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ON WEIGHTED ESTIMATES FOR STEIN'S MAXIMAL FUNCTION

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Let ϕ denote the normalised surface measure on the unit sphere S^{n-1}. We shall be interested in the weighted L^p estimate for Stein's maximal function $M_\phi f$, namely

$$\|M_\phi f\|_{L^p(w)} \leqslant C_{p,w} \|f\|_{L^p(w)}, \quad f \in L^p(w),$$

where w is an A_p weight, especially for $1 < p \leqslant 2$. Using the Mellin transformation approach, we prove that the estimate holds for every weight w^δ where $w \in A_p$ and $0 \leqslant \delta < (p(n-1)-n)/(n(p-1))$, for $n \geqslant 3$ and $n/(n-1) < p \leqslant 2$.

INTRODUCTION

Let ϕ_r be the normalised surface measure on the sphere of radius r, centre 0 in $\mathbf{R}^n$. Consider Stein's maximal function $M_\phi f$, which is defined by

$$M_\phi f(x) = \sup_{r>0} |\phi_r * f(x)|, \quad x \in \mathbf{R}^n,$$

for any nice function f on $\mathbf{R}^n$. Then we have the L^p inequality

$$\|M_\phi f\|_p \leqslant C_p \|f\|_p, \quad f \in L^p,$$

for $n \geqslant 2$ and $n/(n-1) < p \leqslant \infty$, which has been shown to be best possible [**1, 4**]. In this paper, we are interested in the weighted L^p estimate for Stein's maximal function,

$$\|M_\phi f\|_{L^p(w)} \leqslant C_{p,w} \|f\|_{L^p(w)}, \quad f \in L^p(w),$$

where $w \in A_p$, especially for $1 < p \leqslant 2$. (Consult [**3**] about A_p weights.) For $n \geqslant 3$, a positive result can be found in [**3**]; here we shall reprove and extend it.

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Using the Mellin transformation approach of Cowling and Mauceri [**2**], let $K_u(x) = C(u)\,|x|^{-n+iu}$, where $C(u) = \pi^{-(n/2)+iu}\Gamma(n - iu/2)/\Gamma(iu/2)$. ($K_u$ is the distribution on $\mathbf{R}^n$ whose Fourier transform is $\widehat{K_u}(\xi) = |\xi|^{-iu}$.) Then, formally, we have

$$\phi(x) = P_1(x) + \int_{\mathbf{R}} D(u)K_u(x)\,du, \quad x \in \mathbf{R}^n,$$

where P_1 denotes the Poisson kernel at 1 and $D(u)$ satisfies

$$2\pi C(u)D(u) = \int_0^\infty \left(\omega_{n-1}^{-1}\delta_1 - P_1\right)(s)s^{n-1-iu}\,ds, \quad u \in \mathbf{R},$$

with δ_1 being the point mass at 1. One may observe that $C(u) = O\Big((1 + |u|)^{n/2}\Big)$ and $D(u) = O\Big((1 + |u|)^{-(n/2)}\Big)$. Now, for every $r > 0$,

$$\phi_r(x) = P_r(x) + \int_{\mathbf{R}} D(u)K_u(x)r^{-iu}\,du, \quad x \in \mathbf{R}^n,$$

and accordingly, for every smooth function f on $\mathbf{R}^n$,

$$\phi_r * f(x) = P_r * f(x) + \int_{\mathbf{R}} D(u)K_u * f(x)r^{-iu}\,du, \quad x \in \mathbf{R}^n.$$

Hence

$$M_\phi f(x) \leqslant M_{P_1} f(x) + \int_{\mathbf{R}} |D(u)|\,|K_u * f(x)|\,du, \quad x \in \mathbf{R}^n.$$

Here $M_{P_1} f(x) = \sup\limits_{r>0} |P_r * f(x)|$ for x in $\mathbf{R}^n$. Since we know that $M_{P_1} f$ is majorised by the Hardy-Littlewood maximal function $M_{\mathrm{HL}} f$, we obtain

$$\|M_\phi f\|_{L^p(w)} \leqslant \|M_{\mathrm{HL}} f\|_{L^p(w)} + \int_{\mathbf{R}} |D(u)|\;\|K_u * f\|_{L^p(w)}\,du.$$

Thus, to verify the estimate, we need to get a good weighted L^p estimate for $K_u * f$, that is one that makes

$$\int_{\mathbf{R}} |D(u)|\;\|K_u * f\|_{L^p(w)}\,du \leqslant C_{p,w}\,\|f\|_{L^p(w)},$$

for $1 < p \leqslant 2$.

Main Results

We obtain the following results. The first lemma below is standard.

Lemma 1. *For $|x| \geqslant 2|y|$ and for all $\gamma \in (0,1)$,*

$$|K_u(x-y) - K_u(x)| \leqslant C(1+|u|)^{(n/2)+\gamma} |y|^{\gamma} |x|^{-n-\gamma}.$$

Proof: For $|x| \geqslant 2|y|$, we have, as in [**2**], two estimates

$$|K_u(x-y) - K_u(x)| \leqslant C(1+|u|)^{n/2} |x|^{-n}$$

and

$$|K_u(x-y) - K_u(x)| \leqslant C(1+|u|)^{(n/2)+1} |y| \; |x|^{-n-1}.$$

Interpolating these estimates, we get

$$|K_u(x-y) - K_u(x)| \leqslant C(1+|u|)^{(n/2)+\gamma} |y|^{\gamma} |x|^{-n-\gamma},$$

for all $\gamma \in (0,1)$. ∎

Following the work of Watson [**6**], we have

Lemma 2. *For $1 < p \leqslant 2$ and for any $\gamma \in (0,1)$,*

$$\|K_u * f\|_{L^p(w)} \leqslant C_{p,w,\gamma}(1+|u|)^{(n/2)+\gamma} \|f\|_{L^p(w)}, \quad f \in L^p(w),$$

whenever $w \in A_p$.

Proof: First note that $\left|\widehat{K_u}(\xi)\right| = 1$ for all $\xi \in \mathbf{R}^n$. Next, we need to show that the L^r-Hörmander condition : for $R > 2|y| > 0$,

$$\sum_{j=1}^{\infty} (2^j R)^{n/r'} \left(\int_{2^j R<|x|<2^{j+1}R} |K_u(x-y) - K_u(x)|^r \, dx\right)^{1/r} \leqslant C_{\gamma}(1+|u|)^{(n/2)+\gamma},$$

is satisfied for all $r \in (1,\infty)$. (Here r' denotes the dual exponent to r.) Having done this, we can then choose $r \in (1,\infty)$ sufficiently large such that $w^{r'} \in A_p$. Thus, following [**6**], we obtain

$$\|K_u * f\|_{L^p(w)} \leqslant C_{p,w,\gamma}(1+|u|)^{(n/2)+\gamma} \|f\|_{L^p(w)}, \quad f \in L^p(w),$$

as desired. Indeed, using Lemma 1, we observe that for all $r \in (1,\infty)$,

$$\begin{aligned}
&\int_{2^j R<|x|<2^{j+1}R} |K_u(x-y) - K_u(x)|^r \, dx \\
&\leqslant C^r(1+|u|)^{(nr/2)+\gamma r} |y|^{\gamma r} \int_{2^j R<|x|<2^{j+1}R} |x|^{-nr-\gamma r} \, dx \\
&\leqslant C^r(1+|u|)^{(nr/2)+\gamma r} R^{\gamma r} \int_{2^j R<t<2^{j+1}R} t^{-n(r-1)-\gamma r} (dt)/t \\
&\leqslant C^r(1+|u|)^{(nr/2)+\gamma r} R^{\gamma r} (2^j R)^{-n(r-1)-\gamma r} \\
&= \left[C(1+|u|)^{(n/2)+\gamma} (2^j R)^{-(n/r')} 2^{-\gamma j}\right]^r.
\end{aligned}$$

Therefore the condition is satisfied and the lemma is proved. (We have actually proved that the estimate holds whenever $w \in A_p$, for $1 < p < \infty$.) ∎

We are aware that the estimate in Lemma 2 is not good enough. We have, however, the following result of Cowling and Mauceri [2] for the unweighted case.

Lemma 3. (Cowling and Mauceri.) *For $1 < p \leqslant 2$ and for any $\gamma \in (0,1)$,*

$$\|K_u * f\|_p \leqslant C_{p,\gamma}(1+|u|)^{(n/p)-(n/2)+\gamma}\|f\|_p, \quad f \in L^p.$$

Now we have a better estimate for $K_u * f$, namely

Theorem 4. *For $1 < p \leqslant 2$ and for any $\gamma \in (0,1)$,*

$$\|K_u * f\|_{L^p(w^\delta)} \leqslant C_{p,w,\gamma,\delta}(1+|u|)^{(n/p)-(n/2)+\delta n-(\delta n/p)+\gamma}\|f\|_{L^p(w^\delta)}, \quad f \in L^p(w^\delta),$$

whenever $w \in A_p$ and $0 \leqslant \delta \leqslant 1$.

Proof: The proof follows directly from Lemma 2 and Lemma 3 by the Stein-Weiss interpolation theorem [5]. ∎

Theorem 4 leads us to the weighted L^p estimate for Stein's maximal function.

Theorem 5. *For $n \geqslant 3$ and $n/(n-1) < p \leqslant 2$, the weighted L^p estimate*

$$\|M_\phi f\|_{L^p(w^\delta)} \leqslant C_{p,w,\delta}\|f\|_{L^p(w^\delta)}, \quad f \in L^p(w^\delta),$$

holds whenever $w \in A_p$ and $0 \leqslant \delta < (p(n-1)-n)/(n(p-1))$.

Proof: Choose $\gamma \in (0,1)$ sufficiently small such that

$$0 \leqslant \delta < (p(n-1-\gamma)-n)/(n(p-1)).$$

Then, by Theorem 4, we have

$$\begin{aligned}\int_{\mathbf{R}} |D(u)|\,\|K_u * f\|_{L^p(w^\delta)}\,du &\leqslant C_{p,w,\delta}\|f\|_{L^p(w^\delta)}\int_{\mathbf{R}}(1+|u|)^{(n/p)-n+\delta n-(\delta n/p)+\gamma}du \\ &\leqslant C_{p,w,\delta}\|f\|_{L^p(w^\delta)},\end{aligned}$$

and so the theorem follows immediately. ∎

For power weights $w(x) = |x|^a$, we know that $w \in A_p$ for some $p > 1$ if and only if $-n < a < n(p-1)$. So, Theorem 5 implies that the estimate holds for $w(x) = |x|^a$ with $-(p(n-1)-n)/(p-1) < a < p(n-1)-n$. Stating it in another way, the estimate with respect to $w(x) = |x|^a$ holds for $(n+a)/(n-1) < p \leqslant 2$ when $a \geqslant 0$, or for $(n+a)/(n+a-1) < p \leqslant 2$ when $a < 0$. Thus, for $p \leqslant 2$, our result agrees with the one stated in [3, p.571] for the special case where $w(x) = |x|^a$ with $a \geqslant 0$.

Concluding Remarks

We suspect that the same estimate also holds for $p > 2$, but we encounter difficulties in verifying it. Duality arguments will not work since the endpoints of the range of allowable p's are not symmetric. The Stein-Weiss interpolation theorem only gives the estimate for $2 \leqslant p \leqslant \infty$ provided that $w \in A_2$ and $0 \leqslant \delta < (n-2)/n$. Also, since the estimate holds only for some but not all $w \in A_p$ when $n/(n-1) < p \leqslant 2$, we cannot use the existing extrapolation theorem of Rubio de Francia and Garcia-Cuerva. Some novel technique seems to be needed here and we are still working on it.

References

[1] J. Bourgain, 'Averages in the plane over convex curves and maximal operators', *J. Analyse Math.* **47** (1986), 69–85.

[2] M. Cowling and G. Mauceri, 'On maximal functions', *Rend. Sem. Mat. Fis. Milano* **49** (1979), 79–87.

[3] J. Garcia-Cuerva and J.L. Rubio de Francia, *Weighted norm inequalities and related topics* (North-Holland, Amsterdam, 1985).

[4] E.M. Stein, 'Maximal functions: spherical means', *Proc. Nat. Acad. Sci. U.S.A.* **73** (1976), 2174–2175.

[5] E.M. Stein and G. Weiss, 'Interpolation of operators with change of measures', *Trans. Amer. Math. Soc.* **87** (1958), 159–172.

[6] D.K. Watson, 'Weighted estimates for singular integrals via Fourier transform estimates', *Duke Math. J.* **60** (1990), 389–399.

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