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# COMMENT ON

# 'POSITIVELY HOMOGENEOUS LATTICE HOMOMORPHISMS BETWEEN RIESZ SPACES NEED NOT BE LINEAR'

FETHI BEN AMOR

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**Abstract**

This note furnishes an example showing that the main result (Theorem 4) in Toumi ['When lattice homomorphisms of Archimedean vector lattices are Riesz homomorphisms', *J. Aust. Math. Soc.* **87** (2009), 263–273] is false.

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Throughout this note, all Riesz spaces under consideration are assumed to be Archimedean. Let $E$ and $F$ be two Riesz spaces. A mapping $T : E \to F$ is said to be a lattice homomorphism whenever $T(x \vee y) = Tx \vee Ty$ and $T(x \wedge y) = Tx \wedge Ty$ for all $x, y \in E$. The problem of linearity of a lattice homomorphism between Riesz spaces was initiated by Mena and Roth in their paper [3], where they proved that if $X$ and $Y$ are compact Hausdorff spaces and $T : C(X) \to C(Y)$ is a lattice homomorphism such that $T(\alpha\mathbf{1}) = \alpha\mathbf{1}$ for all $\alpha \in \mathbb{R}$, then $T$ is linear. In [4], Thanh generalized this result to the case when $X$ and $Y$ are realcompact spaces. See also [2] by Lochan and Strauss for another generalization. In [1], Ercan and Wickstead extended this investigation to lattice homomorphisms between uniformly complete Riesz spaces with order units. Their main result was the following. If $E$ is a uniformly complete Riesz space with a strong order unit $e$, $F$ is a uniformly complete Riesz space, and $T : E \to F$ is a lattice homomorphism such that $T(\alpha\mathbf{e}) = \alpha T(\mathbf{e})$ for each $\alpha \in \mathbb{R}$, then $T$ is linear. Recently, Toumi [5] gave the following improvement of the above result. Let $E$ be a Riesz space with a strong order unit $e$, $F$ be a Riesz space, and $T : E \to F$ be a lattice homomorphism such that $T(\alpha\mathbf{e}) = \alpha T(\mathbf{e})$ for each $\alpha \in \mathbb{R}^+$; then $T$ is linear. Unfortunately, this result is false, as shown by the following example.

Example 1. Let $E$ and $F$ be nontrivial Riesz spaces. Let $T : E \to F$ be the mapping defined by $T(x) = x^+$ for all $x \in E$. Then $T$ is a nonlinear lattice homomorphism and $T(\alpha x) = \alpha T(x)$ for all $\alpha \in \mathbb{R}^+$ and all $x \in E$.

Proof. It is well known that

$$T(x \vee y) = (x \vee y)^+ = x^+ \vee y^+ = T(x) \vee T(y)$$

and

$$(x \wedge y) = (x \wedge y)^+ = x^+ \wedge y^+ = T(x) \wedge T(y)$$

for all $x, y \in E$. It follows that $T$ is a lattice homomorphism. Moreover,

$$T(\alpha x) = (\alpha x)^+ = \alpha(x)^+ = \alpha T(x)$$

for all $\alpha \in \mathbb{R}^+$ and all $x \in E$. To see that $T$ is not linear, take $x \in E$ with $x > 0$ and observe that

$$T(-x) = (-x)^+ = 0 \quad \text{and} \quad T(x) = x.$$

Thus, $T(-x) \neq -T(x)$ and $T$ is not linear. □

To prove this result, the author used the following affirmation (see the proof of [5, Theorem 4, page 270, line 13]). If $E$ is a Riesz space with a strong order unit $e$, $F$ is a Riesz space, and $T : E \to F$ is a lattice homomorphism such that $T(\alpha\mathbf{e}) = \alpha T(\mathbf{e})$ for each $\alpha \in \mathbb{R}^+$, then $T(x)^- = T(x^-)$ for all $x \in E$. Example 1 above shows that this assertion is false. Indeed, with the same notation in Example 1,

$$T(x)^- = (x^+)^- = 0 \quad \text{and} \quad T(x^-) = (x^-)^+ = x^-$$

for all $x \in E$. Thus, for $x \notin E^+$, $T(x)^- \neq T(x^-)$ and we are done.

## References


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FETHI BEN AMOR, Research Laboratory LATAO,
Department of Mathematics, Faculty of Sciences of Tunis,
University of Tunis El-Manar, 2092 El-Manar, Tunisia
e-mail: fethi.benamor@ipest.rnu.tn


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