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NOTE ON q-DEDEKIND-TYPE SUMS RELATED TO q-EULER POLYNOMIALS

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Abstract. Recently, q-Dedekind-type sums related to q-zeta function and basic L-series are studied by Simsek in [**13**] (Y. Simsek, q-Dedekind type sums related to q-zeta function and basic L-series, *J. Math. Anal. Appl.* 318 (2006), 333–351) and Dedekind-type sums related to Euler numbers and polynomials are introduced in the previous paper [**11**] (T. Kim, Note on Dedekind type DC sums, *Adv. Stud. Contem. Math.* 18 (2009), 249–260). It is the purpose of this paper to construct a p-adic continuous function for an odd prime to contain a p-adic q-analogue of the higher order Dedekind the type sums related to q-Euler polynomials and numbers by using an invariant p-adic q-integrals.

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1. Introduction/preliminaries. Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalised exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = \frac{1}{p}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$ or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, one normally assumes $|1 - q|_p < 1$. Recently, we proposed a definition of a q-extension of p-adic Haar measure as follows: For any positive integer N, we set

$$\mu_q(a + p^N\mathbb{Z}_p) = \frac{(1+q)(-q)^a}{1+q^{p^N}} \text{ (see [\mathbf{1}–\mathbf{13}])}$$

for $0 \le a \le p^N - 1$ and this can be extended to a measure on $\mathbb{Z}_p$. This measure yields an invariant p-adic q-integral for each non-negative integer m and the m-th Carlitz's type q-Euler numbers $\varepsilon_{m,q}$ can be represented by this p-adic q-integral as follows:

$$\varepsilon_{m,q} = \int_{\mathbb{Z}_p} \left(\frac{1-q^a}{1-q}\right)^m d\mu_q(a) = \lim_{N\to\infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} \left(\frac{1-q^a}{1-q}\right)^m (-q)^a,$$

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which has a sense, as we see readily, that the limit is convergent (see [**7**, **10**]). The modified q-Euler numbers are also defined as

$$E_{m,q} = \int_{\mathbb{Z}_p} \left(\frac{1-q^a}{1-q} \right)^m q^{-a} d\mu_q(a), \text{ (see [9]).}$$

Note that $\lim_{q \to 1} E_{n,q} = E_n$, where E_n are the n-th Euler numbers. Now, we also consider the q-Euler polynomials as follows:

$$E_{m,q}(x) = \int_{\mathbb{Z}_p} q^{-t} \left(\frac{1-q^{x+t}}{1-q} \right)^m d\mu_q(t), \qquad \text{for } x \in \mathbb{Z}_p, m \in \mathbb{N}.$$

These numbers $E_{m,q}(x)$ can be represented by

$$E_{m,q}(x) = \sum_{l=0}^{m} \binom{m}{l} q^{xl} E_{l,q} \left(\frac{1-q^x}{1-q} \right)^{m-l}.$$

For any positive integer h, k and m, Dedekind-type DC sums are defined as

$$S_m(h,k) = \sum_{M=1}^{k-1} (-1)^{M-1} \frac{M}{k} \bar{E}_m \left(\frac{hM}{k} \right) \text{ (see [6, 11, 14]),}$$

where $\bar{E}_m(x)$ are the m-th periodic Euler function.

By using an invariant p-adic q-integral on $\mathbb{Z}_p$, we construct a p-adic continuous function for an odd prime to contain a p-adic q-analogue of the higher order Dedekind-type DC sums $k^m S_{m+1}(h,k)$ in this paper. It is the purpose of this paper to give a q-analogue of p-adic Dedekind-type DC sums by using invariant p-adic q-integral on $\mathbb{Z}_p$ approach the p-adic analogue of the higher order Dedekind sums at $q = 1$ as follows.

THEOREM. *Let h, k be positive integer with $(h,k) = 1$, $p \nmid k$. For $s \in \mathbb{Z}_p$, let us define p-adic Dedekind-type DC sums as follows.*

$$S_{p,q}(s : h, k : q^k) = \sum_{M=1}^{k-1} \left(\frac{1-q^M}{1-q} \right) (-1)^{M-1} T_q(s, hM, k : q^k).$$

Then there exists a continuous function $S_{p,q}(s : h, k : q^k)$ on $\mathbb{Z}_p$, which satisfies

$$S_{p,q}(m : h, k : q^k) = \left(\frac{1-q^k}{1-q} \right)^{m+1} S_{m,q}(h, k : q^k)$$
$$- \left(\frac{1-q^k}{1-q} \right)^{m+1} \left(\frac{1-q^{kp}}{1-q^k} \right)^m S_{m,q}((p^{-1}h)_k, k : q^{pk}),$$

where $m+1 \equiv 0 \ (mod\, p-1)$, and $(p^{-1}a)_N$ denotes the integer x with $0 \leq x < N$, $px \equiv a$ (mod N).

2. Proof of theorem. The q-Euler numbers $E_{m,q}$ can be written as

$$E_{0,q} = \frac{1+q}{2} \qquad (qE+1)^n + E_{n,q} = 0 \quad \text{if } n > 0,$$

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which is

$$E_{n,q} = (1+q)\left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^l} \text{ (see [9])},$$

where we use the technique method notation by replacing E^n by $E_{n,q}$ symbolically. Let w denote the Teichüller character (mod p). For $x \in \mathbb{Z}_p^\times = \mathbb{Z}_p \backslash p\mathbb{Z}_p$, we set

$$< x >_q = < x : q > = w^{-1}(x) \left(\frac{1-q^x}{1-q}\right).$$

Let a and N be positive integers with $(p, a) = 1$ and $p|N$. Define

$$T_q(s, a, N : q^N) = w^{-1}(a) < a >_q^s \sum_{j=0}^{\infty} \binom{s}{j} q^{aj} \left(\frac{1-q^N}{1-q^a}\right)^j E_{j,q^N}, \text{ for } s \in \mathbb{Z}_p.$$

In particular, if $m + 1 \equiv 0 \pmod{p-1}$, then

$$\begin{aligned} T_q(m, a, N : q^N) &= \left(\frac{1-q^a}{1-q}\right)^m \sum_{j=0}^{m} \binom{m}{j} q^{aj} \left(\frac{1-q^N}{1-q^a}\right)^j E_{j,q^N} \\ &= \left(\frac{1-q^N}{1-q}\right)^m \int_{\mathbb{Z}_p} \left(\frac{1-q^{Nx+a}}{1-q^N}\right)^m q^{-Nx} d\mu_{q^N}(x). \end{aligned}$$

Therefore, $T_q(m, a, N : q^N)$ is a continuous p-adic extension of $(\frac{1-q^N}{1-q})^m E_{m,q^N}(\frac{a}{N})$.

Let $[\cdot]$ be the Gauss' symbol and let $\{x\} = x - [x]$. Then we consider a q-analogue of the higher order Dedekind-type DC sums $S_{m,q}(h, k : q^l)$ as follows:

$$S_{m,q}(h, k : q^l) = \sum_{M=1}^{k-1} (-1)^{M-1} \left(\frac{1-q^M}{1-q^k}\right) \int_{\mathbb{Z}_p} q^{-lx} \left(\frac{1-q^{l(x+\{\frac{hM}{k}\})}}{1-q^l}\right)^m d\mu_{q^l}(x).$$

If $m + 1 \equiv 0 \pmod{p-1}$, then we have

$$\begin{aligned} &\left(\frac{1-q^k}{1-q}\right)^{m+1} \sum_{M=1}^{k-1} \left(\frac{1-q^M}{1-q^k}\right) (-1)^{M-1} \int_{\mathbb{Z}_p} \left(\frac{1-q^{k(x+\frac{hM}{k})}}{1-q^k}\right)^m q^{-kx} d\mu_{q^{-k}}(x) \\ &= \sum_{M=1}^{k-1} (-1)^{M-1} \left(\frac{1-q^M}{1-q}\right) \left(\frac{1-q^k}{1-q}\right)^m \int_{\mathbb{Z}_p} \left(\frac{1-q^{k(x+\frac{hM}{k})}}{1-q^k}\right)^m q^{-kx} d\mu_{q^k}(x), \qquad (1) \end{aligned}$$

where $p|k$, $(hM, p) = 1$ for each M. From (1), we note that

$$\begin{aligned} &\left(\frac{1-q^k}{1-q}\right)^{m+1} S_{m,q}(h, k : q^k) \\ &= \sum_{M=1}^{k-1} \left(\frac{1-q^M}{1-q}\right) \left(\frac{1-q^k}{1-q}\right)^m (-1)^{M-1} \int_{\mathbb{Z}_p} q^{-kx} \left(\frac{1-q^{k(x+\frac{hM}{k})}}{1-q^k}\right)^m d\mu_{q^k}(x) \\ &= \sum_{M=1}^{k-1} (-1)^{M-1} \left(\frac{1-q^M}{1-q}\right) T_q(m, (hM)_k : q^k), \qquad (2) \end{aligned}$$

where $(y)_k$ denotes the integers x such that $0 \leq x < n$ and $x \equiv \alpha \pmod{k}$.

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It is easy to check that

$$\int_{\mathbb{Z}_p} q^{-t}\left(\frac{1-q^{x+t}}{1-q}\right)^k d\mu_q(t)$$
$$=\left(\frac{1-q^m}{1-q}\right)^k \frac{1+q}{1+q^m}\sum_{i=0}^{m-1}(-1)^i\int_{\mathbb{Z}_p}\left(\frac{1-q^{m(t+\frac{x+i}{m})}}{1-q^m}\right)^k q^{-mt}d\mu_{q^m}(t). \tag{3}$$

By (2) and (3), we see that

$$\left(\frac{1-q^N}{1-q}\right)^m\int_{\mathbb{Z}_p}\left(\frac{1-q^{N(x+\frac{a}{N})}}{1-q^N}\right)^m q^{-Nx}d\mu_{q^N}(x)$$
$$=\frac{1+q^N}{1+q^{Np}}\sum_{i=0}^{p-1}\left(\frac{1-q^{Np}}{1-q}\right)^m(-1)^i\int_{\mathbb{Z}_p}\left(\frac{1-q^{pN(x+\frac{a+iN}{pN})}}{1-q^{pN}}\right)^m q^{-pNx}d\mu_{q^{pN}}(x). \tag{4}$$

From (2), (3) and (4) we note that the p-adic integration is given by

$$T_q(s,a,N:q^N)=\frac{1+q^N}{1+q^{pN}}\sum_{\substack{i=0\\ a+iN\not\equiv 0(mod\,p)}}^{p-1}(-1)^i T_q(s,(a+iN)_{pN},p^N:q^{pN})$$

such that

$$T_q(m,a,N:q^N)$$
$$=\left(\frac{1-q^N}{1-q}\right)^m\int_{\mathbb{Z}_p}\left(\frac{1-q^{N(x+\frac{a}{N})}}{1-q^N}\right)^m q^{-Nx}d\mu_{q^N}(x)$$
$$-\left(\frac{1-q^{pN}}{1-q}\right)^m\int_{\mathbb{Z}_p}\left(\frac{1-q^{pN(x+\frac{(p^{-1}a)_n}{N})}}{1-q^{pN}}\right)^n q^{-pNx}d\mu_{q^{pN}}(x),$$

where $(p^{-1}a)_N$ denotes the integer x with $0\le x<N$, $px\equiv a \pmod N$ and m is integer with $m+1\equiv 0 \pmod{p-1}$.

Hence, we have

$$\sum_{M=1}^{k-1}\left(\frac{1-q^M}{1-q}\right)(-1)^{M-1}T_q(m,hM,k:q^k)=\left(\frac{1-q^k}{1-q}\right)^{m+1}S_{m,q}(h,k:q^k)$$
$$-\left(\frac{1-q^k}{1-q}\right)^{m+1}\left(\frac{1-q^{kp}}{1-q^k}\right)^m S_{m,q}((p^{-1}h)_k,k:q^{pk}), \tag{5}$$

where $p\nmid k$ and $p\nmid hM$ for each M.

For $s\in\mathbb{Z}_p$, let us define p-adic Dedekind-type DC sums as follows:

$$S_{p,q}(s:h,k:q^k)=\sum_{M=1}^{k-1}\left(\frac{1-q^M}{1-q}\right)(-1)^{M-1}T_q(s,hM,k:q^k).$$

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Then there exists a continuous function $S_{p,q}(s : h, k : q^k)$ on $\mathbb{Z}_p$, which satisfies

$$S_{p,q}(m : h, k : q^k) = \left(\frac{1-q^k}{1-q}\right)^{m+1} S_{m,q}(h, k : q^k)$$
$$-\left(\frac{1-q^k}{1-q}\right)^{m+1}\left(\frac{1-q^{kp}}{1-q^k}\right)^m S_{m,q}((p^{-1}h)_k, k : q^{pk}), \text{ where } m+1 \equiv 0 \pmod{p-1}.$$

REMARK. Note that

$$S_m(h, k : q^l) = \sum_{M=1}^{k-1}\left(\frac{1-q^M}{1-q^k}\right)(-1)^{M-1}E_{m,q^l}\left(\left\{\frac{hM}{k}\right\}\right).$$

It is easy to see that $S_{p,1}(s : h, k : 1)$ is the p-adic analogue of the higher order Dedekind-type DC sums $k^m S_{m+1}(h, k)$.

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