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ON INFINITE FULL COLOURINGS OF GRAPHS

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This paper answers affirmatively a question of Pavol Hell [**2**]: if a graph admits a full n-colouring for every finite $n \geqq n_0$, does it admit an infinite full colouring? (A colouring is *full* if every pair of distinct colour classes is joined by at least one edge).

The graphs considered here are undirected graphs without loops or spindles. The edge which joins vertices x and y is denoted by the unordered pair $[x, y]$. A graph is then a pair $X = (V, E)$ where V (or $V(X)$) is a set and E (or $E(X)$) is a set of unordered pairs of distinct elements of V. An equivalence relation R on the vertex set V is a *congruence* of the graph X if $(x, y) \in R$ implies $[x, y] \notin E$ for all vertices x and y. The congruence relations determine quotient objects in the usual way: the vertices of the quotient graph X/R are the equivalence classes $\{Rx: x \in V\}$, two of which are joined whenever some edge joins these classes in X. *Morphisms* are defined as edge-preserving maps on the vertex sets of graphs; thus congruence relations are characterized by the feature that the natural projection to the quotient is a morphism. An n-colouring is a morphism onto a complete graph K_n. A graph satisfying a formally (see [**1**, p. 143]) weaker version of the hypothesis of Hell's question, namely that it is fully n-colourable for arbitrarily large finite n will be called *colourful*.

Colourful graphs. It is readily verified that a quotient X/R is a complete graph if and only if the congruence R is maximal with respect to inclusion; thus a colouring of X is full if and only if X/R is complete where R is that relation induced by the partition of X into its colour classes. In another direction, full colourings may be obtained by coarsening the equivalence relation induced by an arbitrary colouring. (R *refines* S or S *coarsens* R if and only if every S-class is a union of R-classes). Standard maximality arguments applied to the partially ordered set of congruences of a graph ordered by refinement yield the following result.

THEOREM 1. *Every colouring of a graph may be coarsened to a full colouring.*

Any colourful graph is ω-colourable. An inductive construction will demonstrate that not every ω-colouring of a colourful graph may be coarsened to a finite full colouring.

LEMMA 1. *A colourful graph remains colourful after the removal of a finite subgraph.*

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Proof. Let X be colourful and consider a fixed finite subgraph F where $|F| = n$. Let θ be a full k-colouring of X whose colour classes are $\{c_1, c_2, \ldots c_k\}$. The restriction of θ to $X - F$ may fail to colour $X - F$ fully but may be coarsened to a full-colouring χ as follows: first omit altogether any colour class which is unrepresented outside F; if the colour class of an element of F is not joined to an external class in $X - F$, amalgamate that class into *one* such external class. After at most n losses of colour through omission or amalgamation, a full colouring χ of $X - F$ is obtained whose colour classes number at least $k - n$. (The point is that the colour classes not represented in F cannot be reduced in number by coarsening θ.) Since k may be made arbitrarily large and n is fixed, $X - F$ is colourful.

LEMMA 2. *If θ fully n-colours a graph X, then θ fully n-colours some finite subgraph F of X.*

Proof. Choose edges between the colour classes, as guaranteed by fullness. Since the endpoints form a suitable F one may take $|F| \leqq 2 \binom{n}{2}$.

LEMMA 3. *Let X be a colourful graph. There exists a sequence (G_i), $i < \omega$, of finite subgraphs of X and, for each i, a finite full colouring θ_i of G_i whose range is an initial segment of ω satisfying:*

(1) $G_i \subset G_{i+1}$ *for all* $i < \omega$;

(2) θ_{i+1} *agrees with* θ_i *when restricted to* G_i;

(3) p_i, *the number of colour classes of* θ_i, *satisfies* $p_i < p_{i+1}$ *for all* $i < \omega$.

Proof. Choose $p_1 < \omega$ such that X admits a full p_1-colouring. Let F_1 be a finite fully p_1-coloured subgraph, set $G_1 = F_1$ and let θ_1 be a full p_1-colouring of G_1 based on the numbers $\{1, 2, \ldots p_1\}$. Suppose that G_i has been suitably defined for all $i < n + 1$.

Then $X - G_n$ is colourful by Lemma 1 and we may take ψ to be a full q-colouring of $X - G_n$ where $q > p_n$ and range $\psi = \{c_1, c_2, \ldots, c_q\}$. Let F_{n+1} be a finite subgraph of $X - G_n$ which is fully q-coloured by ψ. Let $G_{n+1} = G_n \cup F_{n+1}$ (that is, the subgraph of X induced by the union of these sets of vertices).

Define θ_{n+1} as follows: on G_n, $\theta_{n+1} = \theta_n$; on F_{n+1} the colour classes are lined up in a finite sequence $c_1, c_2, \ldots, c_q$. Looking at each class in turn we write c_1 as 1 if there is *no* edge in G_{n+1} joining the two classes. If there is such an edge, we attempt in the same manner to identify c_1 with the classes $2, 3, \ldots, p_n$ respectively, where possible. If c_1 cannot be identified with any of the classes $1, 2, \ldots, p_n$, it is assigned the integer $p_n + 1$. Repeating the procedure leads to a full colouring with p_{n+1} colours and the inequalities

(*) $\quad p_n < q \leqq p_{n+1} < q + p_n.$

By construction θ_{n+1} is full; the verification of (1), (2), and (3) is routine.

THEOREM 2. *Every colourful graph has an infinite full ω-colourable subgraph.*

Proof. Let X be colourful, and from the preceding lemma take $G = \bigcup_{i<\omega} G_i$ together with its colouring $\theta = \bigcup_{i<\omega} \theta$. θ fully ω-colours G. It is crucial to note that the subgraph G is actually an *induced* subgraph of X. (If vertices x_1 an x_2 belong to G, they belong to a common G_n. By the given construction, any edge of X joining x_1 and x_2 will be an edge of G_{n+1} and so an edge of G.)

THEOREM 3. *Every colourful graph admits an infinite full colouring.*

Proof. Let X be colourful and let θ be the full ω-colouring of subgraph G of Theorem 2. θ may be extended to an ω-colouring $\tilde{\theta}$ of X. No coarsening of $\tilde{\theta}$ is possible which eliminates all but finitely many of the colour classes since $\tilde{\theta}$ restricted to G is full with infinitely many classes.

The preceding theorem answers Hell's question affirmatively. The proof of the analogue of Theorem 3 for infinite cardinals goes through by transfinite induction with only trivial modifications. This gives the following theorem.

THEOREM 4. *If a graph is fully k-colourable for every $k < m$ (m a limit cardinal), then it is fully m-colourable.*

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