COLLINEATION GROUPS OF GENERALIZED ANDRÉ PLANES

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1. Introduction. In a previous paper (**5**), I constructed a class of translation planes, called generalized André planes or λ-planes, and discussed the associated autotopism collineation groups. The main question unanswered in (**5**) is whether or not there exists a collineation η of a λ-plane Π which moves the two axes of Π but does not interchange them.

The answer to this question is "no", except if Π is a Hall plane (or possibly if the order n of Π is 3^4) (Corollary 2.8). This result makes it possible to determine the isomorphisms between λ-planes. More specifically, let Π and Π' be two λ-planes of order n coordinatized by λ-systems Q and Q', respectively. Then, except possibly if $n = 3^4$, Π and Π' are isomorphic if and only if Q and Q' are isotopic or anti-isotopic (Corollary 2.13). In particular, Π is an André plane if and only if Q is an André system (Corollary 2.14).

The λ-planes Π of (**5**) are very similar to the planes constructed by Ostrom (**8**; **10**; **11**; **12**) by "replacing" certain nets in a Desarguesian plane Π^* (see § 3). The results in this paper concerning λ-planes are derived as a result of interpreting λ-planes from Ostrom's point of view. In particular, from a lemma of Ostrom (Theorem 2.2), it follows that any collineation of a λ-plane Π has the form $\eta = \tilde{\sigma}\xi\tilde{\tau}$, where ξ is a collineation of the underlying Desarguesian plane Π^*, and $\tilde{\sigma}$ and $\tilde{\tau}$ are mappings determined by field automorphisms (Theorem 2.5). The results listed above, as well as the results of (**5**) concerning the collineation group of Π, follow directly from this form for η. There is one case ($n = 3^4$) to which the proofs below do not apply. It seems probable that this case is not an exception to the results of this paper (see the remarks following the proof of Theorem 2.7).

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2. The collineation group of $\Pi(Q_\lambda)$. In order to define the class of generalized André systems, or λ-systems (**5**), we introduce the following notation. Let $q = p^s$, for p a prime, let $n = q^d$, and let $f = ds$. Let $F = \mathrm{GF}(n)$ and let $K = \mathrm{GF}(q)$. Let ω be a primitive root of F, let α be the automorphism of F of order f defined by $\alpha \colon x \to x^p$, and let $\rho = \alpha^s$ so that ρ has order d. Let

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V be the vector space of ordered pairs (x, y) of elements of F. Thus, V is a two-dimensional vector space over F, a $2d$-dimensional space over K, and a $2f$-dimensional space over $GF(p)$. If σ is an automorphism of F/K (i.e., if the fixed field of σ contains K), let $\tilde{\sigma}$ be the following mapping of V, $\tilde{\sigma}$: $(x, y) \rightarrow (x, \sigma(y))$.

Let λ be an arbitrary mapping from F^* (the set of non-zero elements of F) into $\langle\rho\rangle$, the automorphism group of F/K. If $\lambda(\omega^k) = \rho^t$, it is sometimes convenient to write $\lambda(k) = t$, i.e., to represent λ as a mapping from I_{n-1} into I_d, where $I_z = \{0, 1, \ldots, z - 1\}$ is the set of least positive residues (mod z). It is also convenient to define the *index* of an element $m \in F^*$ as follows: if $m = \omega^k$ with $k \in I_{n-1}$, then $\operatorname{ind}(m) = k$. For a, $b \in F$, define the binary operation "o" by $a \circ b = a\rho^{\lambda(a)}(b)$, if $a \neq 0$, and $0 \circ b = 0$. Finally, let Q be the system $Q = \{F, +, \circ\}$.

LEMMA 2.1. *Q is a (left) quasi-field (or Veblen-Wedderburn system) if and only if the mapping λ: $I_{n-1} \rightarrow I_d$ satisfies the following conditions:*

(a) $\lambda(0) = 0$;

(b) *If i, $j \in I_{n-1}$, and if $i \equiv j$ (mod $q^t - 1$), where $t = (d, \lambda(i) - \lambda(j))$, then $i = j$.*

Proof. See (**5**, Lemma 2.1).

If λ satisfies the conditions of the lemma, then $Q = Q_\lambda$ will be called a *generalized André system*, or a *λ-system*. It is always possible to assume (after normalizing λ) that the λ-system Q_λ has Kern $K = GF(q)$ (**5**, 2.5). For the λ-system Q_λ, define v to be the least integer in I_{n-1} such that λ is periodic (mod v); and define $u = \operatorname{LCM}(q^t - 1)$, where $t|d$ and $t < d$. Then $v|u$ from Lemma 2.1(b). A λ-system Q_λ with Kern K is an André system if and only if $i \equiv j$ (mod $q - 1$) implies $\lambda(i) = \lambda(j)$, for i, $j \in I_{n-1}$; i.e., if and only if $v|q - 1$ (**5**; § 3, example 1). The vector space V can be made into the affine plane $\Pi(Q)$ coordinatized by a λ-system Q, by defining the points of $\Pi(Q)$ to be the elements (x, y) of V, and the lines of $\Pi(Q)$ to be the solution sets of the equations $y = m \circ x + b$ and $x = c$, for m, b, $c \in Q$; these lines will be denoted by $[m, b]$ and $[c]$, respectively. The class $\mathfrak{C}$ of lines of $\Pi(Q)$ through the origin, $O = (0, 0)$, consists of the lines $[m, 0]$ and $[0]$, for $m \in F$. $\Pi(Q)$ can be completed to a projective plane by adding the points (m) and (∞), $m \in F$, $\infty \notin F$, and the line $l_\infty = [\infty]$, in the usual manner.

Let Π^* be the Desarguesian affine plane defined by V as a two-dimensional vector space over F. That is, the points of Π^* are the elements of V; and the lines of Π^* are the solutions of the equations $y = mx + b$ and $x = c$ (for m, b, $c \in F$), which will be denoted by $[m, b]^*$ and $[c]^*$, respectively. The class $\mathfrak{C}^*$ of lines of Π^* through $(0, 0)$ consists of the lines $[m, 0]^*$ and $[0]^*$, $m \in F$. In the projective completion of Π^*, the points $(m)^*$ and $(\infty)^*$, and the line $l_\infty^* = [\infty]^*$ are added to Π^*.

Note. The "*" will distinguish the Desarguesian plane Π^* and its lines, from the λ-plane $\Pi(Q)$ and its lines.

THEOREM 2.2 (**12**, Lemma 1). *Let η be a non-singular semi-linear transformation of V over K. Let $\mathfrak{M}_\eta$ be the set of those lines of $\mathfrak{C}^*$ which are mapped onto lines of $\mathfrak{C}^*$ under η. Finally, let $\bar{t}$ be the maximal proper divisor of d. If $|\mathfrak{M}_\eta| > q^{\bar{t}} + 1$, in particular if $|\mathfrak{M}_\eta| > q^{d/2} + 1$, then η is a Desarguesian collineation of Π^* (i.e., η is semi-linear over F).*

Proof. The proof follows from Lemmas 2.3 and 2.4.

LEMMA 2.3. *If $|\mathfrak{M}_\eta| \geqq 3$, then $|\mathfrak{M}_\eta| = q^t + 1$ for some t which divides d.*

LEMMA 2.4. *Assume that η (in (2.2)) fixes the Desarguesian lines $[0, 0]^*$, $[1, 0]^*$, and $[0]^*$. Let $M_\eta = \{m \in F\colon [m, 0] \in \mathfrak{M}_\eta\}$. Then M_η is a subfield of F which contains K.*

Proof. Since η fixes the subspaces $y = 0$, $x = 0$, and $y = x$ of V, then η as a $2d$-dimensional semi-linear transformation over K has the form

$$\eta\colon (x, y) \rightarrow (\sigma(x), \sigma(y)),$$

where σ is a d-dimensional semi-linear transformation of F over K. The mapping of F, $\bar{m}\colon x \rightarrow mx$, is a linear transformation of F, for $m \in F$.

For $m \in M_\eta$, let k be defined by $\eta\colon [m, 0]^* \rightarrow [k, 0]^*$. Then $\eta(x, mx) = \eta(x, \bar{m}(x)) = (\sigma(x), \sigma\bar{m}(x)) = (\sigma(x), \bar{k}\sigma(x))$; thus $\bar{k} = \sigma\bar{m}\sigma^{-1}$. If $m_i \in M_\eta$ with $\eta\colon [m_i, 0]^* \rightarrow [k_i, 0]^*$ for $i = 1, 2$, then $m_1 + m_2 \in M_\eta$ and $m_1m_2 \in M_\eta$. For example, to show that $m_1m_2 \in M_\eta$, consider the line $[m_1m_2, 0]^*$. Applying η, $\eta(x, m_1m_2x) = (\sigma(x), \sigma\bar{m}_1\bar{m}_2(x)) = (\sigma(x), \bar{m}_1{}^\sigma\bar{m}_1{}^\sigma\sigma(x))$. Since $\bar{m}_i{}^\sigma = \bar{k}_i$, $i = 1, 2$, then $\bar{m}_1{}^\sigma\bar{m}_2{}^\sigma = \bar{k}_1\bar{k}_2 = \overline{k_1k_2}$. Hence, $\eta\colon [m_1m_2, 0]^* \rightarrow [k_1k_2, 0]^*$; thus $m_1m_2 \in M_\eta$. Therefore, M_η is a subfield of F. Since σ is a semi-linear transformation over K, $K \subseteq M_\eta$.

To prove Lemma 2.3, assume that $|\mathfrak{M}_\eta| \geqq 3$. Let l_i and l_i' for $i = 1, 2, 3$ be three lines of $\mathfrak{M}_\eta$ and their images under η, respectively. Since these six lines are Desarguesian lines, there exist Desarguesian collineations τ_1 and τ_2 of Π^* such that

$$\tau_1\colon [0, 0]^* \rightarrow l_1,\ [0]^* \rightarrow l_2,\ [1, 0]^* \rightarrow l_3$$

and

$$\tau_2\colon l_1' \rightarrow [0, 0]^*,\ l_2' \rightarrow [0]^*,\ l_3' \rightarrow [1, 0]^*.$$

Further, let $\eta' = \tau_2\eta\tau_1$. Thus, η' satisfies the hypothesis of Lemma 2.4. Moreover, η' maps a Desarguesian line l onto some other Desarguesian line if and only if η maps the Desarguesian line $\tau_1(l)$ onto a Desarguesian line; i.e., $|\mathfrak{M}_\eta| = |\mathfrak{M}_\eta'|$. By Lemma 2.4, $|\mathfrak{M}_\eta'| = |M_\eta'| + 1 = q^t + 1$, where $t|d$.

THEOREM 2.5. *Let $\Pi(Q_\lambda)$ be a proper λ-plane of order $n = q^d$, with* Kern $K = \mathrm{GF}(q)$, *and exclude the cases $q = 3$ or 4, and $d = 2$ or 4. Let η be a collineation of $\Pi(Q_\lambda)$ which fixes $(0, 0)$. Then there exists a Desarguesian collineation ξ of Π^* (which fixes $(0, 0)$), and automorphisms σ and τ of F/K, such that $\eta = \tilde{\tau}\xi\tilde{\sigma}$.*

Note. In the exceptional case $n = 3^2$, the collineation group of the nearfield-André-Hall plane of order 9 is discussed in (**1**); see also (**4**, 3.10). For the case $q = 3$ and $d = 4$, see the remarks following the proof of Theorem 2.7. The cases $q = 4$ and $d = 2$ or 4 are not exceptions to any of the results after Lemma 2.6; see the proof of Theorem 2.7.

Proof. The $n + 1$ lines of $\Pi(Q)$ which contain $(0, 0)$ can be divided into d classes $\mathfrak{N}_\mu$, $0 \leqq \mu \leqq d - 1$, as follows.

$$\mathfrak{N}_0 = \{[m, 0]: \lambda(m) = 0\} \cup \{[0, 0], [0]\} \quad \text{and} \quad \mathfrak{N}_\mu = \{[m, 0]: \lambda(m) = \mu\}$$

for $0 < \mu < d$. Here, λ is the mapping of Q_λ, λ: $Q_\lambda^* \to I_d$. Thus, $\mathfrak{N}_\mu$ is the set of lines whose slopes are inverse images of μ under λ; $\mathfrak{N}_0$, in addition, contains the axes. Of course, $\mathfrak{N}_\mu$ may be empty (if $\mu \neq 0$). Suppose that exactly T of the classes $\mathfrak{N}_\mu$ are non-empty.

For each μ_1 and μ_2 $(0 \leqq \mu_1, \mu_2 < d)$, let N_{μ_1,μ_2} be the set of lines of $\mathfrak{N}_{\mu_1}$ which are mapped into $\mathfrak{N}_{\mu_2}$ by η. Choose μ_1 and μ_2 so that $\mathfrak{N}_{\mu_1,\mu_2}$ is maximal, and let $\mathfrak{N} = \mathfrak{N}_{\mu_1,\mu_2}$. Clearly, $|\mathfrak{N}| \geqq (n + 1)/T^2$ from the definition of T.

Lemma 2.6 below shows that, except possibly if $q = 3$ or 4 and $d = 2$ or 4, $|\mathfrak{N}| > q^{\tilde{t}} + 1$, where $\tilde{t}$ is the maximal proper divisor of d. This lemma permits the application of Theorem 2.2, as follows.

Let σ and τ be defined by $\sigma = \rho^{-\mu_1}$ and $\tau = \rho^{\mu_2}$, and let $\xi = \tilde{\tau}^{-1}\eta\tilde{\sigma}^{-1}$. Then ξ maps at least $|\mathfrak{N}|$ Desarguesian lines onto Desarguesian lines. Let $l \in \mathfrak{N}$ and let $l' = \eta(l)$. Suppose that $l = [m, 0]$ and $l' = [k, 0]$ (if l or $l' = [0]$, then a similar argument holds). Consider what ξ does to the Desarguesian line $[\sigma(m), 0]^* = \{(x, \sigma(m)x)\}$:

$$\xi\colon (x, \sigma(m)x) \xrightarrow{\tilde{\sigma}^{-1}} (x, m\sigma^{-1}(x)) = (x, m\rho^{\lambda(m)}(x)) = (x, m \circ x) \xrightarrow{\eta} (\bar{x}, k \circ \bar{x})$$

$$= (\bar{x}, k\tau(\bar{x})) \xrightarrow{\tilde{\tau}^{-1}} (\bar{x}, \tau^{-1}(k)\bar{x}).$$

That is, for each line $[m, 0]$ (or $[0]$) of $\mathfrak{N}$, ξ maps the corresponding Desarguesian line $[\sigma(m), 0]^*$ (or $[0]^*$) onto a Desarguesian line. Since $|\mathfrak{N}| > q^{\tilde{t}} + 1$ by Lemma 2.6, then Theorem 2.2 implies that ξ is a Desarguesian collineation of Π^*, and hence $\eta = \tilde{\tau}\xi\tilde{\sigma}$, as required.

LEMMA 2.6. *Let Q_λ be a proper λ-system of order $n = q^d$, with* Kern $K =$ GF(q). *However, exclude the cases $q = 3$ or 4 and $d = 2$ or 4. Let $\mathfrak{N}$ be the set of lines of* $\Pi(Q_\lambda)$ *defined in the proof of Theorem* 2.5. *Then* $|\mathfrak{N}| > q^{\tilde{t}} + 1$, *where $\tilde{t}$ is the maximal proper divisor of d.*

Proof. (1) Since $T \leqq d$ and $\tilde{t} \leqq d/2$, then $|\mathfrak{N}| \geqq (q^d + 1)/T^2 \geqq (q^d + 1)/d^2 > q^d/d^2$, and $q^{d/2} + 1 \geqq q^{\tilde{t}} + 1$. Hence, it is sufficient tro pove that $q^d/d^2 \geqq q^{d/2} + 1$, i.e., that $q^{d/2}/d^2 \geqq 1 + (1/q^{d/2})$.

(2) Consider the function $f(x) = (q^{x/2})/x^2$, for fixed q and $x > 0$. Then $f'(x) > 0$, i.e., $f(x)$ is increasing, for $x > 4/\ln q$. Moreover, the function $g(x) = 1 + (1/q^{x/2})$ is decreasing for all x. To prove that $f(d) \geqq g(d)$ for fixed q and for $d \geqq d_0$, it is sufficient to check that $d_0 > 4/\ln q$ and that $f(d_0) \geqq g(d_0)$.

(3) Let $d = 2$ and $q \geqq 5$. Then clearly $q^d/d^2 \geqq q + 1$, as required in part (1).

(4) Let $d \geqq d_0 = 4$, and let $q \geqq 5$ in part (2). Then $d_0 > 4/\ln q$; $f(d_0) = q^2/16 \geqq 25/16$; $g(d_0) = 1 + 1/q^2 \leqq 26/25$; and hence $f(d_0) \geqq g(d_0)$. Therefore, $q^{d/2}/d^2 \geqq 1 + (1/q^{d/2})$, as required.

(5) Let Q_λ be an André system and $d > 2$. Then λ, as a mapping from I_{n-1}, is periodic (mod $q - 1$); see § 2 and (**5**, § 3, example 1). Therefore, $T \leqq q - 1$. Hence, as in part (1) above, $|\mathfrak{N}| \geqq (n + 1)/T^2 \geqq (n + 1)/(q - 1)^2$. If $d \geqq 4$, then $|\mathfrak{N}| \geqq (q^d + 1)/(q - 1)^2 > q^{d/2} + 1$ follows from the inequalities:

$$\begin{aligned}(q^d + 1)/(q - 1) > (q^d - 1)/(q - 1) &= q^{d-1} + q^{d-2} + \ldots + q + 1\\ &> q^{d/2+1} - q^{d/2} + q - 1 = (q - 1)(q^{d/2} + 1).\end{aligned}$$

If $d = 3$, then $\tilde{t} = 1$, and $|\mathfrak{N}| \geqq (q^3 + 1)/(q - 1)^2 > q^{\tilde{t}} + 1 = q + 1$.

If Q_λ is a λ-system with d a prime, then Q_λ is also an André system. This is true since λ, as a mapping from I_{n-1} to I_d, is always periodic (mod u), where $u = \mathrm{LCM}(q^t - 1)$ for $t|d$, $0 < t < d$ (**5**, 2.4). If d is a prime, then $u = q - 1$, and therefore λ is periodic (mod $q - 1$) which implies that Q_λ is an André system (**5**, § 3, example 1).

Thus, parts (3)–(5) above prove Lemma 2.6 for the cases $q \geqq 5$ and $d \geqq 2$, and for the cases in which d is a prime, $d > 2$.

(6) Let $q = 2$. Part (2) applies to the case $q = 2$ and $d \geqq 30$; i.e., let $d_0 = 30$ in part (2) and check that $d_0 > 4/\ln 2$ and that $f(d_0) \geqq g(d_0)$. For $d < 30$ and $q = 2$, no λ-system has Kern $K = \mathrm{GF}(2)$ (**5**, Lemma 3.4). For, in order that the Kern be GF(2), d must have at least three distinct prime factors and in particular, $d \geqq 30$. (I do not know whether *any* λ-system exists with Kern GF(2).) Hence, Lemma 2.6 is true for $q = 2$.

(7) Let $q = 4$. Part (2) applies if $d \geqq 5$ and part (5) applies for $d = 3$. The cases $d = 2$ and 4 are excluded. However, the case $q = d = 4$ actually satisfies Theorem 2.5 as follows. If η fixes or interchanges the axes, then Theorem 2.5 follows from the description of the autotopism and anti-autotopism collineations of $\Pi(Q_\lambda)$ in (**5**). Otherwise, Lemma 2.6 can be proved directly; see the proof of Theorem 2.7.

(8) Let $q = 3$. Part (2) above applies to the cases $d \geqq 8$, and part (3) applies to the cases $d = 3$, 5, and 7.

The case $q = 3$ and $d = 6$ can be excluded as follows. If $T \leqq 5$, then $q^d/T^2 \geqq 3^6/25 > 3^3 + 1 = q^{d/2} + 1$, as required.

Now assume that $T = 6$. Since $T = |\mathrm{Image}\,\lambda|$, then $T = 6$ implies that λ maps I_{n-1} onto I_6. Choose i and $j \in I_{n-1}$ such that $\lambda(i) = 1$ and $\lambda(j) = 5$

in I_6. From Lemma 2.1(a), $\lambda(0) = 0$. Applying Lemma 2.1(b) to i and $0 \in I_{n-1}$, it follows that $t = (6, 1 - 0) = 1$, and hence $i \not\equiv 0 \pmod{3^t - 1}$; thus i is an odd integer. Similarly, applying Lemma 2.1(b) to j and 0 implies that j is an odd integer. Next, assume that $\lambda(k) = 0$ for $k \in I_{n-1}$. Applying Lemma 2.1(b) to k and i implies that k is an even integer. Similarly, if $\lambda(k) = 2$ or if $\lambda(k) = 4$, then applying Lemma 2.1(b) to k and i, or to k and j, respectively, implies that k is an even integer. Finally, if $\lambda(k) = 3$ choose $u \in I_{n-1}$ so that $\lambda(u) = 2$. Applying Lemma 2.1(b) to k and u implies that k is an odd integer.

To complete the proof in this case, let k and $k' \in I_{n-1}$ such that $k \equiv k' \pmod 8$. Assume that $\lambda(k) \neq \lambda(k')$. If k and k' are both odd, then $\lambda(k)$ and $\lambda(k')$ are distinct integers in the set $\{1, 3, 5\}$. Applying Lemma 2.1(b) to k and k', where $t = (6, \lambda(k) - \lambda(k')) = 2$, it follows that $k = k'$, and hence $\lambda(k) = \lambda(k')$. Similarly, if k and k' are both even, then $\lambda(k) = \lambda(k')$. Therefore, λ is periodic (mod 8). Hence, λ induces a mapping from I_8 onto I_6, and therefore at least one integer $\mu_1 \in I_6$ is the image of two or more elements in I_8. The corresponding class $\mathfrak{N}_{\mu_1}$ contains at least $2 \cdot (3^6 - 1)/8 = 182$ lines. Therefore, $\mathfrak{N}$ (= maximal $\mathfrak{N}_{\mu_1,\mu_2}$) satisfies $|\mathfrak{N}| \geqq 182/T = 182/6 > 3^{\tilde{t}} + 1 = 28$. Thus Lemma 2.6 holds for $q = 3$ and $d = 6$.

The remaining cases $d = 2$ and 4 are excluded.

THEOREM 2.7. *Let $\Pi(Q_\lambda)$ be an arbitrary proper λ-plane of order $n = q^d$, with* Kern $= GF(q)$. *Exclude the case $q = 3$, $d = 4$. Suppose that η is a collineation of* Π *which fixes* $(0, 0)$ *and which moves* (0) *to* (a) *with $a \neq 0$ or ∞. Then $d = 2$, so that λ is periodic* $\pmod{q - 1}$, *and one of the following conditions is satisfied:*

(*a*) *there are exactly $q - 2$ i's in I_{q-1} with $\lambda(i) = 0$; or*
(*b*) *there are exactly $q - 2$ i's in I_{q-1} with $\lambda(i) = 1$.*

COROLLARY 2.8. *If an arbitrary proper λ-plane $\Pi(Q_\lambda)$ has a collineation moving* (0) *to* (a) *with $a \neq 0$, or ∞, then $\Pi(Q_\lambda)$ is a Hall plane, or possibly $q = 3$ and $d = 4$.*

Proof of Corollary 2.8. From Theorem 2.7, $d = 2$, and hence Q_λ is an André system. Conditions (a) and (b) imply that one net of order $\sqrt{n} + 1$ has been replaced in the Desarguesian planes Π^* and $\tilde{\rho}(\Pi^*)$, respectively, to use Ostrom's terminology. Finally, that Π is a Hall plane follows from Ostrom's proof (**12**; **14**) of the following theorem: *every finite Hall plane is an André plane*; see also (**3**; **5**, 7.6) for other proofs of this theorem. Alternatively, Albert and Hughes (**5**, 7.6) have shown that Π is a Hall plane if Q_λ satisfies condition (a). It is straightforward to show that if Q_λ satisfies (b), then Q_λ is isotopic to a λ-system which satisfies (a), by using (**5**, 7.3), and hence Π is a Hall plane.

Proof of Theorem 2.7. Exclude for the moment the cases $q = 3$ or 4 and $d = 2$ or 4. Let η be a collineation of $\Pi(Q_\lambda)$ which fixes $(0, 0)$ and maps (0)

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to (a) and (∞) to (b), with $a \neq 0$ or ∞. Then, since $\Pi(Q_\lambda)$ has no $((\infty), [0])$-elations, it follows that $b \neq 0$ or ∞ (**5**, 6.1). From Theorem 2.5, $\eta = \tilde{\tau}\xi\tilde{\sigma}$, for ξ a Desarguesian collineation of Π^*, and τ and σ automorphisms of F/K. Since ξ is a non-singular semi-linear transformation of V over F, ξ can be represented as follows:

$$(2.1)\qquad \xi = \gamma\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where (a_{ij}) is a non-singular matrix with elements from F, and γ is an automorphism of F (γ is applied to both components of a vector $(x, y) \in V$). Notice that since η maps $(0) \to (a)$, $(\infty) \to (b)$ with $a, b \neq 0, \infty$, then $a_{ij} \neq 0$, for $1 \leqq i, j \leqq 2$. For ξ as given in (2.1), $\eta = \tilde{\tau}\xi\tilde{\sigma}$ acts as follows on the points of V:

$$(2.2)\qquad \eta(x, y) = (\gamma(a_{11}x + a_{12}\sigma(y)), \tau\gamma(a_{21}x + a_{22}\sigma(y))).$$

Let η: $(m) \to (m')$, for m and $m' \neq \infty$. Then $\eta(x, m \circ x) = (x', m' \circ x')$, and (2.2) implies that:

$$(2.3)\qquad m' \circ x' = m'\rho^{\lambda(m')}\gamma(a_{11}x + a_{12}\sigma(m \circ x)) = \tau\gamma(a_{21}x + a_{22}\sigma(m \circ x)),$$

for $x \in F$. Applying γ^{-1} to (2.3) and rewriting, one derives:

$$(2.4)\qquad b_1\rho^{\lambda(m')}(x) + b_2\rho^{\lambda(m')}\sigma\rho^{\lambda(m)}(x) = b_3\tau(x) + b_4\tau\sigma\rho^{\lambda(m)}(x),$$

for all x, and for all m such that $m, m' \neq (\infty)$, where

$$(2.5)\qquad \begin{aligned} b_1 &= \gamma^{-1}(m')\rho^{\lambda(m')}(a_{11}), \qquad b_2 = \gamma^{-1}(m')\rho^{\lambda(m')}(a_{12})\rho^{\lambda(m')}\sigma(m), \\ b_3 &= \tau(a_{21}), \quad \text{and} \quad b_4 = \tau(a_{22})\tau\sigma(m). \end{aligned}$$

Note. (2.3)–(2.5) hold for m or $m' = 0$ if $\rho^{\lambda(0)}$ is defined to be any automorphism of F/K.

At this point, a well-known result is required.

LEMMA 2.9 (**2**, p. 35, Corollary to Theorem 12). *Let $\sigma_1, \ldots, \sigma_k$ be distinct automorphisms of a field F, and let $b_1, \ldots, b_k \in F$ such that $\sum_{i=1}^k b_i\sigma_i(x) = 0$ for all $x \in F$. Then $b_1 = \ldots = b_k = 0$.*

Now, (2.4) has the form:

$$(2.4')\qquad b_1\sigma_1(x) + b_2\sigma_2(x) - b_3\sigma_3(x) - b_4\sigma_4(x) = 0$$

for $b_i \in F$ and σ_i automorphisms of F $(1 \leqq i \leqq 4)$. Applying Lemma 2.9 to this equation, the following four cases for the b_i's must be considered:

(1) $b_1 = -b_2$, $b_3 = -b_4$, and $\rho^{\lambda(m)} = \sigma^{-1}$;
(2) $b_1 = b_3$, $b_2 = b_4$, and $\rho^{\lambda(m')} = \tau$;
(3) $b_1 = b_4$, $b_2 = b_3$, $\rho^{\lambda(m')} = \tau\sigma\rho^{\lambda(m)}$ and $(\sigma\rho^{\lambda(m)})^2 = I$;
(4) $b_1 + b_2 = b_3 + b_4$, $\rho^{\lambda(m)} = \sigma^{-1}$ and $\rho^{\lambda(m')} = \tau$.

To see this, first note that if m and $m' \neq \infty$, then either all the b_i's are non-zero; or $m = 0$, or $m' = 0$, and in these two cases, exactly two b_i's are zero.

Now, suppose that the σ_i's are all distinct in (2.4′). Then Lemma 2.9 implies that $b_1 = \ldots = b_4 = 0$, which is false. Next, suppose that $\sigma_1 = \sigma_2 = \sigma_3$ and σ_4 is distinct. Then Lemma 2.9 implies that $b_4 = 0$, and hence exactly one other b_i is zero. Thus, one of the cases (1)–(3) applies. Now, assume that $\sigma_1 = \sigma_2$, $\sigma_3 = \sigma_4$, and $\sigma_1 \neq \sigma_3$. Then case (1) applies. Finally, if $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4$, then case (4) applies. Similarly, the other possibilities for (2.4′) which are allowed by Lemma 2.9 are contained in the cases (1)–(4).

LEMMA 2.10. *If $m' \neq 0$ or ∞, then either $\rho^{\lambda(m')} = \tau$ or $\rho^{\lambda(m')} = \tau\psi$, where $\psi^2 = I$.*

Proof. If m' satisfies case (1), then $m' = 0$, $\rho^{\lambda(m)} = \sigma^{-1}$, and $b_3 = -b_4$ implies that $m = \sigma^{-1}(-a_{21}/a_{22})$. For if $m' \neq 0$, then $b_1 = -b_2$ implies that $m = \sigma^{-1}(-a_{11}/a_{12})$, and hence $\det(a_{ij}) = 0$, which is false since η is non-singular. Similarly, case (2) implies that $m = 0$, $\rho^{\lambda(m')} = \tau$, and $m' = \gamma\tau(a_{21}/a_{11})$.

From Lemma 2.9 and (2.2), if $\eta\colon (\infty) \to (m')$, then $m' = \tau\gamma(a_{22}/a_{12})$ and $\rho^{\lambda(m')} = \tau$. Similarly, if $\eta\colon (m) \to (\infty)$, then $m = \sigma^{-1}(-a_{11}/a_{12})$ and $\rho^{\lambda(m)} = \sigma^{-1}$.

Finally, if $\eta\colon (m) \to (m')$ with $m' \neq 0$ or ∞, then cases (3) and (4) apply, and either $\rho^{\lambda(m')} = \tau$ (case (4)) or $\rho^{\lambda(m')} = \tau\psi$ (case (3)), where $\psi^2 = I$.

LEMMA 2.11. $d = 2$.

Proof. Let $J = \{\rho^{\lambda(m)}\colon m \in F^*\}$. From Lemma 2.10, $J = \{\tau, \tau\psi\}$. Since Q is a λ-system, then $I \in J$; in particular, for $m = 1 \in F^*$, $\rho^{\lambda(1)} = I$. Hence, either $\tau = I$ or $\tau = \psi^{-1}$ and $\tau^2 = I$. In either case, the automorphism subgroup $\mathfrak{I}$ of F/K generated by J has order 1 or 2. Since Q is a proper λ-system with Kern $K = \mathrm{GF}(q)$, then $|\mathfrak{I}| = d = 2$.

To complete the proof of Theorem 2.7, let $d = 2$ so that Q is an André system. Since $J = \{\rho^{\lambda(m)}\colon m \in F^*\} = \{\tau, \tau\psi\}$ from Lemma 2.10, and since $|\mathfrak{I}| = 2$ (because Q is a proper λ-system and $I \in J$), we have that $\psi = \rho$. Further, $\rho^{\lambda(m')} = \tau$ except when m' satisfies case (3).

In case (3), $\psi = \sigma\rho^{\lambda(m)}$, $\rho^{\lambda(m')} = \tau\psi$, $b_1 = b_4$, and $b_2 = b_3$. Hence, $\gamma^{-1}(m')\tau\psi(a_{11}) = \tau(a_{22})\tau\sigma(m)$ and $\gamma^{-1}(m')\tau\psi(a_{12})\sigma\tau\psi(m) = \tau(a_{21})$. Cancelling τ and eliminating $\gamma^{-1}(m')$ (since $m' \neq 0$), we derive:

$$\frac{a_{22}\sigma(m)}{\psi(a_{11})} = \frac{a_{21}}{\psi(a_{12})\sigma\psi(m)}\,;$$

i.e.,

$$m\cdot\psi(m) = \sigma^{-1}\left[\frac{a_{21}\psi(a_{11})}{a_{22}\psi(a_{12})}\right] = k;$$

i.e.,

(2.6) $$m^{q+1} = k \quad \text{for } k \text{ as defined above.}$$

Since $|J| = 2$, (2.6) must have solutions for m, in which case there exist exactly $q + 1$ solutions. If $\tau = I$, then since Q is an André system, there exists exactly one $i_0 \in I_{q-1}$ such that $\lambda(i_0) = 1$, and for the other $q - 2$ i's (in I_{q-1}), $\lambda(i) = 0$ (case (a) for Theorem 2.7). Similarly, if $\tau = \rho$, then case (b) applies.

If $q = 3$ or 4 and $d = 2$, then condition (a) or (b) of Theorem 2.7 is satisfied since $q - 1$ is small, and thus in these cases $\Pi(Q)$ is a Hall plane.

Next, let $q = 4$ and $d = 4$. It is sufficient to show that η has the form $\eta = \tilde{\tau}\xi\tilde{\sigma}$ with ξ Desarguesian. Since $q = 4$ and $d = 4$, then $q^{\tilde{i}} + 1 = 17$, and, moreover, from (**5**, § 3, example 5) $u = q^2 - 1 = 15$; therefore λ is periodic (mod 15). Since $15/4 > 3$, then some $\mathfrak{N}_\mu$ (see the proof of Theorem 2.5) contains at least $4 \cdot 17$ lines, excluding [0] and [0, 0]; hence, either $|\mathfrak{N}_{\mu,\nu}| > 17$ for some ν, and Lemma 2.6 is satisfied, or $|\mathfrak{N}_{\mu,\nu}| = 17$ for each ν. Now, choose ν such that η maps [0] or [0, 0] onto $\mathfrak{N}_\nu$. Then $\xi = \tilde{\rho}^{-\nu}\eta\tilde{\rho}^{-\mu}$ maps the 17 Desarguesian lines of $\tilde{\rho}^{-\mu}(\mathfrak{N}_{\mu,\nu})$ onto Desarguesian lines. In addition, the Desarguesian line [0] or [0, 0], which η maps into $\mathfrak{N}_\nu$, is mapped by ξ onto a Desarguesian line. Hence, ξ maps at least $18 > q^{\tilde{i}} + 1$ Desarguesian lines onto Desarguesian lines. By Theorem 2.2, ξ is a Desarguesian collineation, as required.

Finally, let $q = 3$ and $d = 4$. If Theorem 2.7 is false for Q, then techniques similar to those above for $q^d = 4^4$ show that $T = 4$ and that each inverse image under λ has order 20. From (**5**, § 3, example 5), λ is periodic (mod 8), and in fact there are 36 possible systems, and less than 36 isotopism classes, to consider. It seems very unlikely that any of these systems violate Theorem 2.7.

COROLLARY 2.12. *Exclude the case $q = 3$, $d = 4$. Let Π be a proper λ-plane which is not a Hall plane, coordinatized by the λ-system Q. Let G be the group of all collineations of Π, T the translation group, and $\tilde{A}$ the group of autotopism and anti-autotopism collineations of Π. Then $G_{(0,0)} = \tilde{A}$, and $G = T \cdot \tilde{A}$;* cf. (**5**, § 6).

COROLLARY 2.13. *Exclude the case $q = 3$, $d = 4$. Two λ-planes $\Pi(Q)$ and $\Pi(Q')$, are isomorphic if and only if the λ-systems Q and Q' are isotopic or anti-isotopic.*

Proof. If $\Pi(Q)$ is a λ-plane coordinatized by the λ-system Q, then $\Pi(Q)$ has ((0, [0])-homologies and ((∞), [0, 0])-homologies which are not involutions. If $\Pi(Q)$ is not a Hall plane, then any collineation of $\Pi(Q)$ either fixes or interchanges (0) and (∞). Hence, if $\Pi(Q)$ is coordinatized with respect to different axes, then the resulting ternary system is not a λ-system. If σ is an isomorphism from $\Pi(Q)$ to $\Pi(Q')$ and the planes are not Hall planes, then σ must map the axes of Π onto the axes of Π'. Hence, Q and Q' are isotopic or anti-isotopic (**5**, § 4).

If Π and Π' are Hall planes, then Q and Q' each must satisfy either condition (a) or (b) of Theorem 2.7. As in the proof of Corollary 2.8, Q and Q' are isotopic.

COROLLARY 2.14. *Let $\Pi(Q)$ be a proper λ-plane coordinatized by the λ-system Q. Then $\Pi(Q)$ is isomorphic to an André plane if and only if Q is an André system, even for the case $q = 3$ and $d = 4$.*

Proof. Apply Corollary 2.13 and (**5**, 7.3 and 7.4): if the λ-system Q is isotopic or anti-isotopic to an André system, then Q is an André system. If $\Pi(Q)$ is a Hall plane, then $d = 2$ and every λ-system is an André system. Finally, let $q = 3$ and $d = 4$. From the proof of Lemma 2.6 (part (5)), or from the remarks in the proof of Theorem 2.7, if Q_λ is an André system, then no collineation of $\Pi(Q_\lambda)$ can move (0) and (∞), except possibly to interchange them. Hence, as in the proof of Theorem 2.5, if $\Pi(Q_\lambda)$ is isomorphic to $\Pi(Q')$ for Q' a λ-system, then Q' is isotopic or anti-isotopic to Q_λ, and hence Q' is an André system.

3. Miscellaneous examples. Theorem 2.7 implies the following result about the special class of λ-planes $\Pi(Q_g)$ described in (**6**).

LEMMA 3.1. (1) *$\Pi(Q_g)$ is isomorphic to an André plane if and only if $g|q - 1$.*

(2) *Exclude the case $q = 3$, $d = 4$. If $g = d$, then there exist $\phi(d)/\delta \operatorname{ord}_d p$ non-isomorphic planes, $\Pi(Q_d)$, where ϕ is the Euler function, and $\delta = 1$ or 2;* see (**6**, following 3.7). *If $g < d$, then there exist $\phi(d)\phi(g)/2\delta \operatorname{ord}_g p$ non-isomorphic planes $\Pi(Q_g)$. Moreover, if $g_1 \neq g_2$, then $\Pi(Q_{g_1})$ is not isomorphic to $\Pi(Q_{g_2})$.*

Proof. (1) Use Corollary 2.14 and the fact that Q_g is an André system if and only if $g|q - 1$.

(2) Use Corollary 2.13 and the description of the isotopism classes for Q_g (**6**, 3.11, 3.14, and 4.8). Finally, for $g_1 \neq g_2$, Q_{g_1} and Q_{g_2} are not isotopic (**6**, 4.7).

Lemma 3.1 shows that there is a large number of non-isomorphic planes, even in the class $\{\Pi(Q_g)\}$. A better indication of the large number of λ-planes is given by the following lemma.

LEMMA 3.2. *Let $n = p^f = q^d$, and let $d = c^t$, for c a prime. Exclude the case $q = 3$, $d = 4$. Let r be defined by*

$$r = \left(\sum_{k=0}^{t-1} q^{c^k}\right) - 2t.$$

Then there exist at least $c^r/2f(n - 1)^2$ non-isomorphic λ-planes of order n whose Kerns contain GF(q).

Proof. There exist c^r distinct λ-systems of order n with Kerns containing GF(q) (**5**, 3.3). From (**5**, 5.12) there exist at most $2f(n - 1)^2$ systems in an (isotopism, anti-isotopism)-class. Hence, the result follows from Corollary 2.13. As an example, let $q = 5$, $c = 3$, and $t = 2$. In this case, there exist at least 3^{96} non-isomorphic λ-planes of order $n = 5^9$.

Finally, let us indicate the relation of the class of λ-planes to a sub-class of planes which Ostrom constructed by a "homology-type replacement" of certain nets of the Desarguesian plane Π^* (**10**; **11**; **12**). This construction consists of replacing certain Desarguesian lines, $y = mx$ of Π^*, by lines of the form $y = m\sigma_m(x)$, to obtain a new plane Π (here σ_m is an automorphism of F).

The plane Π has the property that the $((0, 0), l_\infty^*)$-homologies of Π^*, namely the mappings $(x, y) \to (ax, ay)$ for $a \neq 0$ in F, are also collineations in Π.

From the form of the new lines, it is not surprising that Π is a λ-plane.

THEOREM 3.3. *Let Π be a plane which can be obtained from the Desarguesian plane Π^* be a series of disjoint, restricted homology-type replacements, as described above. Then Π is isomorphic to a λ-plane.*

Proof. Since the lines of Π have the form $x = 0$, $y = 0$, and $y = m\sigma_m(x)$ for $m \in F^*$, the point $(1, 1)$ is on some line $y = \sigma_1(x)$. Let Π' be the image of Π under the mapping $\tilde{\sigma}_1^{-1}$: $(x, y) \to (x, \sigma^{-1}(y))$. The lines of Π' have the form $x = 0$, $y = 0$, $y = m\tau_m(x)$ for $m \in F^*$ and τ_m an automorphism of F, where $\tau_1 = 1$. Define λ: $F^* \to \langle \rho \rangle$ by $\lambda(m) = \tau_m$, and define $m \circ x = m\rho^{\lambda(m)}(x) = m\tau_m(x)$. Then $\{F, +, \circ\} = Q_\lambda$ is a quasi-field coordinatizing Π', since $1 \circ x = x$. From Lemma 2.1, $\{F, +, \circ\}$ is a λ-system, and hence Π' is a λ-plane. Therefore. Π is isomorphic to the λ-plane Π'.

The $((0, 0) - l_\infty^*)$-homology $(x, y) \to (ax, ay)$ of Π^* induces a collineation in the λ-plane Π' as follows:

(3.1) $\qquad (x, y) \to (ax, \sigma(a)y) \quad$ for σ an automorphism of F.

This condition can be used to determine which λ-planes occur in Ostrom's construction.

THEOREM 3.4. *Exclude the case $q = 3$, $d = 4$. Suppose that the λ-plane $\bar{\Pi}$, coordinatized by the λ-system Q, is isomorphic to a plane Π obtained from the Desarguesian plane Π^* by "restricted homology-type replacement". Then there exists a λ-system $Q' = Q_{\lambda'}$ isotopic to Q such that λ' satisfies the following condition for some μ $(0 \leqq \mu < f)$:*

(3.2) $\qquad \lambda'(k) \equiv \lambda'(k + \nu(q^{\lambda'(k)} - p^\mu)) \pmod{d} \quad$ *for* $k, \nu \in I_{n-1}$.

Proof. Let $\Pi' = \Pi(Q_{\lambda'})$ be the λ-plane isomorphic to Π, as in the proof of Theorem 3.3. Then Π' is isomorphic to $\bar{\Pi}$, and by Corollary 2.13, Q' and Q are isotopic or anti-isotopic. Further, the mappings (3.1), for $a \in F^*$, are collineations of Q', and hence for each $a \neq 0$, $m' \circ (ax) = \sigma(a) \cdot (m \circ x)$ determines m' as a function of m (for all x). This condition is easily shown to be equivalent to (3.2), by using Lemma 2.9.

There are some λ-planes which are not isomorphic to planes constructed from Π^* by homology-type replacement. For example, let $q = 3$, $d = 12$, $g = 4$, and let $Q_\lambda = Q_g$ (**6**); i.e., for $k \in I_{n-1}$, let $k' \equiv k \pmod{g}$ with $k' \in I_g$. Define $\lambda(k) = k'$. Let $\Pi = \Pi(Q_g)$ and let us consider whether Π itself can be obtained from Π^* by homology-type replacement. If so, then two lines of Π which intersect a line of Π^* non-trivially have intersections of the same dimension over K. However, $[\omega, 0]$ and $[\omega^{-25}, 0]$ in Π intersect $[\omega, 0]^*$ in Π^* in subspaces of dimension 1 and 3, respectively, over K. Moreover, in this example, $\Pi(Q_g)$ is not isomorphic to a plane obtained from Π^* by homology-type replacement. For if so, then Theorem 3.4 and (**6**, 4.8) imply that Q_g is isotopic to a λ-system which satisfies (3.2), for some μ. By using the definition of Q_g and (**5**, 5.5), the general condition for two λ-systems to be isotopic, it is easy to check that no isotope of Q_g satisfies (3.2) for any μ (it is sufficient to check $\mu = 0$ and 1).

On the other hand, as Ostrom (**11**) pointed out in a special case, any λ-plane, $\Pi(Q_\lambda)$, can be obtained from Π^* by replacing certain "André-type" nets, but not necessarily by a homology-type replacement.

LEMMA 3.5. *Let $\Pi(Q_\lambda)$ be an arbitrary λ-plane with* Kern K. *Let N^* and N be the nets in Π^* and $\Pi(Q_\lambda)$, respectively, consisting of all lines with slopes m such that* ind $m \equiv \delta \pmod{q-1}$, *for fixed δ. Then N is a replacement for N^* (N and N^* are not necessarily distinct).*

Added in proof. Recently, Theorem 2.7 and Corollary 2.8 (including the case $q = 3$, $d = 4$) have been proved independently by M. L. Narayana Rao (*A conjecture of D. R. Hughes in λ-planes*, to appear), based on a paper of F. W. Wilke (*A class of translation planes and a conjecture of D. R. Hughes*, to appear). Further, Rao and J. L. Zemmer have answered the question in the proof of Lemma 2.6 (part (6)) by constructing λ-systems with GF(2) as the Kerns (*A question of Foulser on λ-systems of characteristic two*, to appear in Illinois J. Math.).

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