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ON THE EXISTENCE OF NASH EQUILIBRIUM POINTS

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Abstract

The existence of Nash equilibrium points of game between n players has been established under the conditions dual to known ones.

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1. Introduction

To start with let us consider the following well-known theorem of Nash (1951):

THEOREM 1.1. *Let $K_1, K_2, \ldots, K_n$ be $n \geqslant 2$ nonempty compact convex sets each in a real Hausdorff topological vector space. Let $f_1, f_2, \ldots, f_n$ be n real-valued continuous functions defined on $K = \prod_{j=1}^{n} K_i$. Further suppose that for each $j = 1, 2, \ldots, n$ and each fixed $\hat{x}_j \in \prod_{i \neq j} K_j = \hat{K}_j$, $f_j(x_j, \hat{x}_j)$ is a quasiconcave function of x_j on K_j (that is, for each real number t and each fixed $\hat{x}_j$, the set $\{x_j \in K_j : f(x_j, \hat{x}_j) > t\}$ is convex). Then there exists a point $u \in K$ such that for all $j = 1, 2, \ldots, n$,*

$$f_j(u) = \max_{y_j \in K_j} f_j(y_j, \hat{u}_j),$$

where $\hat{u}_j$ is the natural projection of u on $\hat{K}_j$.

In terms of a game between n players, $K_1, K_2, \ldots, K_n$ are n sets of mixed strategies (convex hulls of pure strategies) corresponding to n players; f_i is the pay-off function of the ith player. The vector u is called a Nash equilibrium point in the sense that, for each $j = 1, 2, \ldots, n$, $f_j(u) = \max_{y_j \in K_j} f_j(y_j, \hat{u}_j)$ implies that each

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player's mixed strategy maximizes his pay-off if the strategies of other players are kept fixed.

Theorem 1.1 in the above form is due to Fan (1966), Theorem 4 (see also Browder (1968), Theorem 14). Browder (1968) has proved a similar result for an arbitrary family of sets, each in a locally convex Hausdorff topological vector space, while Ma (1969) extended Theorem 1.1 for an arbitrary family of sets each in a Hausdorff topological vector space.

The purpose of this note is to investigate whether the condition in Theorem 1.1 that for each $\hat{x}_j \in \hat{K}_j = \prod_{i \neq j} K_i$, $f_j(x_j, \hat{x}_j)$ is a quasiconcave function of x_j on K_j can be replaced by the type of the condition that, for each fixed $x_j \in K_j$, $f_j(x_j, \hat{x}_j)$ is a quasiconcave function of $\hat{x}_j$ on $\hat{K}_j$. To this end we will use two recent results of ours (Tarafdar and Husain (1978), Theorem 2.1 and Theorem 2.3). Here we write these two results as lemmas.

LEMMA 1.1 (Theorem 1.1). *Let $K_1, K_2, \ldots, K_n$ be $n \geqslant 2$ nonempty compact convex sets, each in a real Hausdorff linear topological space, and let $K = \prod_{j=1}^{n} K_j$. Let $S_1, S_2, \ldots, S_n$ be n subsets of K having the following properties.*

(a) *Let $\hat{K}_j = \prod_{i \neq j} K_i$, and let us denote the points of $\hat{K}_j$ by $\hat{x}_j$. For each $j = 1, 2, \ldots, n$ and for each fixed $x_j \in K_j$, the set $S_j(x_j) = \{\hat{x}_j \in \hat{K}_j : [x_j, \hat{x}_j] \in S_j\}$ is a convex subset of $\hat{K}_j$ (maybe empty).*

(b) *For each $j = 1, 2, \ldots, n$ and for each fixed $\hat{x}_j \in \hat{K}_j$, the set*

$$S_j(\hat{x}_j) = \{x_j \in K_j : [x_j, \hat{x}_j] \in S_j\}$$

is a nonempty open subset of K_j.

(c) *For $x \in (x_1, x_2, \ldots, x_n) \in K$, let $A(x) = \prod_{j=1}^{n} S_j(\hat{x}_j)$ where $\hat{x}_j$ is as before the natural projection of x on $\hat{K}_j$. Assume that $\bigcup_{x \in K} A(x) = K$. Then $\bigcap_{j=1}^{n} S_j \neq \emptyset$.*

LEMMA 1.2 (Theorem 2.3). *Let $\{K_\lambda : \lambda \in I\}$ be a family of nonempty compact convex sets, each in a locally convex Hausdorff topological vector space. Let $K = \prod_{\lambda \in I} K_\lambda$ and $\hat{K}_\lambda = \prod_{\mu \neq \lambda} K_\mu$. Let $\{S_\lambda : \lambda \in I\}$ be a corresponding indexed family of closed subsets of K having the following properties.*

(a) *For each $x = \{x_\lambda\} \in K$ and $\lambda \in I$, the sets $S_\lambda(\hat{x}_\lambda) = \{y_\lambda \in K_\lambda : [y_\lambda, \hat{x}_\lambda] \in S_\lambda\}$ is nonempty, where $\hat{x}_\lambda$ is the natural projection of x on $\hat{K}_\lambda$.*

(b) *For each $x = \{x_\lambda\} \in K$ and each $\lambda \in I$, the set $S_\lambda(x_\lambda) = \{\hat{y}_\lambda \in \hat{K}_\lambda : [x_\lambda, \hat{y}_\lambda] \in S_\lambda\}$ is a convex subset of $\hat{K}_\lambda$ (maybe empty).*

(c) *Let, for each $x = \{x_\lambda\}$, $A(x) = \prod_{\lambda \in I} S_\lambda(\hat{x}_\lambda)$. Assume that $\bigcup_{x \in K} A(x) = K$. Then $\bigcap_{\lambda \in I} S_\lambda \neq \emptyset$.*

REMARK. Lemma 1.1 is dual to Theorem 1 of Fan (1966) (see also Theorem 11 of Browder (1968) and Theorem 8 of Fan (1972)) in the sense that we have interchanged the positions of the words 'open' and 'convex' in (a) and (b). In the same way, Lemma 1.2 is dual to Theorem 2 of Fan (1966) (see also Theorem 13 of Browder (1968)). To prove Lemmas 1.1 and 1.2 we have employed a duality principle, namely, that if $T: K \to 2^K$ is a multivalued mapping (K being any nonempty set) then a point $x_0 \in K$ is a fixed point of T (that is, $x_0 \in T(x_0)$) if and only if x_0 is a fixed point of the mapping.

$$T^{-1}: T(K) \to 2^K \quad \text{where} \quad T(K) = \bigcup_{x \in K} T(x)$$

and T^{-1} is defined by

$$T^{-1}(x) = \{y \in K: x \in T(y)\}.$$

2. Main Results

As an application of our Lemma 1.1 we prove the following two theorems on the existence of Nash equilibrium point in many-person game.

THEOREM 2.1. *Let $K_1, K_2, \ldots, K_n$ be $n \geqslant 2$ nonempty compact convex sets each in a real linear Hausdorff topological space. Let $f_1, f_2, \ldots, f_n$ be n real-valued continuous functions defined on $K = \prod_{j=1}^n K_j$ satisfying the following conditions.*

(i) *For each $j = 1, 2, \ldots, n$ and each $y_j \in K_j$, the set*

$$\{\hat{x}_j \in \hat{K}_j: f_j(y_j, \hat{x}_j) > \max_{u_j \in K_j} f_j(u_j, \hat{x}_j) - t\}$$

is a convex set in $\hat{K}_j$ for each real number $t > 0$. (Note that $\max_{u_j \in K_j} f_j(u_j, \hat{x}_j)$ is finite due to the continuity of f_j and compactness of K_j.)

(ii) *If for $j = 1, 2, \ldots, n$, $t_j(\hat{x}_j)$ is a real valued continuous function on $\hat{K}_j$ such that for each $j = 1, 2, \ldots, n$ and $\hat{x}_j \in \hat{K}_j$, the set $\{y_j \in K_j: f_j\{y_j, \hat{x}_j\} > t_j(\hat{x}_j)$ is nonempty, then $\bigcup_{x \in K} A(x) = K$ where $x = (x_1, x_2, \ldots, x_n)$ and $A(x) = \prod_{j=1}^n \{y_j \in K_j: f_j\{y_j, \hat{x}_j\} > t_j(\hat{x}_j)\}$.*

Then there exists a point $u \in K$ such that for $j = 1, 2, \ldots, n$,

$$f_j(u) = \max_{y_j \in K_j} f_j(y_j, \hat{u}_j).$$

PROOF. Basically, we follow the argument of Browder (1968). For each $\hat{x}_j$ in $\hat{K}_j$, let $g_j(\hat{x}_j) = \max_{y_j \in K_j} f_j(y_j, \hat{x}_j)$. For each $j = 1, 2, \ldots, n$, g_j is a real valued function of $\hat{K}_j$ and is continuous, due to the uniform continuity of f_j on the compact space K. For each $\varepsilon > 0$, we define

$$H_\varepsilon = \{x \in K: f_j(x) \geqslant g_j(\hat{x}_j) - \varepsilon, j = 1, 2, \ldots, n\},$$

in which $\hat{x}_j$ is as before the natural projection of x on $\hat{K}_j$. The continuity of f_j, g_j and the projection mapping on $\hat{K}_j$ for all $j = 1, 2, \ldots, n$, implies that H_ε is a compact subset of K. Also H_ε decreases as ε decreases. Clearly each element $u \in H_0 = \bigcap_{\varepsilon>0} H_\varepsilon$ satisfies the conclusion of Theorem 2.1. Since H_ε is decreasing, H_0 will be nonempty if H_ε is nonempty for each $\varepsilon > 0$. We now prove that for each $\varepsilon > 0$, H_ε is nonempty. Let $\varepsilon > 0$ be chosen arbitrarily and let $S_j = \{x \in K : f_j(x_j, \hat{x}_j) > g_j(\hat{x}_j) - \varepsilon\}$. By continuity of f_j, it follows that for each $\hat{x}_j \in \hat{K}_j$, the set

$$S_j(\hat{x}_j) = \{y_j \in \hat{K}_j : f_j(y_j, \hat{x}_j) > g_j(\hat{x}_j) - \varepsilon\} = \{y_j \in K_j : [y_j, \hat{x}_j] \in S_j\}$$

is an open set in K_j. Also by definition of g_j, $S_j(\hat{x}_j)$ is nonempty. By condition (ii) we see, by taking $t_j(\hat{x}_j) = g_j(\hat{x}_j) - \varepsilon$, that $\bigcup_{x \in K} A(x) = K$, where for each

$$x = (x_1, x_2, \ldots, x_n) \in K,$$

$$A(x) = \prod_{j=1} \{y_j \in K_j : f_j(y_j, \hat{x}_j) > t_j(\hat{x}_j)\} = \prod_{j=1}^{n} S_j(\hat{x}_j).$$

By condition (i), for each j and each $y_j \in K_j$, the set

$$S_j(y_j) = (\hat{x}_j \in \hat{K}_j, f_j(y_j, \hat{x}_j) > g_j(\hat{x}_j) - \varepsilon\}$$

is a convex set in $\hat{K}_j$. Thus all the conditions of Lemma 1.1 are satisfied. Hence, by Lemma 1.1, there is a point $v \in \bigcup_{j=1}^{n} S_j$. Clearly $v \in H_\varepsilon$.

DEFINITION 2.1. Let K be a nonempty convex subset of a linear space E. A real-valued function g defined on K is said to be *convex* if $g(\lambda x + \mu y) \leqslant \lambda g(x) + \mu g(y)$ for all $x, y \in K$ where $0 \leqslant \lambda$, $\mu \leqslant 1$ and $\lambda + \mu = 1$.

DEFINITION 2.2. Let K be a convex subset of a linear topological space. A real-valued function f defined on K is said to be *strongly quasiconcave* if, given any continuous convex function g on K, the set $\{x \in K : f(x) > g(x)\}$ is either empty or a convex set. It is trivial to see that a strongly quasiconcave function is quasiconcave.

EXAMPLES. Let E and F be two linear topological spaces with a continuous bilinear pairing Φ of $E \times F$ into the reals. In the usual notation $\Phi(x, w) = (x, w)$, $x \in E$, $w \in F$.

(i) For each fixed $x \in E$, $\Phi(x, w)$ is strongly quasiconcave on any convex subset of F.

(ii) For each fixed $w \in F$, $\Phi(x, w)$ is strongly quasiconcave on E or on any convex subset of E.

(iii) For each fixed $x \in E$, the function $f(w) = \Phi(x, w) - g(w)$ is strongly quasiconcave on F where g is a real-valued convex function on F.

THEOREM 2.2. *Let $K_1, K_2, \ldots, K_n$ be $n \geqslant 2$ nonempty compact convex sets, each in a real linear Hausdorff space. Let $f_1, f_2, \ldots, f_n$ be n real-valued functions defined on $K = \prod_{j=1}^{n} K_j$ satisfying the following conditions.*

(a) *For each $j = 1, 2, \ldots, n$, the function g_j defined on $\hat{K}_j$ by*

$$g_j(\hat{x}_j) = \max_{y_j \in K} f_j(y, \hat{x}_j)$$

is convex.

(b) *For each $j = 1, 2, \ldots, n$ and for each $y_j \in K_j$, the function $f_j(y_j, \hat{x}_j)$ is a strongly quasiconcave function of $\hat{x}_j$ on $\hat{K}_j$.*

(c) *The condition* (ii) *of Theorem* 2.1 *holds. Then there is a point $u \in K$ such that, for each $j = 1, 2, \ldots, n$, $f_j(u) = \max_{y_j \in K_j} f_j(y, \hat{u}_j)$.*

PROOF. Since for each real number t, the function $g_j(\hat{x}_j) - t$ is convex by condition (a) and continuous by the argument given in Theorem 2.1, condition (i) of Theorem 2.1 holds by virtue of the condition (b), while condition (ii) of Theorem 2.1 is assumed in condition (c). Hence the theorem follows from Theorem 2.1.

The following theorem is dual to Theorem 15 of Browder (1968).

THEOREM 2.3. *Let $\{K_\lambda : \lambda \in I\}$ be a family of nonempty compact convex sets each in a locally convex linear Hausdorff topological space E. Let $\{f_\lambda : \lambda \in I\}$ be a correspondingly indexed family of continuous real valued functions on $K = \prod_{\lambda \in I} K_\lambda$ satisfying the following conditions:*

(i) *For each $\hat{x}_\lambda \in \hat{K}$, consider the nonempty set*

$$B(\hat{x}_\lambda) = \{y_\lambda \in K_\lambda : f_\lambda(y_\lambda, \hat{x}_\lambda) \geqslant \max_{u_\lambda \in K_\lambda} f_\lambda(u_\lambda, \hat{x}_\lambda)\}$$

and for each $x = \{x_\lambda\}$, put $A(x) = \prod_{\lambda \in I} B(\hat{x})$. Assume that $\bigcup_{x \in K} A(x) = K$.

(ii) *For $\lambda \in I$, the function g_λ defined on $\hat{K}_\lambda$ by $g_\lambda(\hat{x}_\lambda) = \max_{y_\lambda \in K_\lambda} f_\lambda(y_\lambda, \hat{x}_\lambda)$ is convex.*

(iii) *For each $\lambda \in I$, and for each fixed $y_\lambda \in K_\lambda$, the function $f_\lambda(y_\lambda, \hat{x}_\lambda)$ is a strongly quasiconcave function of $\hat{x}_\lambda$ on $\hat{K}_\lambda$.*

Then there exists a point $u \in K$ such that for each $\lambda \in I$, $f_\lambda(u) = \max_{y_\lambda \in K_\lambda} f_\lambda(y_\lambda, \hat{u}_\lambda)$

PROOF. We consider for each $\lambda \in I$ the nonempty set

$$S_\lambda = \{u \in K : f_\lambda(u) \geqslant \max_{y_\lambda \in K_\lambda} f_\lambda(y, \hat{u}_\lambda)\}.$$

Since K is compact in the locally convex space $\prod_{\lambda \in I} E_\lambda$, the uniform continuity of f_λ on K implies that the function $g_\lambda(\hat{x}_\lambda)$ as defined above is continuous on $\hat{K}_\lambda$. It then follows that S_λ is a nonempty closed subset of K for each $\lambda \in I$. Now for each $\lambda \in I$ and each $\hat{x}_\lambda$ in $\hat{K}_\lambda$,

$$S_\lambda(\hat{x}_\lambda) = \{y_\lambda \in K_\lambda : f_\lambda(y_\lambda, \hat{x}_\lambda) \geqslant g_\lambda(\hat{x}_\lambda)\} = \{y_\lambda \in K_\lambda : (y_\lambda, \hat{x}_\lambda) \in S_\lambda\}$$

is nonempty by definition of g_λ. Also it is clear that

$$A(x) = \prod_{\lambda \in I} B(\hat{x}_\lambda) = \prod_{\lambda \in I} S_\lambda(\hat{x}_\lambda).$$

Then by condition (i), $\bigcup_{x \in K} A(x) = K$. Finally for each $\lambda \in I$ and for each fixed $x_\lambda \in K_\lambda$, the set

$$S_\lambda(x_\lambda) = \{\hat{y}_\lambda \in \hat{K}_\lambda : (x_\lambda, \hat{y}_\lambda) \in S_\lambda\} = \{\hat{y}_\lambda \in \hat{K}_\lambda : f_\lambda(x_\lambda, \hat{y}_\lambda) \geqslant g_\lambda(\hat{y}_\lambda)\}$$

is convex by conditions (ii) and (iii). Thus all the conditions of Lemma 1.2 are satisfied. Hence there is a point $u \in K$ such that $u \in \bigcap_{\lambda \in I} S_\lambda$, that is

$$f_\lambda(u) = \max_{y_\lambda \in K_\lambda} f_\lambda(y_\lambda, \hat{u}_\lambda)$$

for each $\lambda \in I$.

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