appears when

$$s=2, \text{ and } a_1'=p_1'=\tilde{h}_2=1 \text{ (or equivalently, } s=2,\ (n,a_2)=1 \text{ and } (n,p_2)a_1p_1|n\ ). \tag{5.3.1}$$

In this case clearly $q_1=\tilde{h}_1$ and $p_1=h_1$, but the $\tilde{h}_1$ strings of type $\Gamma(\bar{v}_0)$ and the $h_1$ strings of type $\Gamma(\bar{v}_1)$ are not visible on the minimal graph since their determinants are 1, and hence they are contracted. The graph $\Gamma^{\min}(X_{f,n})$ has the following schematic form, where $\mathsf{g}_{v_1}>0$ and we omit the self-intersections:




The strings that appear on the right correspond to $\Gamma(v_2)$ and $\Gamma(\bar{v}_2)$, but in general, we cannot decide which one is which. From the graph we can read $h_2$ and the genus $\mathsf{g}_{v_1}=(h_1-1)(\tilde{h}_1-1)/2$, and of course, a lot of determinants.

Using $h_2,\tilde{h}_2$, and $D(v_1)$, $D_-(v_2)$, $D_+(v_1)$ and the set $\{D(v_2),D(\bar{v}_2)\}$, we can also recover $a_2$, $q_2$, $n/(h_1\tilde{h}_1)$, $h_1\tilde{h}_1p_2$ and the set $\{p_2,n\}$, where we cannot distinguish $p_2$ from $n$.

Notice that once we know $h_1\tilde{h}_1$, then using the genus formula and $\tilde{h}_1=q_1>p_1=h_1$, we obtain $h_1$ and $\tilde{h}_1$ without any ambiguity, and hence (by the above equations) all the data. But for the three 'variables' $h_1\tilde{h}_1$, $p_2$ and $n$ we know only the values $n/(h_1\tilde{h}_1)$, $h_1\tilde{h}_1p_2$ and the set $\{p_2,n\}$. This, in general, has two possible solutions (which correspond by a permutation of $p_2$ and $n$). If this is the case, then it might happen that there are two different realizations of the same graph for two different pairs $(f,n)$. But for this, both solutions should provide positive integers as candidates for the Newton pairs and $n$. If this does not happen then the graph is uniquely realized (see Example 3 below).

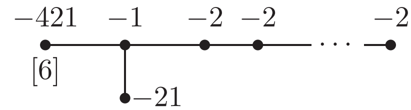
The complete discussion of all the cases when the above equations which involve $D(v_1)$, $D_-(v_2)$, $D_+(v_1)$ and the set $\{D(v_2),D(\bar{v}_2)\}$ associated with the graph provide exactly two 'good' solutions for $(f,n)$ is long and tedious, so we have decided not to give it here (nevertheless we think that Example 3 illuminates the problem completely). What is important is the fact that any graph (in this family) can be realized by at most two possible pairs $(f,n)$, and this coincidence in some cases really occurs. (Moreover, given a pair $(f,n)$, or the graph of $X_{f,n}$, one can write down easily the possible candidate for the numerical data of $(f',n')$, the possible pair of $(f,n)$, with the same graph.)

In the next examples we will write $\{(p_1,q_1),(p_2,q_2);n\}$ for the Newton pairs of $f$ and the integer $n$. Recall that (5.3.1) implies $p_1=h_1$ and $q_1=\tilde{h}_1$.

*Example* 1. The two different solutions $(3,7)$ and $(4,5)$ for the genus formula

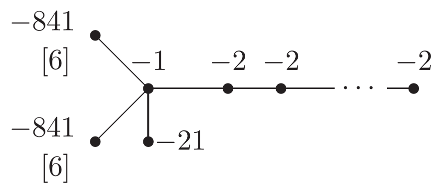
$$(h_1-1)(\tilde{h}_1-1)=2\cdot 6$$

can be completed to the following two sets of invariants: $\{(3,7),(20,1);21\}$ and $\{(4,5),(21,1);20\}$. For them the corresponding two graphs are the same:




Here the number of $(-2)$-curves is 19, and $h_2 = 1$.

*Example* 2. If one wants examples with arbitrary $h_2$, then one of the possibilities is the following: One multiplies in the above data (of Example 1) $p_2$ and $n$ by the wanted $h_2$. For example, the data $\{(3,7),(40,1);42\}$ and $\{(4,5),(42,1);40\}$ provide the same $h_2 = 2$ and the same graph:




Here again, the number of $(-2)$-curves is 19.

*Example* 3. Assume the data $\{(p_1,q_1),(p_2,q_2);n\}$ of $(f,n)$ satisfy (5.3.1); hence $p_1 = h_1$ and $q_1 = \tilde{h}_1$. If $(f,n)$ has 'a pair' $(f_2,n_2)$ (with the same graph) then the data of $(f_2,n_2)$ have the form (cf. the above discussion) $\{(x,y),(n,q_2),p_2\}$, where $x$ and $y$ can be determined by the equations $xy/p_2 = p_1q_1/n$ and $(x-1)(y-1) = (p_1-1)(q_1-1)$. It is easy to write down cases when this has no integral solutions.

For example, the data $\{(2,3),(5,1);6\}$ satisfy (5.3.1), but have no 'pair'. The minimal resolution graph can be realized in a unique way in the form $f + z^n$ ($f$ irreducible) (cf. Theorem 5.4 below).

*Relation with the Milnor number*. Even if the same graph is realized by two different pairs $(f_1,n_1)$ and $(f_2,n_2)$, the corresponding Milnor numbers $\mu_i$ associated with the hypersurface singularities $f_i + z^{n_i}$ $(i = 1,2)$ distinguish the two cases. This follows from the formula $\mu = [2\mathsf{g}_{v_1}p_2 + (p_2-1)(a_2-1)](n-1)$. Since $\mathsf{g}_{v_1} > 0$, $a_2$, $np_2$ and $n + p_2$ are readable from the graph, this relation determines $p_2$, and hence all the numerical data.

Now we are ready to formulate and prove the main result of this paper.

Theorem 5.4. *Let $f : (\mathbb{C}^2,0) \to (\mathbb{C},0)$ be an irreducible plane curve singularity with Newton pairs $\{(p_i,q_i)\}_{i=1}^s$ and let $n$ be an integer $\geqslant 2$. Let $\Gamma^{\min}(X_{f,n})$ be the minimal (good) resolution graph of the hypersurface singularity $(X_{f,n},0) := (\{f(x,y) + z^n = 0\},0)$. Then the following facts hold:*

a) *The integer $s$ is uniquely determined by $\Gamma^{\min}(X_{f,n})$.*

b) *Integer $s = 1$ if and only if $\Gamma^{\min}(X_{f,n})$ is either a string (with all the genera zero), or a star-shaped graph (where only the central vertex might have genus $\mathsf{g}$ nonzero). Moreover, $f(x,y) + z^n$ has the same equisingularity type as the Brieskorn singularity $x^{q_1} + y^{p_1} + z^n$.*

*Also $\Gamma^{\min}(X_{f,n})$ is a string if and only if $\{q_1,p_1,n\} = \{q_1,2,2\}$. If $\Gamma^{\min}(X_{f,n})$ is a star-shaped graph with $\mathsf{g} = 0$, then the set of integers $\{q_1,p_1,n\}$ is uniquely determined. Moreover, the only ambiguity which can appear in the case $\mathsf{g} > 0$ is described in § 5.1.*

c) *If $s = 2$ then it can happen that two pairs $(f_1,n_1)$ and $(f_2,n_2)$ (but not more) provide identical graphs $\Gamma^{\min}(X_{f,n})$. If this is the case then both of them should satisfy the numerical restrictions:*

$$(n,a_2) = 1 \quad \text{and} \quad (n,p_2)a_1p_1 | n \tag{$*$}$$

*(which can be recognized from the graph as well), and (automatically) at least one of the vertices has genus $\mathsf{g} > 0$. This case is described in § 5.3.*

d) *In all other cases (i.e. for any $s \geqslant 3$ or for $s=2$ excluding the exceptional case $(*)$), $\Gamma^{\min}(X_{f,n})$ determines uniquely the Newton pairs of $f$ and the integer $n$ (by a precise algorithm which basically constitutes the next proof).*

e) *In particular, except for the two cases S1 and S2 (cf. §§ 5.1 and 5.3), from the link one can recover completely the Newton pairs of $f$ and the integer $n$ (provided that we disregard the $z$-axis ambiguity; cf. Remark 5.2.) In particular, this is true without any exception provided that the link is a rational homology sphere.*

*On the other hand, in the cases S1 and S2, the link together with the Milnor number of the hypersurface singularity $f+z^n$ determines completely the Newton pairs of $f$ and the integer $n$ (cf. the two paragraphs at the end of §§ 5.1 and 5.3).*

*Proof.* We denote $\Gamma^{\min}(X_{f,n})$ by $\Gamma$ and its rupture vertices by $\mathcal{R}$. It is convenient to separate those cases when $\#\mathcal{R}$ is small.

*Case A. Assume that* $\mathcal{R}=\varnothing$. By Proposition 4.1 the set of rupture vertices of $\Gamma^{\min}(X_{f,n},z)$ is never empty. Hence, by Proposition 4.2, $\mathcal{R}=\varnothing$ if and only if in the 'P-case' we contract $v'$, and $v'$ is the unique rupture vertex of $\Gamma^{\min}(X_{f,n},z)$. But $\Gamma^{\min}(X_{f,n},z)$ has a unique rupture point if and only if $s=1$. Therefore (cf. Proposition 4.2) this situation occurs if and only if $s=1$, $p_1=n=2$ and $(q_1,2)=1$; i.e. $f(x,y)+z^n$ has the equisingularity type of $x^{q_1}+y^2+z^2$. Clearly, $q_1$ can be recovered from the graph: it is its determinant; cf. also § 5.1.

*Case B. Assume that* $\#\mathcal{R}=1$. From Proposition 4.2 it is clear that in the 'P-case' the number of rupture vertices of $\Gamma$ is even. Hence, this case is excluded, and the number of rupture vertices of $\Gamma^{\min}(X_{f,n},z)$ is also 1. By Proposition 4.1, this can happen only of $s=1$. In particular, $f+z^n$ is of Brieskorn type: $x^{q_1}+y^{p_1}+z^n$, with $q_1>p_1\geqslant 2$, $(p_1,q_1)=1$ and $n\geqslant 2$ (where the case $p_1=n=2$ is excluded, see above). This case is completely covered by § 5.1.

*Case C. Assume that* $\#\mathcal{R}>1$. By Propositions 4.1 and 4.2, $\#\mathcal{R}>1$ if and only if $s\geqslant 2$. The proof (algorithm) consists of several steps; each step recovers some data.

1) *The set* $\mathcal{R}_1$ can be determined from § 4.6 and Proposition 4.7. Indeed, we start with the set $\mathcal{L}(\Gamma(\mathcal{R}))$ (where $\Gamma=\Gamma^{\min}(X_{f,n})$). Then, if there exists at least one $v\in\mathcal{L}(\Gamma(\mathcal{R}))$ with $\mathsf{g}_v\neq 0$, then $\mathcal{R}_1=\{v\in\mathcal{L}(\Gamma(\mathcal{R})):\mathsf{g}_v\neq 0\}$ (cf. Proposition 4.7, part a). If $\mathsf{g}_v=0$ for all $v$, then we consider their $\beta$-invariants $\beta(v)$. If they are all equal, then by Proposition 4.7, part b, one gets $\mathcal{R}_1=\mathcal{L}(\Gamma(\mathcal{R}))$. If they are not all equal, then only one can be $\leqslant 1/2$ (corresponding to $\{v\}=\mathcal{R}_s$), and all the others are $>2$ (and equal to each other) corresponding to $\mathcal{R}_1$ (cf. Proposition 4.7, part b).

2) *The sets $\pi(q^{-1}(v_k))$ $(1\leqslant k\leqslant s)$.* Define a *distance* on the set $\mathcal{R}$. If $w_1,w_2\in\mathcal{R}$, and on the shortest path in $\Gamma$ connecting them there are exactly $l$ rupture vertices of $\Gamma$ (including $w_1$ and $w_2$), then we say that $d(w_1,w_2):=l-1\geqslant 0$. Moreover, for any subset $\mathcal{R}'\subset\mathcal{R}$ and $w\in\mathcal{R}$ we define $d(\mathcal{R}',w)$ as usual by $\min\{d(w',w):w'\in\mathcal{R}'\}$.

Then, for any $k\geqslant 1$, we write $\mathcal{R}_k:=\{v\in\mathcal{R}:d(\mathcal{R}_1,v)=k-1\}$. Let

$$s':=\max\{k:\mathcal{R}_k\neq\varnothing\}.$$

We distinguish two cases:

a) $\#\mathcal{R}_{s'}=1$. Then $s=s'$ and $\pi(q^{-1}(v_k))=\mathcal{R}_k$ for $1\leqslant k\leqslant s$.

b) $\#\mathcal{R}_{s'}>1$. This can happen exactly in the 'P-case' (cf. Proposition 4.2). In this situation, $s=s'+1$, $\pi(q^{-1}(v_k))=\mathcal{R}_k$ for $1\leqslant k\leqslant s-1$, and the (unique) vertex $v'=q^{-1}(v_s)$ of $\Gamma^{\min}(X_{f,n},z)$ is 'missing' in $\Gamma$, i.e. $\pi(\{v'\})=\varnothing$ (cf. Proposition 4.2).

For a moment we postpone the 'P-case', and we assume case a. We will come back to the 'P-case' in step 11.

3) *The integers* $\{h_k\}_{k=2}^s$ are determined by the identities $h_k = \#\mathcal{R}_{k-1}/\#\mathcal{R}_k$ ($h_1$ will be determined later).

4) *The sets* $\{\pi(q^{-1}(\bar{v}_k))\}_{k=2}^{s-1}$ *and the integers* $\{\tilde{h}_k\}_{k=2}^{s-1}$, $\{p_k\}_{k=2}^{s-1}$ *(for* $s \geqslant 3$*)*. Fix $2 \leqslant k \leqslant s-1$ and some $v \in \mathcal{R}_k$. Consider the set of strings $\mathcal{S}t(v)$ supported by $v$ (cf. definition 3 in § 4.4).

If $\mathcal{S}t(v) \neq \varnothing$, then $\#\mathcal{S}t(v) = \tilde{h}_k$ and det : $\mathcal{S}t(v) \to \mathbb{Z}$ is constant with value $p'_k$. Then $p_k = p'_k \cdot h_k$. Then this happens for any choice of $v$, and $\pi(q^{-1}(\bar{v}_k))$ is the set of leaf vertices of $\Gamma$ situated on the strings of type $\bigcup_v \mathcal{S}t(v)$, $v \in \mathcal{R}_k$.

If $\mathcal{S}t(v) = \varnothing$ then all the strings of type $\Gamma(\bar{v}_k)$ are collapsed in $\Gamma$; in particular $\pi(q^{-1}(\bar{v}_k)) = \varnothing$. Hence their determinants $p'_k = p_k/h_k = 1$. In particular, $p_k = h_k \geqslant 2$. Then $\tilde{h}_k$ is given by the genus formula $\tilde{h}_k - 1 = 2\mathsf{g}_v/(h_k - 1)$.

For the 'ends' $k = 1$ and $k = s$ we need more special computations (since we have to separate the two different types of strings which may or may not be 'missing' from $\Gamma$). In step 5 we recover $\tilde{h}_s$; in steps 6 and 7, $n$ and $p_s$ and the arrowhead of $\Gamma^{\min}(X_{f,n}, z)$ (excepting the case $(*)$). In step 8 we treat the invariants with index $k = 1$.

5) *The integer* $\tilde{h}_s$. If $h_s > 1$ then the genus formula for $\mathsf{g}_{v_s}$ gives $\tilde{h}_s$. If $h_s = 1$, the strings of type $\Gamma(\bar{v}_s)$ cannot be collapsed; hence $\mathcal{S}t(v) \neq \varnothing$ for $\{v\} = \mathcal{R}_s$. Then $\tilde{h}_s = \alpha(v)$; cf. definition 3 in § 4.4 and Corollary 3.7, part b.

6) $p_s$ *and* $n$ *and the arrowhead of* $\Gamma^{\min}(X_{f,n}, z)$ *in the following cases:*

i) either $s \geqslant 3$, or

ii) $s = 2$ but $\{a'_1, p'_1\} \neq \{1\}$.

First we show that in both cases we can compute the product $h_{s-1}\tilde{h}_{s-1}$. Indeed, in the case i this follows from steps 3 and 4. If $s = 2$ then we proceed as follows. Since $a'_1$ and $p'_1$ are not both 1, $\mathcal{S}t(v) \neq \varnothing$ for $v \in \mathcal{R}_1$. If the determinant has two values on this set, then $h_1\tilde{h}_1 = \alpha(v)$; cf. definition 3 in § 4.4 and Corollary 3.7, part a. If all the determinants are the same, then either $\Gamma(\bar{v}_0)$ or $\Gamma(\bar{v}_1)$ is collapsed. If $\Gamma(\bar{v}_0)$ is collapsed, then $a'_1 = 1$, and hence $\tilde{h}_1 \geqslant 2$. In the second case $h_1 \geqslant 2$. Hence in both cases the following procedure works: take $c_1 := \#\mathcal{S}t(v)$ (which automatically is $\geqslant 2$), compute $c_2$ by the genus formula $2\mathsf{g}_{v_1} = (c_1 - 1)(c_2 - 1)$ and set $h_1\tilde{h}_1 = c_1c_2$.

Now, we go back to $p_s$ and $n$ and the position of the arrowhead.

Notice that by Proposition 4.1 and step 2, the determinants of type $D(v_{s-1})$ and $D_+(v_{s-1})$ are well defined in $\Gamma$, and their values do not change by the minimalization procedure. For example, $D_+(v_{s-1})$ can be computed from Proposition 3.9, part b. Notice also that $D_{St}(v_s) = n(p'_s)^{\tilde{h}_s}/h_s\tilde{h}_s$ (cf. Proposition 3.5). Therefore

$$\frac{D(v_{s-1})^{h_s-1}D_{St}(v_s)}{D_+(v_{s-1})} = \frac{h_{s-1}\tilde{h}_{s-1}p_s}{h_s\tilde{h}_s}.$$

Hence this value can be determined from the graph, a fact which is true for $h_{s-1}\tilde{h}_{s-1}$ (see above) and $h_s$ (cf. step 3) and $\tilde{h}_s$ (cf. step 5) as well. Hence we get $p_s$. In particular, we can compute the string determinants $D(\bar{v}_s) = p'_s$ and (using $D_{St}(v_s)$) $D(v_s) = n/(h_s\tilde{h}_s)$ as well. This gives $n$ too. If $D(v_s) \neq 1$ then we put the arrow on the string with this determinant (cf. Corollary 3.7); if $D(v_s) = 1$ then we put the arrowhead on $\pi(q^{-1}(v_s))$. In this way we recover the arrow of $\Gamma^{\min}(X_{f,n}, z)$.

7) $p_s$ *and* $n$ *and the arrowhead of* $\Gamma^{\min}(X_{f,n}, z)$ *if*

iii) $\tilde{h}_s \neq 1$.

Consider $\mathcal{S}t(v)$ for $\{v\} = \mathcal{R}_s$, cf. definition 3 in § 4.4. Then by a verification $\mathcal{D}^{\text{red}}_{St}(v) = (p'_s)^{\tilde{h}_s - 1}$. Since $\tilde{h}_s$ is determined in step 5, and it is $\neq 1$, one gets $p'_s$. Then we repeat the arguments of step 6.

8) *The integers* $a_1,\ p_1,\ h_1$ *and* $\tilde h_1$ *in the cases when one of the conditions i or ii or iii is valid.* Fix a vertex $v\in\mathcal{R}_1$ and consider $\mathcal{S}t(v)$ as in definition 3 in § 4.4.

If $\mathcal{S}t(v)\neq\varnothing$, then $\#I(v)\leqslant 2$. If $\#I(v)=2$, then compute the two numbers $D_1\cdot\#_2$ and $D_2\cdot\#_1$. They are the candidates for $a_1=q_1$ and $p_1$; cf. Proposition 3.5. Since $q_1>p_1$, these two numbers cannot be the same. If, say, $D_1\cdot\#_2>D_2\cdot\#_1$, then $\mathcal{S}t_1$ is the index set of $\pi(q^{-1}(\bar v_0))$ and $\mathcal{S}t_2$ of $\pi(q^{-1}(\bar v_1))$. Hence $h_1=\#_1$, $\tilde h_1=\#_2$, $q_1=D_1\cdot\#_2$ and $p_1=D_2\cdot\#_1$.

If there is only one level set with data $D_1$ and $\#_1$, then in the above argument we write $D_2=1$ and we determine $\#_2$ using the genus formula $2\mathsf{g}_{v_1}=(\#_1-1)(\#_2-1)$ (which is possible since $D_2\#_1=\#_1\geqslant 2$). We repeat the above argument.

Now we assume that $\mathcal{S}t(v)=\varnothing$. This can happen only if $a_1'=p_1'=1$; hence $q_1=\tilde h_1$ and $p_1=h_1$. First we determine $H:=h_1\tilde h_1$.

$D(v_1)$ gives an equation of type $q_2=H\cdot A$, where $A$ is a positive number which can be determined from the graph by the previous steps. Moreover, $D_-(v_2)=a_2'$, and hence $a_2=D_-(v_2)\tilde h_2$ is known from the graph. Finally, $a_1p_1p_2=Hp_2$, where $p_2$ too is known from the graph. Then the identity $a_2=q_2+a_1p_1p_2$ gives a nontrivial linear equation for $H$.

Then $h_1\tilde h_1=H$ and $(h_1-1)(\tilde h_1-1)=2\mathsf{g}_{v_1}$ provides $h_1$ and $\tilde h_1$ modulo their permutation. But $\tilde h_1=q_1>a_1=h_1$, and hence we get $h_1$ and $\tilde h_1$.

9) *The integers* $\{a_k\}_{k=2}^s$ *when one of the conditions i or ii or iii is valid.* Once we have the position of the arrow, we have all the multiplicities $\{\mathsf{m}_{v_k}\}_k$; hence Corollary 3.2, part b gives all the integers $a_k'$. An alternative way is to use Remark 3.10 inductively.

10) *Assume that the conditions i, ii and iii are not valid.* This means that $s=2$ and $a_1'=p_1'=\tilde h_2=1$. This is exactly the case of S2-coincidence treated in § 5.3.

11) *The 'P-case'.* Now we go back to step 2, case b. In this case $\#\mathcal{R}_{s'}=2$, so write $\mathcal{R}_{s'}=\{w_1,w_2\}$. Take the shortest path in $\Gamma$ connecting $w_1$ and $w_2$. Take the edge 'at the middle of the path', blow it up, and put an arrow on it. This new graph is exactly $\Gamma^{\min}(X_{f,n},z)$. Set $\mathcal{R}_s:=\{v'\}$, where $v'$ is the new vertex. Then we can repeat all the above arguments.

Notice that step 6 works, since if $s=2$ and $a_1'=p_1'=1$, then $h_1=p_1\geqslant 2$ and $\tilde h_1=a_1\geqslant 2$, and hence $\mathsf{g}_{v_1}>0$. But in the 'P-case' all the genera are zero. (Hence step 7 is not needed.) [In fact, since in this case we already have the position of the arrow, we can compute some of the invariants much faster using the multiplicities and Corollary 3.2, part b.] □

### 5.5 The symmetry of the graph

In the above proof we were rather meticulous in separating the possible sets $\pi(q^{-1}(v))$. The fruit of this is the following corollary (whose proof is left to the reader, and basically it is incorporated in the previous proof of the main theorem).

First recall that the cyclic covering $X_{f,n}\to X$ has a $\mathbb{Z}_n$-Galois action. This lifts to the level of the resolution; hence $\Gamma^{\min}(X_{f,n})$ inherits a natural $\mathbb{Z}_n$-action as well. The question is: Has the graph $\Gamma^{\min}(X_{f,n})$ any extra symmetry?

Take for example the Brieskorn case of Example 3.3. Then the Galois action permutes cyclically (via its image $\mathbb{Z}_h$) the $h$ arms with Seifert invariants $a$. On the other hand, the total symmetry group of the graph is the total permutation group of these arms. So, in this sense, the symmetry group of the graph is definitely larger than the (image) of the Galois action. On the other hand, their orbits are the same. This fact is valid in general.

COROLLARY 5.6. *Assume that* $\sigma$ *is a (decorated graph) automorphism of* $\Gamma^{\min}(X_{f,n})$ *which identifies two vertices, say,* $v_1$ *and* $v_2$*. Then* $v_1$ *and* $v_2$ *are in the same orbit with respect to the Galois action.*

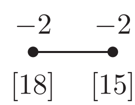
### 5.7 Final remarks about the reducible case

1) We emphasize again that it is rather surprising (at least for the authors) that from the link of $X_{f,n}$ (without any additional assumption about the link) one can recover the equisingularity type of $f$ and the integer $n$ *excepting only two families*. Still, we expect that even for reducible $f$, the link of $X_{f,n}$ contains all the information about the equisingularity type of $f$ and the integer $n$ 'generically', i.e. excepting a set of coincidences which is 'small' inside of suspension singularities. At this moment it is extremely difficult to guess how large is the set of these exceptions. Coincidences with star-shaped graphs are known, and can be classified. But definitely there are other type of coincidences as well.

For example, the minimal resolution graphs of the suspension singularities

$$\{(x^{19}+y^{38})(x^{32}+y^{16})+z^3=0\} \quad \text{and} \quad \{(x^5+y^6)(x^{10}+y^3)+z^{45}=0\}$$

are the same, namely:




Moreover, the above S2-coincidence warns us that we can expect rather strange coincidences as well. We also notice that if $f$ is reducible, the graph of $X_{f,n}$ in general is not a tree, a fact which enlarges the possible graphs considerably.

On the other hand, we conjecture that even *for reducible $f$, if the link of $X_{f,n}$ is a rational homology sphere, then it determines the equisingularity type of $f$ and the integer $n$ in a unique way.*

2) Analyzing the above results, we realize that some of them can be generalized in a natural way to the reducible case; but in some other cases we cannot even formulate the possible generalizations (e.g. in the crucial Proposition 4.7).

3) Corollary 5.6 cannot be extended to the general case of nonirreducible germs. For example, $\Gamma^{\min}(\mathbb{C}^2,0)$ can have a symmetry (take e.g. $f=(x^2+y^3)(x^3+y^2)$) which lifts to an automorphism of $\Gamma^{\min}(X_{f,n})$, which does not come from the Galois covering.

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