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RESEARCH ARTICLE

The expected nodal volume of non-Gaussian random band-limited functions, and their doubling index

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Dedicated to the memory of Steven Morris Zelditch

Abstract
The asymptotic law for the expected nodal volume of random non-Gaussian monochromatic band-limited functions is determined in vast generality. Our methods combine microlocal analytic techniques and modern probability theory. A particularly challenging obstacle that we need to overcome is the possible concentration of nodal volume on a small portion of the manifold, requiring solutions in both disciplines and, in particular, the study of the distribution of the doubling index of random band-limited functions. As for the fine aspects of the distribution of the nodal volume, such as its variance, it is expected that the non-Gaussian monochromatic functions behave qualitatively differently compared to their Gaussian counterpart. Some conjectures pertaining to these are put forward within this manuscript.

1. Introduction

1.1. Band-limited functions

In recent years a lot of effort has been put into understanding the geometry of Laplace eigenfunctions on smooth manifolds. Let (M, g) be a smooth compact Riemannian manifold of dimension n, and $\Delta = \Delta_g$ the Laplace-Beltrami operator on M. Denote $\{\lambda_i\}_{i\geq 1}$ to be the (purely discrete) spectrum of Δ, with the corresponding orthonormal system of Laplace eigenfunctions ϕ_i satisfying

$$\Delta\phi_i + \lambda_i^2\phi_i = 0.$$

An important qualitative descriptor of the geometry of ϕ_i is its *nodal set* $\phi_i^{-1}(0)$ and, in particular, the nodal volume $\mathcal{V}(\phi_i) = \mathcal{H}^{n-1}(\phi_i^{-1}(0))$ – that is, the $(n-1)$-dimensional Hausdorff measure of $\phi_i^{-1}(0)$.

The highly influential *Yau's conjecture* [57] asserts that the nodal volume of ϕ_i is commensurable with λ_i: there exist constants $C_M > c_M > 0$ so that

$$c_M \cdot \lambda_i \leq \mathcal{V}(\phi_i) \leq C_M \cdot \lambda_i.$$

Yau's conjecture was established for the real analytic manifolds [13, 14, 21], whereas, more recently, the optimal lower bound and polynomial upper bound were proved [36, 37, 38] in the smooth case.

In his seminal work [9], Berry proposed to compare the (deterministic) Laplace eigenfunctions on manifolds, whose geodesic flow is ergodic, to the *random* monochromatic isotropic waves – that is, a Gaussian stationary isotropic random field $F_\mu : \mathbb{R}^n \to \mathbb{R}$, whose spectral measure μ is the hypersurface measure on the sphere $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$, normalized by unit total volume. Equivalently, $F_\mu(\cdot)$ is uniquely defined via its covariance function

$$K_\infty(x, y) := \mathbb{E}[F_\mu(x) \cdot F_\mu(y)] = \int_{\mathbb{S}^{n-1}} e^{i\langle x-y,\xi\rangle} d\mu(\xi). \tag{1.1}$$

For example, in $2d$, the covariance function of $F_\mu : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$\mathbb{E}[F_\mu(x) \cdot F_\mu(y)] = J_0(|x-y|),$$

where J_0 is the Bessel J function of order 0. Berry's conjecture should be understood in some random sense (e.g., when averaged over the energy level). Alternatively, one can consider some random ensemble of eigenfunctions or their random linear combination (Gaussian or non-Gaussian).

A concrete ensemble of the said type is that of *band-limited* functions [53]

$$f_T(x) = f(x) = v(T)^{-1/2} \sum_{\lambda_i \in [T-\rho(T),T]} a_i \phi_i(x), \tag{1.2}$$

where a_i are centred unit variance i.i.d. random variables (Gaussian or non-Gaussian), $T \to \infty$ is the *spectral parameter*, and the summation is over the *energy window* $[T-\rho(T), T]$ of width $\rho = \rho(T) \geq 1$. Observe that, because the set of the energies is discrete, in reality, the spectral parameter T is also discrete. The convenience pre-factor

$$v(T) := \frac{(2\pi)^n}{\omega(n) \cdot \mathrm{Vol}(M)} \rho(T) T^{n-1} = c_M \rho(T) T^{n-1}, \tag{1.3}$$

with $\omega_n = \frac{\pi^{n/2}}{\Gamma(n/2+1)}$ being the volume of the unit ball in $\mathbb{R}^n$, is introduced to ensure that $f_T(x)$ is of asymptotic unit variance as $T \to \infty$ at each $x \in M$ and has no impact on the nodal structure of $f_T(\cdot)$. Regardless of whether or not $f_T(\cdot)$ in (1.2) is Gaussian, its covariance kernel is the function $K_T : M \times M \to \mathbb{R}$ given by

$$K_T(x, y) := \mathbb{E}[f_T(x) \cdot f_T(y)] = \frac{1}{v(T)} \sum_{\lambda_i \in [T-\rho(T),T]} \phi_i(x) \cdot \phi_i(y), \tag{1.4}$$

coinciding with the *spectral projector* in $L^2(M)$ onto the subspace spanned by the eigenfunctions $\{\phi_i\}_{\lambda_i \in [T-\rho,T]}$ (recall that a_i are unit variance).

In what follows, we will focus on the most interesting – and, in some aspects, most difficult – *monochromatic* regime $1 \leq \rho(T) = o_{T\to\infty}(T)$. In this case, it is well-known that, under suitable assumptions on M and on ρ (explicated below), the covariance (1.4), after scaling the variables by T, is asymptotic to (1.1), around (almost) every reference point x, in the following sense. Let $\exp_x : T_x M \to M$ be the exponential map – that is, a bijection between a ball $B(r) \subseteq \mathbb{R}^n$ centred at $0 \in \mathbb{R}^n$ and some neighborhood in M of x, with $r > 0$ depending only on M, independent of $x \in M$. Then we have

$$K_T\left(\exp_x(y/T), \exp_x(y'/T)\right) \underset{T\to\infty}{\longrightarrow} K_\infty(y, y') \tag{1.5}$$

uniformly for $\|y'\|, \|y\| \leq 1$, with $K_\infty(\cdot,\cdot)$ as in (1.1), with the convergence (1.5) holding together with an arbitrary number of derivatives [16, 17, 51]. The convergence (1.5) hints that one would expect, in the high energy limit, the nodal volume distribution of f_T in (1.2) to exhibit some aspects of universality.

For the linear combinations (1.2) of Laplace eigenfunctions on real analytic M, the deterministic upper bound analogue

$$\mathcal{V}(f_T) \leq C_M \cdot T \tag{1.6}$$

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of Yau's conjecture remains valid, thanks to the work of Jerison-Lebeau [41, Section 14] (see also the work of Lin [35]). The principal results of this manuscript determine the precise asymptotic growth, in the high energy limit, in the monochromatic regime, of the expected nodal volume of monochromatic random band-limited functions on generic real analytic manifolds with no boundary, under the mere assumption that the a_i have a finite third moment.

1.2. Statement of a principal result

Let the dimensional constant

$$c_n := \left(\frac{1}{\pi n}\right)^{1/2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}, \tag{1.7}$$

where $n \geq 2$, and the exponent

$$\vartheta_n := \begin{cases} \frac{-n^2+4n+1}{2(n+1)} & 2 \leq n \leq 4 \\ 0 & n \geq 5 \end{cases} \tag{1.8}$$

(i.e., $\vartheta_2 = \frac{5}{6}$, $\vartheta_3 = \frac{1}{2}$, $\vartheta_4 = \frac{1}{10}$ and $\vartheta_n = 0$ for $n \geq 5$).

Theorem 1.1. *Let $n \geq 2$ and (M, g) be a real analytic compact n-manifold with empty boundary. Suppose that a_i are i.i.d. random variables so that*

$$\mathbb{E}\left[|a_i|^3\right] < +\infty,$$

and let $f_T(\cdot)$ be the band-limited functions (1.2) with

$$\rho(T) = \rho_n(T) = T^{\vartheta_n}(\log T)^2. \tag{1.9}$$

Then one has

$$\mathbb{E}[\mathcal{V}(f_T)] = c_n \operatorname{Vol}(M) \cdot T + o_{T\to\infty}(T),$$

with c_n given by (1.7).

Although Theorem 1.1 gives some explicit power saving on the monochromatic bound $\rho(T) = o(T)$, one wishes to take ρ as small as possible in order to resemble a single eigenfunction to the highest extent. We are able to address this question in dimension $n \geq 5$ under some (likely redundant) geometric assumption on M, as we will describe in the next section.

1.3. Constant energy windows in high dimensions

The following definition is useful, as we will need to further restrict the class of manifolds to allow to decrease the energy window to constant width.

Definition 1.2. Let (M, g) be a smooth compact manifold with empty boundary, S^*M the cotangent sphere bundle on M, and $G^t : S^*M \to S^*M$ the geodesic flow on M.

(1) The set of loop directions based at x is

$$\mathcal{L}_x = \{\xi \in S^*_x M : \exists t > 0.\ \exp_x(t\xi) = x\}.$$

(2) The set of closed geodesics based at x is

$$\mathcal{CL}_x = \{\xi \in S^*_x M : \exists t > 0.\ G^t(x, \xi) = (x, \xi)\}.$$

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(3) A point $x \in M$ is **self-focal**, if $|\mathcal{L}_x| > 0$, where $|\cdot|$ is the natural measure on S_x^* induced by the metric $g_x(\cdot,\cdot)$.
(4) The geodesic flow on M is **periodic** if the set of its closed geodesics is of full Liouville measure in S^*M. The geodesic flow on M is **aperiodic** if the set of its periodic closed geodesics is of Liouville measure 0.

We observe that for M real analytic, the set of its periodic geodesics is of either full or 0 Liouville measure in S^*M (see either [51, Lemma 1.3.8] or Lemma 8.3 below). Hence, in the real analytic case, **the geodesic flow on M is either periodic or aperiodic**. The following principal result prescribes the precise asymptotic law, as $T \to \infty$, of the expected nodal volume for random band-limited functions with energy window of constant width for 'generic' manifolds of dimension $n \geq 5$.

Theorem 1.3. *Let $n \geq 5$ and (M,g) be a real analytic compact n-manifold with empty boundary, so that either the geodesic flow on M is periodic or the geodesic flow on M is aperiodic and the set of self-focal points of M is of measure 0. There exists a sufficiently large constant $\rho_0 = \rho_0(M,g) \geq 1$ such that the following holds. Suppose that a_i are i.i.d. random variables so that*

$$\mathbb{E}\left[|a_i|^3\right] < +\infty,$$

and let $f_T(\cdot)$ be the band-limited functions (1.2) with $\rho(T) \equiv \rho_0$. Then one has

$$\mathbb{E}[\mathcal{V}(f_T)] = c_n \operatorname{Vol}(M) \cdot T + o_{T\to\infty}(T),$$

with c_n given by (1.7).

As we were circulating this manuscript, we were informed by S. Zelditch that, as part of a work in progress, he proved that the set of self-focal points of *every* real analytic manifold with empty boundary, whose geodesic flow is aperiodic, is of measure 0. That means that the assumptions of Theorem 1.1 imply the assumptions of Theorem 1.3. Hence, for $n \geq 5$, the energy window in (1.9) could be made of constant width $\rho \equiv \rho_0$.

The principal results of this manuscript, Theorem 1.1 and Theorem 1.3, stated for a particular $\rho = \rho(T)$, remain valid, along with all our arguments, when ρ grows faster (but not slower) than as explicitly stated, so long as it obeys the monochromatic condition $\rho(T) = o(T)$. For example, under the scenario of Theorem 1.3, ρ is allowed to grow arbitrarily slowly, as long as $\rho(T) = o(T)$. For the non-monochromatic regime

$$\rho \underset{T\to\infty}{\sim} \alpha \cdot T,$$

with some $\alpha \in (0,1]$, not pursued within this manuscript, our proofs show that the statement of Theorem 1.1 holds except that the limit random field is different, resulting in a different, but explicit, constant depending on α.

It is plausible that the 3rd moment assumption $\mathbb{E}\left[|a_i|^3\right] < +\infty$ in Theorem 1.1 and Theorem 1.3 could be weakened, possibly to $\mathbb{E}\left[|a_i|^{2+\varepsilon}\right] < +\infty$ or even to $\mathbb{E}\left[|a_i|^2\right] < +\infty$. Indeed, the finiteness of the third moment of the a_i is used exclusively for applying the Berry-Esseen theorem on f_T in Lemma 6.5. It is conceivable that the assumptions of Lemma 6.5 could be weakened by a more careful study of the characteristic function of f_T, leading to the said refinement. However, it was decided to keep the statements of the principal results in their present form for the sake of brevity of the arguments and better readability of the manuscript.

1.4. Doubling index

We wish to spend a couple of paragraphs on the doubling index, a novel aspect of the proofs of the main results. The local universality suggested by the convergence of the covariance function in (1.5) does not give sufficient local information on the distribution of the nodal volume of the band-limited functions.

This is due to the (possible) concentration of nodal volume: small probability events contributing positively to the expectation of the nodal volume. To control such events, the (local) nodal volume can be further analysed by studying the doubling index, a local measure of the growth of eigenfunctions [21, 35, 36, 37]:

$$\mathcal{N}_{f_T}(x) := \log \frac{\sup_{B_g(x,2/T)} |f_T|}{\sup_{B_g(x,1/T)} |f_T|},$$

where $B_g(x,r)$ is the geodesic ball centred at $x \in M$ of radius $r > 0$.

The study of the *distribution* of the doubling index as a function of $x \in M$ is a key tool in understanding the zero set of Laplace eigenfunctions. In particular, Donnelly and Fefferman demonstrated that $\mathcal{N}(\cdot)$ is bounded for 'most' $x \in M$ [21] and used this result to deduce a lower bound in Yau's conjecture. Upper bounds on large values of $\mathcal{N}(\cdot)$ have also been used to derive a lower bound in the smooth case by Logunov [37]. In this paper, we focus on the distribution of the doubling index for random band-limited functions. One important result, which is instrumental for the rest of the paper, is proving that, for *random* functions, with high probability, large values of the doubling index are very rare, beyond the deterministic results of Donnelly and Fefferman. See Section 6 for more details.

Some conventions

We write $A \lesssim B$ to designate the existence of some constant $C > 0$ such that $A \le CB$; if C depends on some auxiliary parameter β, then in this case we write $A \lesssim_\beta B$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \asymp B$. We also write $C, c > 0$ for constants whose value may change from line to line. Further, for two functions $A, B : \mathbb{R} \to \mathbb{R}$, we will use the asymptotic notation $A = o(B)$ if $A(t)/B(t) \to 0$ as $t \to \infty$; in particular, $o(1)$ denotes a function tending to zero. Every constant implied in the notation may depend on (M, g), which will be suppressed.

The notation $B(x,r)$ and $B_g(\cdot)$ will stand for the (Euclidean) ball centred at x of radius $r > 0$ and the geodesic ball on M, respectively, and the shorthand $B_0 = B(0,1) \subseteq \mathbb{R}^n$ will be employed. Given a ball B – Euclidean or geodesic – and some number $r > 0$, its closure is $\overline{B}$, whereas rB will stand for the concentric ball of r-times the radius.

We use the multi-index notation $D^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_2}^{\alpha_2}$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$. Furthermore, given a (C^3) function $g : B(x,r) \to \mathbb{R}$ and some $r > 0$, we let

$$\mathcal{V}(g, B(x,r)) = \mathcal{H}^{n-1}\{x \in B(x,r) : g(x) = 0\}$$

be the nodal volume of g in $B(x,r)$. Finally, we denote by $(\Omega, \mathbb{P})$ the abstract probability space where every random object is defined, by $\mathbb{E}[\cdot]$ the expectation with respect to $d\mathbb{P}$, and by

$$d\sigma := \frac{d\,\mathrm{Vol}}{\mathrm{Vol}(M)} \otimes d\mathbb{P}$$

the (normalized) probability measure on the space $M \times \Omega$.

2. Outline of the proofs of the main results

2.1. Reconstructing the total nodal length from local patches

The starting point of the proofs is the following observation: because the nodal volume is a *local* quantity (i.e., it is additive w.r.t. (disjoint) subsets of M), one may asymptotically reconstruct it based on averaging the local nodal volume of f_T restricted to small balls w.r.t. their centres. That is,

$$\mathcal{V}(f_T) = T(1 + o(1)) \int_M \mathcal{V}(F_x) d\,\mathrm{Vol}_g(x), \tag{2.1}$$

where F_x is a scaled local version

$$F_{x,T}(y) = F_x(y) = f_T(\exp_x(y/T))$$

of f_T in the vicinity of $x \in M$, defined on the unit Euclidean ball $y \in B_0(1)$. Thus, to evaluate $\mathcal{V}(f)$, it is sufficient to understand the nodal volume $\mathcal{V}(F_x)$ *on average* w.r.t. $x \in M$.

With this notion in mind, rather than working with f_T as a random field defined on a probability space Ω (where the random variables a_i are defined), we may think of $F_x(\cdot)$ as a random field indexed by $B_0(1)$, defined on the probability space $M \times \Omega$. Thus, the local nodal volume $\mathcal{V}(F_x)$ is, in this sense, a random variable on the product space $M \times \Omega$ w.r.t. the normalized probability measure $\frac{d\,\mathrm{Vol}_g}{\mathrm{Vol}(M)} \otimes d\mathbb{P}$. In light of the above, to use the observation (2.1), the proofs of Theorem 1.1 and Theorem 1.3 will borrow from two important preliminary steps: local asymptotic Gaussianity of f_T in Proposition 4.1 (regarding the growing energy windows case) and Proposition 9.1 (regarding the constant energy windows case) and an anti-concentration estimate for $\mathcal{V}(F_x)$ in Proposition 6.1. We now explicate the meaning of these preliminary steps and give a sketch of their proofs.

2.2. Asymptotic Gaussianity

Because, under the assumptions on ρ of either Theorem 1.1 or Theorem 1.3, the number of the summands within (1.2) is growing and the a_i are i.i.d., the scaled version $F_{x,T}$ of f_T should asymptotically behave like a Gaussian random field with correlations given by (1.1). It will be rigorously proved for either the growing window regime as in Theorem 1.1 or the constant width regime $\rho \equiv \rho_0$, though for the former case, our arguments are significantly simplified.

Let us first explain the proof under the assumptions of Theorem 1.1. In this case, the asymptotic behavior of the correlation function of $F_{x,T}$, postulated in (1.5), is given by the local Weyl's law of Hörmander; see Section 4.2 below. It also follows that all the summands in (1.2) have size $o(v(T))$. Therefore, an application of Linderberg's Central Limit theorem, together with the Continuous Mapping Theorem, implies

$$\mathcal{V}(F_x) \xrightarrow{d} \mathcal{V}(F_\mu) \qquad T \to \infty,$$

where the convergence is in distribution *uniformly* w.r.t. $x \in M$ (that is, for all continuous and bounded functions $h : \mathbb{R} \to \mathbb{R}$, $\mathbb{E}[h(\mathcal{V}(F_x))] \to \mathbb{E}[h(\mathcal{V}(F_\mu))]$ uniformly for all $x \in M$). This is the content of Proposition 4.1 below. Thus, in this case, for the asymptotic Gaussianity, there is no need to make use of the extra averaging with respect to $x \in M$ (but this will be required for the rest of the proof). Due to such a simplification, we will present the proof of Proposition 4.1 first in Section 4 so that the probabilistic arguments are easier to describe and can be separated from the more precise microlocal analysis techniques required in the constant energy window case, which we are going to discuss next.

In the case of constant energy windows $\rho \equiv \rho_0$, there are at least two obstacles to the described approach:

(i) Around some 'bad' points $x \in M$, the asymptotic behavior of the covariance kernel of F_x may not coincide with (1.1).

(ii) Around some other 'bad' points, some of the summands in (1.2) could be as large, by order of magnitude, as $v(T)^{1/2}$, occurring in reality, for example, on the sphere. Around these points the Central Limit Theorem is not applicable.

To overcome obstacle (i), the spectral projector operator

$$L^2(M) \to \mathrm{Sp}\{\phi_i\}_{\lambda_i \in [T-\rho, T]}$$

is carefully studied in Section 8. We show that, under the assumptions of Theorem 1.3 on the self-focal points of M, the asymptotics (1.5) hold outside[1] a set of points $x \in M$ of *small* measure. Sogge's

[1]As a by-product of our analysis, it will follow that around every 'good' point in the complement of the 'bad', there is a $1/T$-neighbourhood, where (1.5) is satisfied with no quantitative error term; see Proposition 8.2.

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bound [54] is used to prove that all the summands in (1.2) are of size $o(v(T)^{1/2})$ for x outside of another set of small measure, though crucially depending on T. Since, other than the vanishing measure of the bad sets, no other useful property of the family of bad sets is established (it would be useful if, for example, this family would be monotone decreasing with T growing), the Central Limit Theorem is not applicable with any *fixed* $x \in M$.

Instead, a 'triangular' version of the Central Limit Theorem, allowing for the random variables to depend on a parameter, is applied with x varying with T; as it was explained above, the elegant way to express the outcome of its application as a single consolidated result is by thinking of x random uniform on the good set and, a forteriori, using the asymptotic vanishing of the excised set for x random uniform on M. Hence, the convergence of $F_x(\cdot)$ to the limit monochromatic random field is as a random field w.r.t. the probability measure $\frac{d\,\mathrm{Vol}_g}{\mathrm{Vol}(M)} \otimes d\mathbb{P}$ on $M \times \Omega$ (and the convergence of $F_x(\cdot)$ w.r.t. $d\mathbb{P}$ on Ω is not asserted for any given $x \in M$).

To the best of our knowledge, this aspect of our proofs, inspired by the de-randomization techniques [12, 15], is novel in the context of the study of the geometry of random fields, and different from the rest of the literature on the subject, where the Central Limit Theorem is normally applied for every $x \in M$ fixed. The convergence, in distribution, of the random variables $\mathcal{V}(F_x)$, also w.r.t. the probability measure $\frac{d\,\mathrm{Vol}_g}{\mathrm{Vol}(M)} \otimes d\mathbb{P}$ on $M \times \Omega$, to $\mathcal{V}(F_\mu)$, follows directly from the convergence of the random fields F_x to the limit random field F_μ, via the Continuous Mapping Theorem. This is the content of Proposition 9.1.

2.3. *Anti-concentration*

Since the outcome of Proposition 4.1 and Proposition 9.1 are valid outside a set of small probability (and outside a set of $x \in M$ of small volume), it is essential to demonstrate that the contribution of the exceptional set to (2.1) is negligible. In other words, we need to show that it is unlikely that a large proportion of the nodal set concentrates in a small portion of space, hence the term 'anti-concentration'. This is precisely the purpose of the anti-concentration Proposition 6.1, whose proofs will be now discussed.

The required anti-concentration result is the existence of some function $h : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, so that

$$\frac{h(t)}{t} \to \infty,$$

and that satisfies the estimate

$$\int_M \mathbb{E}[h(\mathcal{V}(F_x))]\, d\,\mathrm{Vol}_g(x) < C \tag{2.2}$$

for some constant $C = C(M, g) > 1$, independent of T.

In Proposition 6.1, we will show that (2.2) holds true with $h(t) = t \cdot \log t$ provided that the energy window satisfies

$$\rho = \rho(T) \geq \begin{cases} T^{\vartheta_n} (\log T)^2 & 2 \leq n \leq 4 \\ \rho_0 & n \geq 5 \end{cases},$$

with ϑ_n given by (1.8).

Following the approach of Donnelly-Fefferman [21], Lin [35] and Jerison-Lebeau [41, Section 14], the nodal volume of $f(\cdot) = f_T(\cdot)$ can be controlled via the *doubling index* (of the 'harmonic lift' of f). The doubling index is defined for any function

$$h : 3B = B(x, 3r) \subseteq M \to \mathbb{R}$$

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as

$$\mathcal{N}_h(x,r) = \mathcal{N}_h(B) = \log \frac{\sup_{2B}|h|}{\sup_B |h|}.$$

In Section 5, we will show, appealing to the analyticity of M, that the nodal volume of f in a ball of radius $r > 0$ can be bounded as

$$\mathcal{V}(f, B_r) \lesssim r^{n-1}\mathcal{N}_{f^H}(\widetilde{B}_{8r}) = r^{n-1}\mathcal{N}(\widetilde{B}_{8r}),$$

where f^H is the harmonic lift of f to the manifold $M \times \mathbb{R}$ (at this stage it is instructive, although slightly imprecise, to think of the harmonic lift as $f^H(x,t) = f(x)\cdot\exp(T\cdot t)$), and $\widetilde{B}_{8r}$ is the 'ball'

$$\widetilde{B}_{8r} = B_{8r} \times [-8r, 8r].$$

The well-known bounds on the growth rate of eigenfunctions, as in [21], give

$$\mathcal{N}_{f^H}((x,0),c) =: \mathcal{N}(x,c) \lesssim T$$

for some constant $c = c(M,g) > 0$. The monotonicity of the doubling index (w.r.t. the radius $r > 0$) for harmonic function [35] implies

$$\mathcal{N}(x, 8/T) \lesssim \mathcal{N}(x,c) \lesssim T.$$

Thus, the statement (2.2) of Proposition 6.1 is equivalent, in essence, to an estimate of the type

$$(\mathrm{Vol}\otimes\mathbb{P})(\{(x,\omega) : \mathcal{V}(F_x) > H\}) \le \frac{1}{H(\log H)^{2+\varepsilon}}, \tag{2.3}$$

for all $1 < H \lesssim T$. The asymptotic estimate (2.1), together with the global bound $\mathcal{V}(f_T) \lesssim T$, give

$$\mathrm{Vol}(x \in M : \mathcal{V}(F_x) > H) \lesssim H^{-1}.$$

Therefore, the aspired bound (2.3), holding with high probability w.r.t. the product space, is a logarithmic gain only over a bound holding deterministically for *every* band-limited function. This is the most delicate and, to our best knowledge, novel aspect of the proof of Proposition 6.1, described immediately below.

As discussed above, a large value of $\mathcal{V}(F_x)$ also implies a large value of the doubling index on $\tilde{B} = B(x, 8/T)\times[-8/T, 8/T]$, which, in turn, may only happen if either the function has a large value on $2\tilde{B}$ or has a small value on $\tilde{B}$, roughly speaking. The former case can be dealt with via the second moment method. However, controlling the small values of f (or rather, of f^H) is more delicate. Quantifying the Gaussian convergence obtained in Section 4, it is possible to control the probability of $f(x)$ being of 'small' depending on the L^3-norm (cubed) of the eigenfunctions. After some computations, this leads to the bound

$$\sup_{x\in M} \mathbb{P}(\mathcal{N}(x, 8/T) > H) \lesssim \exp(-H) + \sup_{\lambda_i \in [T-\rho, T]} ||\phi_i||_{L^3}^3 v(T)^{-1/2}. \tag{2.4}$$

Therefore, in order to obtain (2.3), it would be sufficient to control the second term on the r.h.s. of (2.4). Unfortunately, appealing to Sogge's bound [54] turns out to be not quite sufficient to yield Proposition 6.1. Thus, we will use one last 'trick', and by using the Gaussian convergence at various scales and the monotonicity of the doubling index, we will show that

$$\mathbb{P}(\sup_x \mathcal{N}(x, 3/T) \le T^{1-c}) \ge 1 + O((\log T)^{-1})$$

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for some constant $c = c(n) > 0$. This will reduce the range of H in (2.3), and thus the bound (2.4) will suffice to prove Propostion 6.1 in the appropriate range of ρ specified above. This concludes the sketch of the proofs.

3. Discussion

3.1. *Survey of non-Gaussian literature*

To our best knowledge, the results presented within this manuscript are the first universality results applicable in the asserted vastly general scenario in terms of both the underlying manifold M and the random coefficients $\{a_i\}$. Our approach is based on a blend of microlocal analytic techniques, missing from the existing non-Gaussian literature, and purely probabilistic methods. The closest analogue to Theorem 1.1 (and Theorem 1.3) we are aware of in the existing literature is [2], dealing with 2d random non-Gaussian trigonometric polynomials. These are related to the random band-limited Laplace eigenfunctions on the standard 2d torus corresponding to the long energy window $\rho(T) = T$ (here, the energies ordering is somewhat different to allow for separation of variables). The asymptotics for the expected nodal length was asserted for centred unit variance random variables in perfect harmony to Theorem 1.1 (though with a different leading constant, a by-product of a non-monochromatic scaling limit).

Even though we did not meticulously validate all the details, we believe that their arguments translate verbatim for the 'pure' 2d toral Laplace eigenfunctions

$$g_m(x) = \sum_{\substack{\mu \in \mathbb{Z}^2 \\ \|\mu\|^2 = m}} a_\mu \cdot e(\langle \mu, x \rangle), \tag{3.1}$$

where the a_μ are i.i.d., save for the relation $a_{-\mu} = \overline{a_\mu}$ making g_m real-valued, and the summation on the r.h.s. of (3.1) is w.r.t. to all standard lattice points lying on the radius-$\sqrt{m}$ centred circle. In the Gaussian context, the g_m are usually referred to as 'arithmetic random waves' (ARW), see, for example, [31, 46, 50]; they are the band-limited functions for the standard flat torus corresponding to the 'very short energy window' $\rho(T) \equiv 1$ (in fact, in this case, the energy window width could be made infinitesimal). Other than the result for 2d random trigonometric polynomials, all the literature concerning real zeros of non-Gaussian ensembles is 1-dimensional in essence: real zeros of random algebraic polynomials or Taylor series (see, for example, [29, 30, 45] and the references therein), random trigonometric polynomials on the circle [4], and the restrictions of 2d random toral Laplace eigenfunctions (3.1) to a smooth curve [19].

3.2. *Gaussian vs. non-Gaussian monochromatic functions: cases of study*

Unlike the non-Gaussian state of art concerning the zeros of the band-limited functions, the Gaussian literature is vast and rapidly expanding thanks to the powerful Kac-Rice method tailored to this case at times combined with the Wiener chaos expansion. Here the literature varies from the very precise and detailed results concerning the zero volume distribution (its expectation, variance and limit law) restricted to some particularly important ensembles, such as random spherical harmonics [40, 56] or the arithmetic random waves [31, 39], to somewhat less detailed results, but of far more general nature [18, 59], to almost sure asymptotic result [26] w.r.t. a randomly independently drawn sequence of functions $\{f_T\}_T$.

It is plausible, if not likely, that under a slightly more restrictive assumptions on the random variables, our techniques yield a power saving upper bound for the nodal length variance of the type

$$\mathrm{Var}\left(\frac{f_T}{T}\right) = O\left(T^{-\delta}\right)$$

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for some $\delta > 0$, but certainly not a *precise* asymptotic law for the variance, even for the particular cases of non-Gaussian random spherical harmonics or the non-Gaussian Arithmetic Random Waves. In the Gaussian case, even some important *non-local* properties of the nodal set were addressed: the expected number of nodal components [42, 43], their fluctuations [5, 44], their fine topology and geometry, and their relative position [6, 53].

The aforementioned random ensemble of Gaussian spherical harmonics is the sequence of functions $f_\ell : \mathbb{S}^2 \to \mathbb{R}$, $\ell \geq 1$, where

$$f_\ell(x) = \frac{1}{\sqrt{2\ell+1}} \sum_{m=-\ell}^{\ell} a_{\ell,m} Y_{\ell,m}(x),$$

with $\{Y_{\ell,m}\}_{-\ell \leq m \leq \ell}$ the standard basis of degree-ℓ spherical harmonics and $a_{\ell,m}$ i.i.d. standard Gaussian random variables. An application of the Kac-Rice formula yields [7] the expected nodal length of $f_\ell(\cdot)$ to be given precisely by

$$\mathbb{E}[\mathcal{V}(f_\ell)] = \sqrt{2}\pi \cdot \sqrt{\ell(\ell+1)} \sim \sqrt{2}\pi\ell,$$

whereas a significantly heavier machinery, also appealing to the Kac-Rice method, yields [56] a precise asymptotic law

$$\operatorname{Var}(\mathcal{V}(f_\ell)) \underset{\ell\to\infty}{\sim} \frac{1}{32} \log \ell,$$

smaller than what would have been thought the natural scaling $\approx c \cdot \ell$ would be ('Berry's cancellation phenomenon').

In light of the nonuniversality result of [4], it is *not unlikely* that in the non-Gaussian case (i.e., the $a_{\ell,m}$ are centred unit variance i.i.d. random variable), the variance satisfies the 2-term asymptotics

$$\operatorname{Var}(\mathcal{V}(f_\ell)) = c_1 \cdot \ell + c_2 \cdot \log \ell + O(1),$$

with c_1, c_2 depending on the law of $a_{\ell,m}$ and c_1 vanishing for a peculiar family of distributions, including the Gaussian one. It seems less likely, though *conceivable*, that $c_1 \equiv 0$.

For the 2d Gaussian arithmetic random waves (3.1), it was found that the expected nodal length is given precisely by $\mathbb{E}[\mathcal{V}(g_m)] = \frac{\pi}{\sqrt{2}} \cdot \sqrt{m}$, whereas the variance is asymptotic to

$$\operatorname{Var}(\mathcal{V}(g_m)) \sim 4\pi^2 b_m \cdot \frac{m}{r_2(m)^2},$$

where $r_2(m)$ is the number of summands in (3.1). Here the numbers b_m are genuinely fluctuating in $[1/512, 1/256]$, depending on the angular distribution of the lattice points in the summation on the r.h.s. of (3.1) and the leading term corresponding to $\frac{m}{r_2(m)}$ 'miraculously' cancelling out precisely ('arithmetic Berry's cancellation').

Using the same reasoning as for the spherical harmonics, for the non-Gaussian case (i.e., a_μ are centred unit variance i.i.d. random variables), it is *expected* that the 2-term asymptotics

$$\operatorname{Var}(\mathcal{V}(g_m)) \sim \widetilde{c_1} \frac{m}{r_2(m)} + \widetilde{c_2} \frac{m}{r_2(m)^2}$$

hold with $\widetilde{c_1}, \widetilde{c_2}$ possibly depending on both the law of a_μ and the angular distribution of the lattice points $\{\mu\}$ in (3.1), with $\widetilde{c_1}$ vanishing for a_μ a peculiar class of distribution laws, including the Gaussian (whence $\widetilde{c_1}$ vanishes independent of the angular distribution of the lattice points $\{\mu\}$). The dependence of $\widetilde{c_1}$ and $\widetilde{c_2}$ on both the distribution law of a_μ and the angular distribution of $\{\mu\}$ is of interest – in particular, whether the vanishing of $\widetilde{c_1}$ depends on the angular distribution of $\{\mu\}$ at all (which is not the case if a_μ is Gaussian). Again, it is *conceivable* that $\widetilde{c_1} \equiv 0$. We leave all of the above questions to be addressed elsewhere.

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4. Asymptotics Gaussianity

The aim of this section is to show that nodal length of f_T, as in (1.2), has a universal limit law in balls of radius $\asymp T^{-1}$. In order to state this result precisely, we need to introduce some notation that will be used throughout the rest of the article.

4.1. Notation and goal of Section 4

First, we will define the rescaled version of f_T, as in (1.2), in geodesic balls of radius T^{-1}. Let $x \in M$ and let F_x be f_T rescaled to the ball $B_g(x, 1/T)$ in normal coordinates. More precisely, we define:

$$F_{T,x}(y) = F_x(y) = f(\exp_x(y/T)) \tag{4.1}$$

for $y \in B(0,1) =: B_0 \subseteq \mathbb{R}^n$, where $\exp_x : \mathbb{R}^n \cong T_xM \to M$ is the exponential map. Notice that, in the definition of the exponential map, we have tacitly identified $\mathbb{R}^n$ with T_xM via an Euclidean isometry. Moreover, we observe that since (M, g) is analytic, the injectivity radius of M is uniformly bounded from below [20]; thus, from now on, we assume that $1/T$ is smaller than the injectivity radius so that the exponential map is a diffeomorphism. Furthermore, thanks to [43, Section 8.1.2] due to Nazarov and Sodin (see also [49, Section 2]), the map

$$(x, \omega) \in M \times \Omega \to F_x(\omega, \cdot) \in C^\infty(B_0)$$

is measurable.

We now define the universal scaling limit for the nodal length of F_x. We denote F_μ to be the monochromatic isotropic Gaussian field on $B_0 \subseteq \mathbb{R}^n$ with spectral measure μ, the (normalised) Lebesgue measure on the $n-1$ dimensional sphere $\mathbb{S}^{n-1}$. Equivalently, F_μ has the covariance function

$$\mathbb{E}[F_\mu(y) \cdot F_\mu(y')] = \int_{|\xi|=1} \exp(i\langle y - y', \xi\rangle) d\mu(\xi) = (2\pi)^\Lambda \frac{J_\Lambda(|y - y'|)}{|y - y'|^\Lambda}, \tag{4.2}$$

with $\Lambda = (n-2)/2$ and where $J_\Lambda(\cdot)$ is the usual Bessel J function of order Λ. In what follows we will use the shorthands

$$\mathcal{V}(F_x) := \mathcal{V}\left(F_x, \frac{1}{2}B_0\right) \qquad \text{and} \qquad \mathcal{V}(F_\mu) := \mathcal{V}\left(F_\mu, \frac{1}{2}B_0\right).$$

The aim of this section is to prove the following proposition:

Proposition 4.1. *Let F_x be as in (4.1), F_μ be as above. Then, under the assumptions of Theorem 1.1 on the energy window width $\rho = \rho(T)$, uniformly for all $x \in M$, we have*

$$\mathcal{V}(F_x) \xrightarrow{d} \mathcal{V}(F_\mu) \qquad T \to \infty$$

convergence in distribution.

Observe that in Proposition 4.1, the convergence to the Gaussian random field is claimed for *fixed* $x \in M$, stronger than the average statement w.r.t. $x \in M$, required for the proof of Theorem 1.1 (cf. Proposition 9.1 that is used for the proof of Theorem 1.3). This is where the growing energy window assumption of Theorem 1.1 is also used. In particular, Proposition 4.1 implies that

$$\mathcal{V}(F_x) \xrightarrow{d} \mathcal{V}(F_\mu) \qquad T \to \infty \tag{4.3}$$

converges in distribution as a random variable on $(M \times \Omega, d\sigma)$, where

$$d\sigma = (\mathrm{Vol}_g(M))^{-1} d\,\mathrm{Vol}_g \otimes d\mathbb{P}$$

(cf. Proposition 9.1).

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Remark 4.2. The proof of Proposition 4.1 holds verbatim under the much weaker assumption that the energy window $\rho(T) \to \infty$ as $T \to \infty$. The full strength of the assumptions of Theorem 1.1 will be needed only in Section 6 below. Nevertheless, we prefer to state the assumptions of Proposition 4.1 in the precise form it will be used.

4.2. Hörmander's local Weyl's law

In order to study F_x as in (4.1), we will need the well-known local Weyl's law of Hörmander [28, Theorem 4.4], which we do not present in its full generality but in a form convenient for our purposes. In particular, there are no restrictions on the width of the spectral windows ρ in the following result (for ρ too small, the error term dominates):

Proposition 4.3. *Let (M, g) be a compact, real analytic manifold with empty boundary. Let $x \in M$ and consider a (sufficiently small) coordinate patch Ω_x around x in normal coordinates. Then*

$$\sup_{y,y'\in\Omega_x}\left|\sum_{\lambda_i\in[T-\rho,T]}\phi_i(y)\phi_i(y') - c_M T^n \mathcal{J}_{\Upsilon(T)}(Td_g(y,y'))\right| = O_{M,g}(T^{n-1}),$$

where $d_g(y, y')$ is the geodesic distance between y, y', $c_M > 0$ is given in (1.3), $\Upsilon(T) = 1 - \frac{\rho}{T}$ and

$$\mathcal{J}_{\Upsilon(T)}(w) = \int_{\Upsilon(T)\le|\xi|\le 1} \exp(i\langle w, \xi\rangle)d\xi. \tag{4.4}$$

Moreover, we can also differentiate both sides of (8.1) an arbitrary finite number of times; that is,

$$\sup_{y,y'\in\Omega_x}\frac{\left|\sum_{\lambda_i\in[T-\rho,T]} D_y^\alpha\phi_i(y)D_{y'}^{\alpha'}\phi_i(y') - \frac{c_M T^n D_y^\alpha D_{y'}^{\alpha'}\mathcal{J}_{\Upsilon(T)}(Td_g(y,y'))}{(2\pi)^n}\right|}{T^{|\alpha|+|\alpha'|}} = O_{M,g,\alpha}(T^{n-1}),$$

where α, α' are multi-indices and $\xi^\alpha = (\xi_1^{\alpha_1}, \ldots, \xi_n^{\alpha_n})$ and the derivatives are understood after taking normal coordinates around the point x.

The following corollary will be quite useful later:

Corollary 4.4. *Let (M, g) be a compact, real analytic manifold with empty boundary. There exists some $\rho_0 = \rho_0(M, g)$ such that for all $\rho \ge \rho_0$, the following holds. Let $v(T)$ be as in (1.3). Then*

$$\sum_{\lambda_i\in[T-\rho,T]}|\phi_i(x)|^2 \asymp v(T),$$

where $A \asymp B$ means that there exist two constant $0 < c < C$, depending only on (M, g), such that $cA \le B \le CA$.

Although ρ_0 does not appear in the proof of Proposition 4.1 (and Theorem 1.1), the value of ρ_0 is fixed from now until the end of the article, for the results building up to Theorem 1.3. As a direct consequence of Proposition 4.3 and a straightforward calculation, we also have the following result:

Lemma 4.5. *Let (M, g) be a real analytic compact manifold with empty boundary of dimension n, let $f_T(\cdot)$ be as in (1.2), let $\rho(T)$ be the width of the energy window and let c_M and $v(T)$ be as in (1.3). Then, under the assumption of Theorem 1.1 on the width of the spectral window $\rho(T)$, we have*

$$\mathbb{E}\left[|f_T(x)|^2\right] = \frac{1}{v(T)}\sum_{\lambda_i\in[T-\rho,T]}|\phi_i(x)|^2 = (1 + o(1)),$$

where the error term is uniform for all $x \in M$. Moreover, for F_x as in (4.1), we have

$$\sup_{\substack{x\in M\\ y,y'\in B_0}} \left| \mathbb{E}[F_x(y)\cdot F_x(y')] - (2\pi)^{\Lambda}\frac{J_\Lambda(|y-y'|)}{|y-y'|^{\Lambda}} \right| \to 0T \to \infty,$$

with $\Lambda = (n-2)/2$ and $J_\Lambda(\cdot)$ the Λ-th Bessel function. Further, one can differentiate both sides an arbitrary finite number of times; that is,

$$\mathbb{E}[D^{\alpha}F_x(y)\cdot D^{\alpha'}F_x(y')] = (-1)^{|\alpha'|}i^{|\alpha|+|\alpha'|}\int_{|\xi|=1}\xi^{\alpha+\alpha'}\exp(i\langle y-y',\xi\rangle)d\mu(\xi) + o_{T\to\infty}(1),$$

valid uniformly for all $x \in M,\ y, y' \in B_0$, where α, α' are fixed multi-indices and $\xi^{\alpha} = (\xi_1^{\alpha_1}, \ldots, \xi_n^{\alpha_n})$.

Proof. By the first claim in Proposition 4.3 and the compactness of M, we have

$$\sum_{\lambda_i\in[T-\rho,T]}|\phi_i(x)|^2 = c_M\rho(T)T^{n-1} + O(T^{n-1}),$$

where the error term is uniform for all $x \in M$. Thus, the first claim in Lemma 4.5 follows by dividing both sides by $v(T)$. In order to see the second claim in Lemma 4.5, let us take $y, y' \in B_0$ with $y \neq y'$ and let us rewrite the integral in (4.4) in the spherical coordinates and use the identity

$$\int_{S^{n-1}}\exp(i\langle u,\xi\rangle)d\mu(\xi) = (2\pi)^{\Lambda}\frac{J_\Lambda(|u|)}{|u|^{\Lambda}},$$

to obtain

$$\begin{aligned}\sum_{\lambda_i\in[T-\rho,T]}\phi_i(y')\phi_i(y) &= c_M\rho(T)T^{n-1}\int_{|\xi|=1}\exp(i\langle Td_g(y',y),\xi\rangle)d\mu(\xi)\\ &\quad + O\Big(\rho(T)T^{n-1}d_g(y',y)\Big) + O(T^{n-1})\\ &= c_M\rho T^{n-1}(2\pi)^{\Lambda}\frac{J_\Lambda(|Td_g(y',y)|)}{|Td_g(y',y)|^{\Lambda}}\\ &\quad + O\Big(\rho(T)T^{n-2}\Big) + O(T^{n-1}),\end{aligned} \tag{4.5}$$

where $\Lambda = (n-2)/2$. Thus, the second claim in Lemma 4.5 follows by dividing both sides of (4.5) by $v(T)$ and compactness of M. The third claim in Lemma 4.5 follows by the second claim in Proposition 4.3 and similar computation to (4.5) (and again the compactness of M). □

4.3. Convergence of finite-dimensional distributions

In this section, we state and prove the following lemma about the convergence of finite-dimensional distributions of F_x to the finite-dimensional distributions of F_μ.

Lemma 4.6 (Convergence of finite-dimensional distributions). *Let m be some positive integer, $B_0 = B(0,1)$, let F_x be as in (4.1) and let F_μ be the random monochromatic wave as in (4.2). Then, under the assumptions of Theorem 1.1 on $\rho(T)$, for every $y_1, \ldots y_m \in B_0 \subseteq \mathbb{R}^n$, we have*

$$(F_x(y_1),\ldots,F_x(y_m)) \xrightarrow{d} (F_\mu(y_1),\ldots,F_\mu(y_m)) \qquad T\to\infty,$$

where the convergence is in distribution uniformly for all $x \in M$. Moreover, for every $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $|\alpha| \le 2$, one has

$$(D^\alpha F_x(y_1), \ldots, D^\alpha F_x(y_m)) \xrightarrow{d} (D^\alpha F_\mu(y_1), \ldots, D^\alpha F_\mu(y_m)) \qquad T \to \infty.$$

In order to prove Lemma 4.6, we will need a simple (not sharp) bound on the maximum value of an eigenfunction in terms of its eigenvalue.

Claim 4.7. Let (M, g) be a smooth compact Riemannian manifold of dimension $n \ge 2$ and let ϕ_λ be a solution to the eigenvalue problem

$$\Delta_g \phi_\lambda + \lambda^2 \phi_\lambda = 0.$$

Then, we have

$$\sup_{x \in M} |\phi_\lambda|^2 \lesssim \lambda^{n-1} \log \lambda$$

and

$$\sup_{x \in M} \lambda^{-2\alpha} |D^\alpha \phi_\lambda|^2 \lesssim \lambda^{n-1} \log \lambda$$

for all multi indices $|\alpha| \le 2$.

Proof. Observe that, by the first part of Lemma 4.5, we have

$$\sup_{x \in M} |\phi_\lambda|^2 \le \sum_{\lambda_j \in [\lambda - \log \lambda, \lambda]} |\phi_j|^2 = c_M (\log \lambda) \lambda^{n-1} + O(\lambda^{n-1})$$

and the bound on the supremum of ϕ_λ follows. The bound on the derivatives can be obtained similarly using the second part of Lemma 4.5. □

We also recall, for the convenience of the reader, the following multidimensional version of Lindeberg-CLT (see, for example, [24, Proposition 6.2] and [10, Theorem 27.2]):

Lemma 4.8 (CLT). *Let $d > 0$ be a positive integer and let $\{V_{n,k}\}_{n,k}$ be a triangular array of $\mathbb{R}^d$-valued random variables, so that the random vectors lying on each of its rows are independent and of zero mean. That is, for any n, k, $V_{n,k} = (V^i_{n,k})^d_{i=1}$ is a d-dimensional random vector with zero mean, and for every n fixed and every $k_1 \ne k_2$, the vectors V_{n,k_1} and V_{n,k_2} are independent. The random variables $V^i_{n,k}$ are normalized by setting*

$$(s^i_n)^2 = \sum_k \mathbb{E}[(V^i_{n,k})^2]$$

and

$$\tilde{V}^i_{n,k} = (s^i_n)^{-1} V^i_{n,k}.$$

We make the following two assumptions:

(1) *The covariance matrices*

$$(\Sigma_{n,k}))_{ij} = \mathbb{E}[\tilde{V}^i_{n,k} \tilde{V}^j_{n,k}]$$

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of the k-th vector of $\{\tilde{V}_{n,k}\}_{n,k}$ *satisfy*

$$\lim_{n\to\infty}\sum_k \Sigma_{n,k} = \Sigma_0$$

for some positive definite $d \times d$*-positive matrix.*

(2) *One has*

$$\max_{i=1,\dots,d}\frac{1}{(s_n^i)^2}\sum_k \mathbb{E}\Big[(\tilde{V}_{n,k}^i)^2 \mathbb{1}_{\tilde{V}_{n,k}^i > \varepsilon s_n^i}\Big] \to 0, \qquad n\to\infty$$

for any positive $\varepsilon > 0$*, where* $\mathbb{1}$ *is the indicator function.*

Then we have

$$W_n := \sum_k \tilde{V}_{n,k} \xrightarrow{d} N(0,\Sigma_0) \qquad n\to\infty,$$

where the convergence is in distribution, and the rate of convergence depends on the rates of convergence in (1) and (2) only. That is, for every $h:\mathbb{R}^d\to\mathbb{R}$ *bounded continuous,*

$$\mathbb{E}[h(W_n)] \to \mathbb{E}[h(Z)],$$

where $Z\sim N(0,\Sigma_0)$*, with the rate of convergence depending on h and the rate of convergence in (1) and (2).*

We are now ready to prove Lemma 4.6.

Proof of Lemma 4.6. First, we need a piece of notation that we will use throughout the proof. Let $\phi_{i,x}$ be the scaled restriction of ϕ_i to $B_g(x,4/T)$ via the exponential map; that is,

$$\phi_{i,x}(y) = \phi_i(\exp_x(y/T)),$$

for $y\in B(0,4)$ (here we tacitly assume that T is sufficiently large so that $4/T$ is less than the injectivity radius). Before embarking on the proof of Lemma 4.6, we also observe that by Claim 4.7, we have

$$\max_{\lambda_i\in[T-\rho,T]}\sup_{x\in M}\sup_{B_0}|\phi_i|^2 \lesssim T^{n-1}\log T. \tag{4.6}$$

Similarly, given a multi-index $|\alpha|\le 2$, we also have

$$\max_{\lambda_i\in[T-\rho,T]}\sup_{x\in M}\sup_{B_0}|D^\alpha\phi_{i,x}|^2 \lesssim T^{n-1}\log T. \tag{4.7}$$

We are going to first consider the distribution of the vector $(F_x(y_1),\dots,F_x(y_m))$ for $x\in M$. Thanks to Lemma 4.5, we have

$$\sup_{\substack{i,j\in\{1,\dots,m\}\\ x\in M}}\Big|\mathbb{E}[F_x(y_i)\cdot F_x(y_j)] - \mathbb{E}[F_\mu(y_i)\cdot F_\mu(y_j)]\Big| \to 0 \qquad T\to\infty. \tag{4.8}$$

Therefore, by the multidimensional version of Lindeberg's Central Limit Theorem (Lemma 4.8), and upon using (4.8), it suffices to prove that for every $\varepsilon>0$, we have

$$\sup_{\substack{y\in B_0\\ x\in M}}\frac{1}{v(T)}\sum_{\lambda_i}\mathbb{E}[|a_i\phi_{i,x}(y)|^2\mathbb{1}_{|a_i\phi_{i,x}(y)|>\varepsilon v(T)^{1/2}}] \to 0 \qquad T\to\infty, \tag{4.9}$$

where $\mathbb{1}$ is the indicator function and $v(T) = c_M\rho T^{n-1}(1+o(1))$.

Now we prove (4.9). Thanks to Lemma 4.5, we have

$$\frac{1}{v(T)}\sum_{\lambda_i}\mathbb{E}[|a_i\phi_{i,x}(y)|^2\mathbb{1}_{|a_i\phi_{i,x}(y)|>\varepsilon v(T)^{1/2}}] = \frac{1}{v(T)}\sum_{\lambda_i}|\phi_{i,x}(y)|^2\mathbb{E}[|a_i|^2\mathbb{1}_{|a_i\phi_{i,x}(y)|>\varepsilon v(T)^{1/2}}]$$
$$\lesssim \sup_{\lambda_i\in[T-\rho,T]}\mathbb{E}[|a_i|^2\mathbb{1}_{|a_i\phi_{i,x}(y)|>\varepsilon v(T)^{1/2}}].$$

Therefore, to prove (4.9), it is sufficient to show that

$$\sup\mathbb{E}[|a_i|^2\mathbb{1}_{|a_i\phi_{i,x}(y)|>\varepsilon v(T)^{1/2}}]\to 0 \qquad T\to\infty, \tag{4.10}$$

where the supremum is over all $\lambda_i\in[T-\rho,T]$, all $y\in B_0$ and all $x\in M$. Thanks to (4.6) and the fact that $v(T)\gtrsim T^{n-1}(\log T)^2$, we have

$$\mathbb{1}_{|a_i\phi_{i,x}(y)|>\varepsilon v(T)^{1/2}}\leq \mathbb{1}_{|a_i|\gtrsim\varepsilon\log T}.$$

Hence, since the a_i are i.i.d. with common distribution a_0 (say), we have

$$\lim_{T\to\infty}\sup\mathbb{E}[|a_i|^2\mathbb{1}_{|a_i\phi_{i,x}(y)|>\varepsilon v(T)^{1/2}}]\leq \lim_{M\to\infty}\lim_{T\to\infty}\int_{\varepsilon\log T}^{M}t^2 d\mathbb{P}(|a_0|>t)=0,$$

where we used Fubini and $\mathbb{E}[|a_0|^2]=1$ to switch the order of the limits. This concludes the proof of (4.10) and thus of (4.9).

In order to prove the convergence of the derivative vector and upon recalling the second part of Proposition 4.3, again by the multidimensional version of Lindeberg's Central Limit Theorem, it is sufficient to prove that for any $\varepsilon>0$ and $|\alpha|\leq 2$, we have

$$\sup\frac{1}{v(T)}\sum_{\lambda_i}\mathbb{E}[|a_iD^\alpha\phi_{i,x}(y)|^2\mathbb{1}_{|a_iD^\alpha\phi_{i,x}(y)|>\varepsilon v(T)^{1/2}}]\to 0 \qquad T\to\infty. \tag{4.11}$$

Similar to the above argument, (4.7) implies (4.11) if $|\alpha|\leq 2$, thus concluding the proof of Lemma 4.6. □

4.4. Tightness

The aim of this section is to show that Lemma 4.6 implies that F_x converges as a random function to F_μ. To formally state the results of this section, let us first introduce some notation. Let $V=\overline{B_0}$ and let ν_T be the sequence of probability measures on $C^2(V)$ induced by the pushforward measure of F_x (recall that, since the law of f_T is locally constant, we may assume that T varies along a sequence); that is, for an open set $H\subseteq C^2(V)$, we set

$$\nu_T(H):=(F_x)_*\mathbb{P}(H)=\mathbb{P}(F_x(\omega,\cdot)\in H). \tag{4.12}$$

Lemma 4.6 says that there exists a subsequence T_k such that ν_{T_k} converges to ν_∞, the pushforward of F_μ onto $C^2(V)$. Thus, to obtain the convergence of the whole sequence, it is enough to show that the sequence ν_T is tight.

A sequence of probability measures $\{\nu_k\}_{k=0}^\infty$ on some topological space X is *tight* if for every $\epsilon>0$, there exists a compact set $K=K(\epsilon)\subseteq X$ such that

$$\nu_k(X\backslash K)\leq\epsilon,$$

uniformly for all $k\geq 0$. We will need the following lemma, borrowed from [47, Lemma 1] (see also [11, Chapter 6 and 7]), which characterises the tightness in the space of continuously twice differentiable functions:

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Lemma 4.9 (Tightness). *Let V be a compact subset of $\mathbb{R}^n$ and $\{\nu_k\}$ a sequence of probability measures on the space $C^2(V)$ of continuously twice differentiable functions on V. Then $\{\nu_k\}$ is tight if the following conditions hold:*

(1) *There exists some $y \in V$ such that for every $\varepsilon > 0$, there exists $K > 0$ with*

$$\max_{|\alpha|\leq 2} \quad \nu_k(g \in C^2(V) : |D^\alpha g(y)| > K) \leq \varepsilon$$

for all $k \geq 0$.

(2) *For every $|\alpha| \leq 2$ and $\varepsilon > 0$, we have*

$$\lim_{\delta\to 0} \limsup_{k\to\infty} \nu_k \left(g \in C^2(V) : \sup_{|y-y'|\leq\delta} |D^\alpha g(y) - D^\alpha g(y')| > \varepsilon \right) = 0.$$

Lemma 4.10. *Let $V = \overline{B_0}$ and let ν_T be as in (4.12). Then the sequence ν_T is tight.*

Proof. For condition (1) of Lemma 4.9, we observe that Lemma 4.5 implies the bound

$$\mathbb{E}\left[|D^\alpha F_x(0)|^2\right] \lesssim 1$$

for $|\alpha| \leq 2$ and uniformly for all $x \in M$. Thus, Chebyshev's inequality yields

$$\mathbb{P}(|D^\alpha F_x(0)| > K) \lesssim K^{-2},$$

and condition (1) follows by taking $K = \epsilon^{-1/2}$.

To check condition (2) of Lemma 4.9, we note that since F_x is almost surely analytic, we have

$$\sup_{|y-y'|\leq\delta} |D^\alpha F_x(y) - D^\alpha F_x(y')| \lesssim \sup_{B_0} |\nabla D^\alpha F_x| \delta. \tag{4.13}$$

Therefore, it is sufficient to prove the following claim:

$$\mathbb{P}(\sup_{B_0} |\nabla D^\alpha F_x| > K) \lesssim K^{-2} \tag{4.14}$$

uniformly for all $x \in M$. Indeed, as above, (4.14) together with (4.13) imply condition (2) by choosing $K = \epsilon\delta^{-1}$.

We are now going to prove (4.14). By Sobolev embedding, there exists some $t = t(n) > 1$, sufficently large depending on n only, so that $C^3(B_0)$ embeds in $H^t(B_0)$, the Sobolev space. Thus, using Lemma 4.5, uniformly for all $x \in M$, we have

$$\mathbb{E}\left[||F_x||^2_{C^3(B_0)}\right] \lesssim_n \mathbb{E}\left[||F_x||^2_{H^t(B_0)}\right] \lesssim 1, \tag{4.15}$$

where the constant implied in the '$\lesssim$'-notation is independent of T (and $x \in M$). Now, inequality (4.15) together with Chebyshev's inequality implies (4.14), and this concludes the proof of Lemma 4.10. □

As mentioned above, combining Lemma 4.6 and Lemma 4.9, we proved the following lemma; see, for example, [11, Theorem 7.1]:

Lemma 4.11. *Let $V = \overline{B_0}$, ν_T be as in (4.12) and let ν_∞ be the pushforward of F_μ on $C^2(V)$, where F_μ is as in (4.2). Then, under the assumptions of Theorem 1.1, ν_T weak* converges to ν_∞ in the space of probability measures on $C^2(V)$.*

4.5. Concluding the proof of Proposition 4.1

To conclude the proof of Proposition 4.1, we just need the following Lemma (see, for example, [48, Lemma 6.2]), which shows that $\mathcal{V}(\cdot)$ – that is, the nodal volume – is a continuous map on the appropriate space of functions:

Lemma 4.12. *Let $B \subseteq \mathbb{R}^n$ be a ball and define the (open) set*

$$C^2_*(2B) = \{h \in C^2(2B) : |h| + |\nabla h| > 0\}.$$

Then $\mathcal{V}(\cdot, B)$ is a continuous functional on $C^2_*(2B)$.

We are now in the position to prove Proposition 4.1.

Proof of Proposition 4.1. An application of Bulinskaya's lemma (see, for example, [43, Lemma 6]) on F_μ restricted to $V = \overline{B_0}$ yields that $F_\mu \in C^2_*(V)$ almost surely. Therefore, Lemma 4.11 and the Continuous Mapping Theorem [11, Theorem 2.7] imply

$$\mathcal{V}(F_x) \xrightarrow{d} \mathcal{V}(F_\mu) \quad T \to \infty,$$

as required. □

5. Nodal volume and the doubling index

Having shown convergence in distribution of the random variable $\mathcal{V}(F_x)$ in Proposition 4.1, we wish to pass to the convergence of expectations. In order to do this, we will need to show that the random variable $\mathcal{V}(F_x)$ is uniformly integrable. Unfortunately, we will not be able to achieve this for *fixed* $x \in M$, averaging with respect to $\omega \in \Omega$. However, we will be able to show (Proposition 6.1 below) that $\mathcal{V}(F_x)$ is uniformly integrable as a random variable defined on $M \times \Omega$ – that is, averaging with respect to *both* $x \in M$ and $\omega \in \Omega$. This will be enough for our purposes as Proposition 4.1 directly implies that $\mathcal{V}(F_x)$ has a universal limit as a random variable defined on $M \times \Omega$.

In this section, we collect some results that will allow us to control the nodal volume of f_T as in (1.2) in terms of the *doubling index* of the *harmonic lift* of f_T, defined below. In doing so, we follow the work of Jerison and Lebeau [41, Section 14] and Lin [35]; see also Kukavica [32, 33, 34] for a different approach. For the sake of the reader's convenience, most of the proofs are reproduced here. However, the proof of the Cauchy uniqueness result (Lemma 5.7 below) is beyond the scope of this article, and we refer the reader directly to [41, Section 14].

5.1. Bounding the nodal volume of sums of eigenfunctions

For a start, we introduce a few notions. Following [21] and [37, 36], the *doubling index* of a function $h : M \to \mathbb{R}$ on a ball $B = B_g(x, r) \subseteq M$ is defined as

$$\mathcal{N}_h(B) = \mathcal{N}(x, r) := \log \frac{\sup_{2B} |h|}{\sup_B |h|}. \tag{5.1}$$

The *harmonic lift* of f_T in (1.2) is defined [41, Page 231] (see also [35, Section 4]) as the unique solution $f^H : M \times \mathbb{R} \to \mathbb{R}$ of

$$(\Delta + \partial_t^2) f^H(x, t) = 0 \qquad f^H(x, 0) = 0 \qquad \partial_t f^H(x, 0) = f_T. \tag{5.2}$$

One may express f^H explicitly as

$$f^H(x, t) = v^{-1/2}(T) \sum_{\lambda_i \in [T-\rho, T]} a_i \frac{\sinh(\lambda_i t)}{\lambda_i} \phi_i(x), \tag{5.3}$$

where a_i and ϕ_i are as in (1.2) and $v(T)$ is as in (1.3). We also introduce the following piece of notation that we will use throughout this section:

$$\tilde{B}(x,r) := B_g(x,r) \times [-r,r] \subseteq M \times \mathbb{R}$$

will stand for the 'ball' of radius $r > 0$ centred at a point $x \in M \cong M \times \{0\}$, and the doubling index of f^H on $\tilde{B}$ is defined via (5.1) as above, with $\tilde{B}$ in place of B. Finally, we recall that for any $s > 0$,

$$s\tilde{B} := B_g(x, sr) \times [-sr, sr]$$

is the radius-sr ball centred at the same point as B.

The aim of this section is to prove the following result:

Proposition 5.1. *Let f_T and f^H be as in (1.2) and (5.3), respectively. Then there exists some $\eta = \eta(M,g) > 0$ with the following property: for every ball*

$$\tilde{B}_r := B_g(x,r) \times [-r,r] \subseteq M \times \mathbb{R}$$

centred at a point $x \in M \cong M \times \{0\}$ of radius $0 < r < \eta/10$, we have

$$\mathcal{V}\big(f_T, \tilde{B}_{r/2} \cap M\big) \cdot r^{-n+1} \lesssim \mathcal{N}(f^H, \tilde{B}_{8r}),$$

where the constant implied in the $\lesssim$ notation depends only on (M,g).

Before embarking on the proof of Proposition 5.1, we will recall some standard properties of the doubling index, which will be used throughout the rest of the paper.

5.2. *Monotonicity of the doubling index and a few consequences*

The fundamental property of the doubling index of an harmonic function, shown in [25], is that $\mathcal{N}(\cdot)$ is an almost monotonic function of the radial variable in the sense that

$$\mathcal{N}(\cdot, r_1) - C \le (1+\varepsilon) \cdot \mathcal{N}(\cdot, r_2) + C$$

for $r_2 \ge 2r_1$ and some $C = C(M,g) \ge 1$. Formally, we have the following (see [36, Lemma 1.3]):

Lemma 5.2. *Let $(\tilde{M}, g)$ be a smooth manifold. For any $0 < \varepsilon < 1$ and any point $O \in \tilde{M}$, there exists some $C = C(\tilde{M}, g, O, \varepsilon) > 0$ and $r_0 = r_0(\tilde{M}, g, O, \varepsilon) > 0$ such that*

$$t^{\mathcal{N}(x,r)(1-\varepsilon)-C} \le \frac{\sup_{B_g(x,tr)} |u|}{\sup_{B_g(x,r)} |u|} \le t^{\mathcal{N}(x,tr)(1+\varepsilon)+C}$$

uniformly for all harmonic functions $u : \tilde{M} \to \mathbb{R}$, for all $x \in \tilde{M}$ and numbers $r > 0$, $t > 2$ satisfying $B_g(x,tr) \subseteq B(O, r_0)$.

We apply Lemma 5.2 in the following convenient settings. Fix $\varepsilon = 1/2$ and $\tilde{M} = M \times [-10, 10]$ in Lemma 5.2, covering $M \times [-10, 10]$ by balls of radius r_0. Upon using the compactness of $\tilde{M}$, we obtain the following:

Corollary 5.3. *Let $f^H : M \times [-10, 10] \to \mathbb{R}$ be as in (5.2). There exists some $C = C(M,g) > 0$, independent of f^H, such that*

$$t^{\mathcal{N}(x,r)/2-C} \le \frac{\sup\limits_{\tilde{B}(x,tr)} |f^H|}{\sup\limits_{\tilde{B}(x,r)} |f^H|} \le t^{2\mathcal{N}(x,tr)+C}$$

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for all $x \in M \times [-10, 10]$ *and numbers* $r > 0$, $t > 2$ *satisfying*

$$B_g(x, tr) \times [-tr, tr] \subseteq M \times [-10, 10].$$

We conclude this section with a useful consequence of the monotonicity formula for the doubling index.

Lemma 5.4. *Let* $f^H : M \times [-10, 10] \to \mathbb{R}$ *be as in (5.2). There exist constants* $C_1, C_2 \geq 1$ *depending only on* M, g, *such that*

$$\sup_{d_g(x,y) \leq r/8} \mathcal{N}(y, r/4) \leq C_1 \cdot \mathcal{N}(x, r) + C_2,$$

where $y \in M \cong M \times \{0\}$, *uniformly for all* $x \in M \times [-10, 10]$ *with*

$$\tilde{B}(x, 2r) \subseteq M \times [-10, 10].$$

Proof. Since, in the relevant range, $d_g(x, y) \leq r/8$, and by Corollary 5.3 applied with $t = 8$ (say), we have

$$\sup_{\tilde{B}(y,r/2)} |f^H| \leq \sup_{\tilde{B}(x,r)} |f^H| \leq 8^{2\mathcal{N}(x,r)+C} \sup_{\tilde{B}(x,r/8)} |f^H| \leq \exp(C_1 \mathcal{N}(x, r) + C_2) \cdot \sup_{\tilde{B}(y,r/4)} |f^H|,$$

as required. □

5.3. Complexification of f

Since (M, g) is real analytic and compact, by the Bruhat-Whitney Theorem [55], there exists a complex manifold $M^{\mathbb{C}}$ where M embeds as a totally real manifold. Moreover, it is possible to analytically continue any Laplace eigenfunction ϕ_i to a holomorphic function $\phi_i^{\mathbb{C}}$ defined on a maximal uniform *Grauert tube*; that is, there exists some $\eta_0 = \eta_0(M, g) > 0$ such that $\phi_i^{\mathbb{C}}$ is an holomorphic function on

$$M_{\eta_0}^{\mathbb{C}} := \{\zeta \in M^{\mathbb{C}} : \sqrt{\gamma}(\zeta) < \eta\},$$

where $\sqrt{\gamma}(\cdot)$ is the Grauert tube function; see [60, Chapter 14] for details. For notational brevity, and in light of the fact that the precise value of η_0 will be unimportant, from now on we write $M^{\mathbb{C}}$ in place of $M_{\eta_0}^{\mathbb{C}}$ and let $f^{\mathbb{C}}$, defined on $M^{\mathbb{C}}$, be the complexification of f. The nodal volume of f can be controlled via the order of growth of $f^{\mathbb{C}}$ using the following classical fact, borrowed from [41, Theorem 14.7] and [21, Proposition 6.7]:

Lemma 5.5. *Let* $B^{\mathbb{C}} \subseteq \mathbb{C}^n$ *be a ball of radius* 1 *and let* H *be a holomorphic function on* $3B^{\mathbb{C}}$. *If, for some* $N > 1$,

$$|H|_{L^\infty(2B^{\mathbb{C}})} \leq e^N \cdot |H|_{L^\infty(B^{\mathbb{C}} \cap \mathbb{R}^n)},$$

then

$$\mathcal{H}^{n-1}\left(\{H = 0\} \cap \frac{1}{2} B^{\mathbb{C}} \cap \mathbb{R}^n\right) \lesssim_n N.$$

5.4. Growth of f^H and $f^{\mathbb{C}}$

In this section, we wish to quantify the growth of $f^{\mathbb{C}}$ in terms of the doubling index of f^H. Our proof will proceed by using f^H to control the derivatives of f so that we can bound the growth of $f^{\mathbb{C}}$ by bounding each term in its power series. Unfortunately, this approach requires introducing an extra (small) constant

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$c = c(M, g) > 0$ in the next result in order to control the radius of convergence of the power series. We will then get rid of this extra technicality in the proof of Proposition 5.1 via a covering argument. All in all, the aim of this section is to prove the following result:

Lemma 5.6. *There exist some (small) numbers $\eta_1 = \eta_1(M, g) > 0$ and $c = c(M, g) > 0$ such that the following holds. Let f be as in (1.2), f^H be as in (5.2) and $f^{\mathbb{C}}$ be the complexification of f. Moreover, let $\tilde{B} \subseteq M \times \mathbb{R}$ be a ball centred at a point lying on $M \cong M \times \{0\}$ of radius less than $\eta_1/10$. Suppose that, for some (large) $N > 1$, one has*

$$||f^H||_{L^\infty(\tilde{B})} \leq e^N \cdot ||f^H||_{L^\infty(\frac{c}{2}\tilde{B})}. \tag{5.4}$$

Then we have

$$||f^{\mathbb{C}}||_{L^\infty((2c\tilde{B}\cap M)^{\mathbb{C}})} \leq C' e^{CN} \cdot ||f||_{L^\infty(c\tilde{B}\cap M)}$$

for some constants $C, C' > 1$ depending on M, g only.

To prove Lemma 5.6, we first need the following result on the unique continuation of f^H, borrowed from [41, Page 231]; see also [35, Lemma 4.3].

Lemma 5.7. *Let $x \in M$. There exist constants $r_0 = r_0(M, g, x) > 0$, $C_0 = C_0(M, g, x, r_0) > 0$ and $0 < \beta = \beta(M, g, x, r_0) < 1$ so that the following holds. Recall that the harmonic lift f^H is the function defined in (5.2). Then one has*

$$\left\|f^H\right\|_{L^\infty(\tilde{B}^+)} \leq C_0 \left\|r \cdot \partial_t f^H\right\|^{\beta}_{L^\infty(2\tilde{B}\cap M)} \cdot \left\|f^H\right\|^{1-\beta}_{L^\infty(2\tilde{B}^+)}$$

uniformly w.r.t. balls $\tilde{B}$ of radius $r > 0$ and centred at a point lying in $M \cong M \times \{0\}$ such that

$$\tilde{B} \subseteq B_g(x, r_0/4) \times [-r_0/4, r_0/4] \subseteq M \times \mathbb{R},$$

where $\tilde{B}^+ = \tilde{B} \cap (M \times [0, \infty))$.

Note that, although not explicated in [41], the constant β in Lemma 5.7 depends only on a particular coordinate patch around the point $x \in M$, provided this is sufficiently small. Therefore, β is uniform with respect to all the balls contained in the said coordinate patch and well-separated from the boundaries, as stated in Lemma 5.7. We refer the reader to [1, Theorem 1.7] for the details (in a much more general scenario). We are now ready to prove Lemma 5.6.

Proof of Lemma 5.6. First, given $x \in M$, let $r_0 = r_0(M, g, x)$ be given by Lemma 5.7. Covering M by balls of radius r_0 and using the compactness of M, we find that there exists some $\eta_1 > 0$, depending only on M and g, such that the conclusion of Lemma 5.7 is applicable on every ball

$$B_g(x, \eta_1/2) \times [-\eta_1/2, \eta_1/2].$$

Moreover, we may assume that $\eta_1 \leq \eta_0/2$, with η_0 as constructed in Section 5.3. Now observe that, appealing to the compactness of M again, it is sufficient to prove Lemma 5.6 in every coordinate patch of radius η_1. That is, it is sufficient to prove that, for every $x \in M$, there exists some $c > 0$ and some $C', C \geq 1$, depending on M, g, x, η_1, such that if (5.4) is satisfied, then one has

$$||f^{\mathbb{C}}||_{L^\infty(2(c\tilde{B}\cap M)^{\mathbb{C}})} \leq C' e^{CN} \cdot ||f||_{L^\infty(c\tilde{B}\cap M)}$$

uniformly w.r.t. balls $\tilde{B}$ of radius $r > 0$ and centred at a point lying in $M \cong M \times \{0\}$, such that

$$4\tilde{B} \subseteq B_g(x, \eta_1/2) \times [-\eta_1/2, \eta_1/2].$$

In what follows, this claim is established.

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Since the supremum norm is scale invariant, we may rescale the metric and assume that $\tilde{B}$ has radius $r = 1$. Since f^H satisfies

$$(\partial_t^2 + \Delta) f^H = 0,$$

the elliptic estimates for the operator $\partial_t^2 + \Delta$ (see, for example, [27, Lemma 7.5.1 and equation (4.4.1)] or [23, Page 330]) imply that there exists some constants $C_1, C_2 = C_1, C_2(M, g, \eta_1, x)$ such that for any $k > 0$, one has

$$\left\|f^H\right\|_{C^k(\frac{1}{2}\tilde{B})} \leq C_1^k k! \cdot \left\|f^H\right\|_{L^2(\frac{3}{4}\tilde{B})} \leq C_2^k k! \cdot \left\|f^H\right\|_{L^\infty(\tilde{B})}.$$

Moreover, by the definition (5.2) of f^H, for any multi-index α so that $|\alpha| = k$, we have

$$\sup_{\frac{1}{2}(\tilde{B}\cap M)} |D^\alpha f| \leq 2 \cdot \|f^H\|_{C^k(\tilde{B})}.$$

Therefore, we obtain the bound

$$\sup_{\frac{1}{2}(\tilde{B}\cap M)} \frac{|D^\alpha f|}{|\alpha|!} \leq 2C_2^k \cdot \left\|f^H\right\|_{L^\infty(\tilde{B})}. \tag{5.5}$$

Now we are going to bound the r.h.s. of (5.5) using the assumed doubling property (5.4). First, observe that since $\sinh(\cdot)$ is an odd function, we have

$$\left\|f^H\right\|_{L^\infty(\tilde{B}^+)} = \left\|f^H\right\|_{L^\infty(\tilde{B})}.$$

Thus, using the assumption (5.4) on the doubling of f^H (for some $c > 0$ to be chosen later), the assumption $r = 1$, and the equality

$$\left\|\partial_t f^H\right\|_{L^\infty(r'\tilde{B}\cap M)} = \|f\|_{L^\infty(r'\tilde{B}\cap M)}$$

that follows from (5.2) for any $r' > 0$, Lemma 5.7 implies

$$\begin{aligned}\left\|f^H\right\|_{L^\infty(\tilde{B}^+)} &= \left\|f^H\right\|_{L^\infty(\tilde{B})} \leq e^N \cdot \left\|f^H\right\|_{L^\infty(\frac{c}{2}\tilde{B})} = e^N \cdot \left\|f^H\right\|_{L^\infty(\frac{c}{2}\tilde{B}^+)} \\ &\lesssim_M e^{C_3 N} \cdot \|f\|^\beta_{L^\infty(c\tilde{B}\cap M)} \cdot \left\|f^H\right\|^{1-\beta}_{L^\infty(cB^+)} \leq e^{C_3 N} \cdot \|f\|^\beta_{L^\infty(c\tilde{B}\cap M)} \cdot \left\|f^H\right\|^{1-\beta}_{L^\infty(\tilde{B}^+)}\end{aligned} \tag{5.6}$$

for some $0 < \beta = \beta(M, g, x, \eta_1) < 1$ and some $C_3 = C_3(M, g, x, \eta_1) > 1$. Since f^H is an analytic function, we have $||f^H||_{L^\infty(\tilde{B}^+)} \neq 0$; thus, (5.6) implies

$$||f^H||_{L^\infty(\tilde{B}^+)} \lesssim_\beta e^{C_4 N} \cdot \|f\|_{L^\infty(c\tilde{B}\cap M)} \tag{5.7}$$

for some $C_4 = C_4(\beta, M, g) > 1$. Therefore, combining (5.5) and (5.7), we obtain

$$\sup_{\frac{1}{2}(\tilde{B}\cap M)} \frac{|D^\alpha f|}{|\alpha|!} \lesssim e^{C_5 N + C_6|\alpha|} \sup_{c\tilde{B}\cap M} |f| \tag{5.8}$$

for some $C_5, C_6 > 1$ depending only on M, g, x, η_1. Since f is real analytic, we may expand

$$f^{\mathbb{C}}(z) = \sum_\alpha \frac{D^\alpha f(y)}{|\alpha|!} z^{|\alpha|}$$

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into an absolutely convergent Taylor series in $(2c\tilde{B} \cap M)^{\mathbb{C}}$ for some sufficiently small $0 < c = c(M, g, x, \eta_1) < 1/2$. Then, (5.8) gives

$$\sup_{(2c\tilde{B}\cap M)^{\mathbb{C}}} |f^{\mathbb{C}}| \leq C_7 \exp(C_8 N) \sup_{c\tilde{B}\cap M} |f|$$

for some constants $C_7, C_8 > 1$ depending only on M, g, x, η_1, as required. □

5.5. *Concluding the proof of Proposition 5.1*

We are finally in a position to prove Proposition 5.1:

Proof. First, we take $\eta = c \min\{\eta_0, \eta_1, r_0(M)\}$, where η_0 is given at the beginning of Section 5.3, η_1 is given by Lemma 5.6 and $r_0(M)$ is the injectivity radius of M and $c = c(M, g) > 0$ is as in Lemma 5.6. Next, denote

$$\tilde{B}_r = B_g(x, r) \times [-r, r]$$

to be the ball as in the statement of Proposition 5.1 and let $c_1 = c_1(M, g) = c/8$ with $c = c(M, g)$ given as in Lemma 5.6. We cover the ball $\frac{1}{2}\tilde{B}_r$ by balls $\tilde{B}^i$ of radius $c_1 r/2$ and centre x^i so that

$$\mathcal{V}(f, \tilde{B}_{r/2} \cap M) \lesssim \max_i \mathcal{V}(f, \tilde{B}^i \cap M),$$

where the constant implied in the $\lesssim$ notation depends only on (M, g). Now let $f^i_{c_1 r}$ be a version of f rescaled by a factor of $c_1 r$ in the ball $\tilde{B}^i$ in the normal coordinates; that is,

$$f^i_{c_1 r} = f(\exp_{x^i}(c_1 r y))$$

for $y \in B_0 \subset \mathbb{R}^n$, where B_0 is the unit ball. The scaling property of the nodal volume gives

$$\mathcal{V}(f, \tilde{B}^i \cap M) \lesssim r^{n-1}\mathcal{V}\left(f^i_{c_1 r}, \frac{1}{2}B_0\right), \tag{5.9}$$

where the constant implied in the $\lesssim$ notation depends only on (M, g). Thus, Lemma 5.5 and invariance of the L^∞-norm w.r.t. scaling imply

$$\mathcal{V}\left(f^i_{c_1 r}, \frac{1}{2}B_0\right) \lesssim \log \frac{\sup_{2(B_0)^{\mathbb{C}}} |(f^i)^{\mathbb{C}}_{c_1 r}|}{\sup_{B_0} |f^i_{c_1 r}|} \lesssim \log \frac{\sup_{2(\tilde{B}^i\cap M)^{\mathbb{C}}} |f^{\mathbb{C}}|}{\sup_{(\tilde{B}^i\cap M)} |f|}, \tag{5.10}$$

where $f^{\mathbb{C}}$ is the complexification of f. Then, denoting $N^i = \mathcal{N}_{f^H}(2c_1^{-1}\tilde{B}^i)$ with f^H as in (5.2), Lemma 5.6, applied under the assumption $r < \eta/10$, and Corollary 5.3 yield

$$\log \frac{\sup_{2(\tilde{B}^i\cap M)^{\mathbb{C}}} |f^{\mathbb{C}}|}{\sup_{(\tilde{B}^i\cap M)} |f|} \lesssim \log \frac{\sup_{c^{-1}\tilde{B}^i} |f^H|}{\sup_{\frac{1}{2}\tilde{B}^i_r} |f^H|} \lesssim N^i. \tag{5.11}$$

Since $\tilde{B}^i$ have, by construction, radius $cr/8$, we find that

$$N^i = \mathcal{N}(x^i, r/4) \lesssim \mathcal{N}_{f^H}(\tilde{B}_{8r}),$$

where the second inequality follows from Lemma 5.4. Hence, the statement of Proposition 5.1 follows by combining (5.9), (5.10) and (5.11). □

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5.6. Estimates for the local nodal volume

In this section, we deduce a bound on $\mathcal{V}(F_x)$ from Proposition 5.1. We begin with the following estimate (see [41, page 231]):

Lemma 5.8. *Let f^H be as in (5.2) and let $\tilde{B} \subseteq M \times \mathbb{R}$ be a ball centred at some point on $M \cong M \times \{0\}$ and of any radius less than $\eta/10$, where η is given by Proposition 5.1. Then*

$$||f^H||_{L^\infty(2\tilde{B})} \lesssim e^{CT} ||f^H||_{L^\infty(\tilde{B})} \tag{5.12}$$

with some $C = C(M, g) > 1$.

Although we do not wish to reproduce the proof of Lemma 5.8 in full detail, for the sake of completeness, we quickly indicate how Lemma 5.8 follows from Lemma 5.7. Indeed, applying Lemma 5.7 to $B = M$ with L^2-norm instead of L^∞-norm (which is possible by elliptic estimates), upon observing that $||f^H||_{L^2(M\times[-a,a])} \asymp \exp(T) \sum_i |\phi_i|^2$ for $a = 1, 2$, we obtain

$$||f^H||_{L^\infty(2\tilde{B})} \lesssim e^{CT} ||f^H||_{L^\infty(\tilde{B})}$$

at macroscopic scales. Now, Lemma 5.8 follows by Corollary 5.3. As a direct consequence of Lemma 5.8, we have the following bound:

Lemma 5.9. *For F_x as in (4.1), one has*

$$\sup_{x\in M} \mathcal{V}(F_x) \lesssim T.$$

Proof. Applying Proposition 5.1 on f as in (1.2) with

$$\tilde{B} = B_g(x, 1/T) \times (-1/T, 1/T)$$

(where we tacitly assume that T is sufficiently large so that $1/T \le \eta/80$ with η as in Proposition 5.1), we obtain

$$\mathcal{V}\left(f, \frac{1}{2}\tilde{B} \cap M\right) T^{n-1} \lesssim \mathcal{N}_{f^H}(8\tilde{B}).$$

Lemma 5.8 gives $\mathcal{N}_{f^H}(8\tilde{B}) \lesssim T$. Therefore, Lemma 5.9 follows upon noticing that the definition (4.1) of F_x, being the scaled version of f_T, implies that

$$\mathcal{V}(F_x) \lesssim T^{n-1}\mathcal{V}\left(f, \frac{1}{2}\tilde{B} \cap M\right) \lesssim \mathcal{N}_{f^H}(8\tilde{B}) \lesssim T. \qquad \square$$

6. Anti-concentration

The aim of this section is to show that $\mathcal{V}(F_x(\omega, \cdot))$ is uniformly integrable as a random variable on $M \times \Omega$ equipped with the measure $d\sigma = d\,\mathrm{Vol} \otimes d\mathbb{P}/\mathrm{Vol}(M)$; that is, we prove the following result:

Proposition 6.1. *Let F_x be as in (4.1), $v(T)$ be as in (1.3), ϑ_n as in (1.8), $\rho(T)$ the energy window width, ρ_0 as in Corollary 4.4, and $h(t) = t \log t$.*

(i) *For $n \le 4$, assume that*

$$\rho(T) \ge T^{\vartheta_n} (\log T)^2.$$

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Then there exists a constant $C = C(M, g) > 1$, independent of T, such that

$$\int_{M\times\Omega} h(\mathcal{V}(F_x))d\sigma < C.$$

(ii) *For $n \geq 5$, the conclusions of (i) hold for all $\rho \geq \rho_0$; that is, there exists a constant $C = C(M, g) > 1$, independent of T, such that*

$$\int_{M\times\Omega} h(\mathcal{V}(F_x))d\sigma < C.$$

As discussed in Section 2, the required estimates for a proof of Proposition 6.1 are of the form

$$\sigma(\{(x,\omega) \in M \times \Omega : \mathcal{V}(F_x) > H\}) \lesssim \frac{1}{H(\log H)^c},$$

with $c > 2$ an absolute constant. We begin with a deterministic, weak L^1-type estimate that will be improved later for *random f*.

6.1. Weak L^1 estimate for the local nodal volume

We collect here a well-known result about the locality of the nodal volume, which will also be useful later in the proof, allowing for an L^1-weak type estimate for the volume of $x \in M$ with $\mathcal{V}(F_x)$ large.

Lemma 6.2. *Let F_x and f be as in (4.1) and (1.2), respectively, and let ω_n be the unit n-ball volume. Then one has:*

$$\mathcal{V}(f) = \frac{2^n T}{\omega_n}(1 + o_{T\to\infty}(1)) \cdot \int_M \mathcal{V}(F_x)d\operatorname{Vol}(x).$$

Proof. First, we observe that we may write

$$\mathcal{V}\left(f, B_g\left(x, \frac{1}{2T}\right)\right) = \int_{f^{-1}(0)} \mathbb{1}_{B_g(x,1/(2T))}(y)d\mathcal{H}^{n-1}(y), \tag{6.1}$$

where $\mathbb{1}$ is the indicator function and $\mathcal{H}^{n-1}$ is the Hausdorff measure. Then, integrating both sides of (6.1) and using Fubini's Theorem, we have

$$\int_M \mathcal{V}\left(f, B_g\left(x, \frac{1}{2T}\right)\right) d\operatorname{Vol}(x) = \int_{f^{-1}(0)} \operatorname{Vol}_g\left(B_g\left(y, \frac{1}{2T}\right)\right) d\mathcal{H}^{n-1}(y). \tag{6.2}$$

Now, we observe that in light of the definition of F_x in Section 4.1, since $1/T$ is smaller than the injectivity radius of M, we have

$$\mathcal{V}(F_x) = \mathcal{V}\left(f, B_g\left(x, \frac{1}{2T}\right)\right) \cdot T^{n-1}(1 + o_{T\to\infty}(1)),$$

where we have used the scaling property of the nodal volume

$$\mathcal{V}(F_x) = \mathcal{V}\left(F_x(\cdot), \frac{1}{2}B_0\right) = T^{n-1}\mathcal{V}\left(F_x(T\cdot), B\left(\frac{1}{2T}\right)\right).$$

Thus, the l.h.s. of (6.2) is

$$\int_M \mathcal{V}\left(f, B_g\left(x, \frac{1}{2T}\right)\right) d\operatorname{Vol}(x) = \frac{1}{T^{n-1}}(1 + o_{T\to\infty}(1)) \int_M \mathcal{V}(F_x)d\operatorname{Vol}_g(x). \tag{6.3}$$

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Moreover, for all $y \in M$, we also have

$$\mathrm{Vol}_g\left(B_g\left(y, \frac{1}{2T}\right)\right) = \mathrm{Vol}_{\mathbb{R}^n}\left(B\left(0, \frac{1}{2T}\right)\right)\left(1 + O\left(T^{-1}\right)\right) = \frac{\omega_n}{(2T)^n}(1 + O(T^{-1})).$$

Thus, the r.h.s. of (6.2) is

$$\int_{f^{-1}(0)} \mathrm{Vol}_g\left(B_g\left(y, \frac{1}{2T}\right)\right) d\mathcal{H}^{n-1}(y) = \frac{\omega_n}{(2T)^n}\left(1 + O\left(T^{-1}\right)\right) \cdot \mathcal{V}(f) \tag{6.4}$$

Hence, Lemma 6.2 follows upon inserting (6.3) and (6.4) into (6.2). □

As a direct consequence of Lemma 6.2, we have the following result:

Corollary 6.3. *Let F_x be as in (4.1). Then, uniformly for all $t > 0$,*

$$\mathrm{Vol}_g(x : \mathcal{V}(F_x) > t) \lesssim t^{-1}.$$

Proof. We first aim to prove (1.6), that is claimed no novelty of, but was decided to be included for the sake of completeness. Let $\eta > 0$ be as prescribed by Proposition 5.1. Then, by Proposition 5.1 and Lemma 5.8, we have

$$\mathcal{V}(f_T, B_\eta) \lesssim T$$

for any ball $B_\eta \subseteq M$ of radius $\eta/20$. Covering M by finitely many such balls, we obtain

$$\mathcal{V}(f_T) \lesssim T, \tag{6.5}$$

which is (1.6). Inserting (6.5) into Lemma 6.2, we obtain

$$\int_M \mathcal{V}(F_x) d\,\mathrm{Vol}(x) \lesssim 1.$$

Hence, Corollary 6.3 follows from Markov's inequality. □

6.2. *A probabilistic anti-concentration inequality for the doubling index*

The aim of this section is to prove the following result that, unlike Proposition 6.1 for $n \le 4$, will not require the growth of $\rho(T)$:

Lemma 6.4. *Let f^H be as in (5.3), $\eta > 0$ as in Proposition 5.1, $v(T)$ as in (1.3) and ρ_0 as in Corollary 4.4. Moreover, given a parameter $100/\eta \le A \le 50T$, let*

$$\tilde{B}_A = B_g(x, A^{-1}) \times \left[-\frac{1}{A}, \frac{1}{A}\right] \subseteq M \times [-10, 10]$$

be a ball centred at some $(x, 0) \in M \times \{0\}$. If $\rho(T) \ge \rho_0$, then for all $Q \ge 100$, one has

$$\mathbb{P}\left(\mathcal{N}_{f^H}(\tilde{B}_A) > \frac{Q \cdot T}{A}\right) \lesssim \exp\left(-\frac{Q \cdot T}{50A}\right) + E_A(x),$$

where

$$E_A(x) := v(T)^{-3/2} \cdot \sum_{\lambda_i \in [T-\rho, T]} A^n \int_{B_g(x,(2A)^{-1})} |\phi_i(y)|^3 d\,\mathrm{Vol}_g(y)$$

and the constant involved in the '$\lesssim$'-notation may depend on M, g, but not on Q, A, T or x.

Before giving a proof to Lemma 6.4, we would like to describe the intuition behind its proof. By the definition (5.1) of the doubling index, a large doubling index of f^H indicates a rapid growth of f^H on concentric balls. This could happen in two possible scenarios: either f^H is large on the larger ball or f^H is small on the smaller ball. The probability of the former event can be controlled using some L^2-bounds and Chebyshev's inequality, whereas the probability of the latter one is controlled with the following lemma:

Lemma 6.5. *Let $f^H(x)$ be as in (5.3), $v(T)$ as in (1.3), and ρ_0 as in Corollary 4.4. Denote*

$$\Psi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left(-\frac{z^2}{2}\right) dz$$

to be the standard Gaussian cumulative distribution function, and, given $x \in M$, denote $\tilde{x} = (x, (100T)^{-1}) \in M \times [-10, 10]$. Then if $\rho(T) \geq \rho_0$, one has

$$\sup_{t\in\mathbb{R}} \left| \mathbb{P}\left(\frac{f^H(\tilde{x})}{\mathbb{E}[|f^H(\tilde{x})|^2]^{1/2}} \leq t \right) - \Psi(t) \right| \lesssim E(x),$$

where the constant implied in the '$\lesssim$'-notation depends only on (M, g) and

$$E(x) := v(T)^{-3/2} \sum_{\lambda_i \in [T-\rho, T]} |\phi_i(x)|^3.$$

Proof. For $\lambda_i \in [T - \rho, T]$, let us write

$$X_i = X_i(\tilde{x}) := \frac{a_i}{\lambda_i^{-1}} \sinh\left(\frac{\lambda_i}{100T}\right) \cdot \phi_i(x),$$

and, on recalling that $\lambda_i/T = 1 + o_{T\to\infty}(1)$,

$$\sigma_i = \sigma_i(\tilde{x}) := \mathbb{E}\left[|X_i|^2\right]^{1/2} = \lambda_i^{-1} \left| \sinh\left(\frac{\lambda_i}{100T}\right) \cdot \phi_i(x) \right| \asymp T^{-1} \cdot |\phi_i(x)|, \tag{6.6}$$

where the constant in the '$\asymp$'-notation is absolute. Moreover, let

$$\tau_i = \tau_i(\tilde{x}) := \mathbb{E}\left[|X_i(\tilde{x})|^3\right] \asymp \sigma_i^3(\tilde{x}).$$

On recalling (5.3), observe that we have

$$\mathbb{E}\left[|f^H(\tilde{x})|^2\right] = \sum_{\lambda_i \in [T-\rho, T]} \sigma_i^2(\tilde{x}).$$

By the well-known Berry-Esseen Theorem [8, 22] applied to the sum of the X_i's, we have

$$\begin{aligned} \sup_{t\in\mathbb{R}} \left| \mathbb{P}\left(\frac{f^H(\tilde{x})}{\mathbb{E}[f^H(\tilde{x})]^{1/2}} < t \right) - \Psi(t) \right| &\lesssim \left(\sum_{\lambda_i \in [T-\rho, T]} |\sigma_i(\tilde{x})|^2 \right)^{-3/2} \sum_{\lambda_i \in [T-\rho, T]} \tau_i(\tilde{x}) \\ &\lesssim \left(\sum_{\lambda_i \in [T-\rho, T]} |\sigma_i|^2 \right)^{-3/2} \sum_{\lambda_i \in [T-\rho, T]} |\sigma_i|^3, \end{aligned} \tag{6.7}$$

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where the constant implied in the '$\lesssim$'-notation is absolute. Using (6.6), the r.h.s of (6.7) may be bounded as

$$\left(\sum_{\lambda_i\in[T-\rho,T]}|\sigma_i(\tilde{x})|^2\right)^{-3/2}\sum_{\lambda_i\in[T-\rho,T]}|\sigma_i(\tilde{x})|^3\lesssim\left(\sum_{\lambda_i\in[T-\rho,T]}|\phi_i(x)|^2\right)^{-3/2}\sum_{\lambda_i\in[T-\rho,T]}|\phi_i(x)|^3.$$

Hence, Lemma 6.5 follows from Corollary 4.4; that is,

$$\sum_{\lambda_i\in[T-\rho,T]}|\phi_i(x)|^2\asymp v(T).$$

□

Lemma 6.5 gives the following bound on the supremum of f^H of the ball $\tilde{B}_A$ as in Lemma 6.7:

Corollary 6.6. *Let f^H be as in (5.3), $\eta>0$ as in Proposition 5.1, and ρ_0 as in Corollary 4.4. Given a parameter $100/\eta\le A\le 50T$, let*

$$\tilde{B}_A=B_g(x,A^{-1})\times\left[-\frac{1}{A},\frac{1}{A}\right]\subseteq M\times[-10,10]$$

be a ball centred at some $(x,0)\in M\times\{0\}$. Moreover, let us write $c(\tilde{x}):=\mathbb{E}[|f^H(\tilde{x})|^2]$, where $\tilde{x}:=(x,(100T)^{-1})$. Suppose that $\rho(T)\ge\rho_0$. Then, for all $\tau>0$ (which may depend on A), we have

$$\mathbb{P}\left(\sup_{\tilde{B}_A}\left|\frac{f^H}{c(\tilde{x})^{1/2}}\right|\le\tau\right)\lesssim\tau+E_A(x),$$

where

$$E_A(x):=v(T)^{-3/2}\cdot\sum_{\lambda_i\in[T-\rho,T]}A^n\int_{B_g(x,(2A)^{-1})}|\phi_i(y)|^3d\,\mathrm{Vol}_g(y),$$

and the constant involved in the '$\lesssim$'-notation may depend on M,g but not on A,T or x.

Proof. Since for every $\tilde{y}:=(y,(100T)^{-1})$ with $y\in B_g(x,A^{-1})$, we have

$$\mathbb{P}\left(\sup_{\tilde{B}_A}\left|\frac{f^H}{c(\tilde{x})^{1/2}}\right|\le\tau\right)\le\mathbb{P}\left(\left|\frac{f^H(\tilde{y})}{c(\tilde{x})^{1/2}}\right|\le\tau\right),$$

we have the bound

$$\mathbb{P}\left(\sup_{\tilde{B}_A}\left|\frac{f^H}{c(\tilde{x})^{1/2}}\right|\le\tau\right)\le\inf_{y\in B_g(x,A^{-1})}\mathbb{P}\left(\left|\frac{f^H(\tilde{y})}{c(\tilde{x})^{1/2}}\right|\le\tau\right).$$

Bounding the infimum by the average (over, say, a slightly smaller ball), we obtain

$$\begin{aligned}\mathbb{P}\left(\sup_{\tilde{B}_A}\left|\frac{f^H}{c(\tilde{x})^{1/2}}\right|\le\tau\right)&\lesssim A^n\int_{B_g(x,(2A)^{-1})}\mathbb{P}\left(\left|\frac{f^H(\tilde{y})}{c(\tilde{x})^{1/2}}\right|\le\tau\right)d\,\mathrm{Vol}_g(y)\\&\lesssim A^n\int_{B_g(x,(2A)^{-1})}\mathbb{P}\left(\left|\frac{f^H(\tilde{y})}{c(\tilde{y})^{1/2}}\right|\le\frac{c(\tilde{x})^{1/2}}{c(\tilde{y})^{1/2}}\tau\right)d\,\mathrm{Vol}_g(y).\end{aligned}$$

Therefore, Lemma 6.5 yields

$$\mathbb{P}\left(\sup_{\tilde{B}_A}\left|\frac{f^H}{c(\tilde{x})^{1/2}}\right| \leq \tau\right) \lesssim c(\tilde{x})^{1/2}\tau A^n \int_{B_g(x,(2A)^{-1})} \frac{1}{c(\tilde{y})^{1/2}} d\,\mathrm{Vol}_g(y) + E_A(x). \tag{6.8}$$

Now, using Lemma 4.5, for all $y \in B_g(x, A^{-1})$ (and so in particular for x), we have

$$\begin{aligned} c(\tilde{y}) = \mathbb{E}\big[|f^H(y,(100T)^{-1})|^2\big] &= \sum_{\lambda_i \in [T-\rho,T]} \left|\lambda_i^{-1}\sinh\left(\frac{\lambda_i}{100T}\right)\phi_i(x)\right|^2 \\ &\asymp T^{-2} \sum_{\lambda_i \in [T-\rho,T]} |\phi_i(x)|^2 \asymp T^{-2}. \end{aligned}$$

Thus, the first term on the r.h.s. of (6.8) is

$$c(\tilde{x})^{1/2}\tau A^n \int_{B_g(x,(2A)^{-1})} \frac{1}{c(\tilde{y})^{1/2}} \asymp \tau,$$

and this concludes the proof of Corollary 6.6. □

We are finally ready to prove Lemma 6.4:

Proof of Lemma 6.4. To simplify notation, we will use the following shorthand: $\tilde{x} = (x, (100T)^{-1})$ and $\tilde{B} = \tilde{B}_A$. First, we may renormalize f^H by dividing it by the nonvanishing number $c(\tilde{x}) := \mathbb{E}[|f^H(x,(100T)^{-1})|^2]$; that is, by a slight abuse of notation, we write f^H in place of

$$\frac{f^H}{\mathbb{E}[|f^H(x,(100T)^{-1})|^2]^{1/2}} = \frac{f^H}{c(\tilde{x})^{1/2}} = \frac{1}{c(\tilde{x})^{1/2}v(T)^{1/2}} \sum_{\lambda_i \in [T-\rho,T]} a_i \frac{\sinh(\lambda_i t)}{\lambda_i}\phi_i(x), \tag{6.9}$$

throughout the proof of Lemma 6.4. We are now in a position to commence the proof of Lemma 6.4.

To bound the probability that the doubling index us large, we note that it could occur under two possible scenarios: either $\sup_{\tilde{B}}|f^H|$ is small or $\sup_{2\tilde{B}}|f^H|$ is large. Given some $\tau > 0$ to be determined later, we write

$$\begin{aligned} \mathbb{P}\left(\mathcal{N}(\tilde{x}, A^{-1}) > \frac{Q\cdot T}{A}\right) &= \mathbb{P}\left(\mathcal{N}(\tilde{x}, A^{-1}) > \frac{Q\cdot T}{A} \quad \text{and} \quad \sup_{\tilde{B}}|f^H| < \tau\right) \\ &\quad + \mathbb{P}\left(\mathcal{N}(\tilde{x}, A^{-1}) > \frac{Q\cdot T}{A} \quad \text{and} \quad \sup_{\tilde{B}}|f^H| \geq \tau\right). \end{aligned} \tag{6.10}$$

The first term on the r.h.s. of (6.10) can be bounded as

$$\mathbb{P}\left(\mathcal{N}(\tilde{x}, A^{-1}) > \frac{Q\cdot T}{A} \quad \text{and} \quad \sup_{\tilde{B}}|f^H| < \tau\right) \leq \mathbb{P}(\sup_{\tilde{B}}|f^H| \leq \tau).$$

Thus, Corollary 6.6 gives

$$\mathbb{P}\left(\mathcal{N}(\tilde{x}, A^{-1}) > \frac{Q\cdot T}{A} \quad \text{and} \quad \sup_{\tilde{B}}|f^H| < \tau\right) \lesssim \tau + E_A(x). \tag{6.11}$$

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Now we bound the second term on the r.h.s. of (6.10). To this end, we use the definition (5.1) of the doubling index, and since, under the relevant event, $\sup_{\tilde{B}} |f^H| \geq \tau$, we may write under the same event

$$\frac{Q \cdot T}{A} < \mathcal{N}(\tilde{x}, A^{-1}) \leq \log \frac{||f^H||_{L^\infty(2\tilde{B})}}{\tau}.$$

Thus, we obtain

$$||f^H||_{L^\infty(2\tilde{B})} \geq \exp\left(\frac{Q \cdot T}{A}\right)\tau. \tag{6.12}$$

Now we claim the following:

$$\mathbb{E}\left[||f^H||^2_{L^\infty(2\tilde{B})}\right] \lesssim \exp\left(8\frac{T}{A}\right), \tag{6.13}$$

where the constant implied in the '$\lesssim$'-notation may depend on (M, g) only. Upon using the elliptic regularity [23, Page 330], we have

$$||f^H||^2_{L^\infty(2\tilde{B})} \lesssim A^{n+1}||f^H||^2_{L^2(4\tilde{B})},$$

where the constant implies in the '$\lesssim$'-notation depends only on (M, g). Therefore, using the formula (5.3), exchanging the order of the expectation and the summation, and upon bearing in mind that f^H is normalized via (6.9), we have

$$\begin{aligned}\mathbb{E}\left[||f^H||^2_{L^\infty(2\tilde{B})}\right] &\lesssim A^{n+1}\mathbb{E}\left[||f^H||^2_{L^2(4\tilde{B})}\right] \\ &\lesssim c(\tilde{x})^{-1}v(T)^{-1}\sum_{\lambda_i}\frac{\sinh(8\lambda_i/A)}{\lambda_i^2}A^n\int_{B_g(x,4/A)}|\phi_i(x)|^2 d\,\mathrm{Vol}_g(x),\end{aligned}$$

where the constant implied in the '$\lesssim$'-notation may depend on (M, g) only. Switching the sum with the integral, using Lemma 4.5, the obvious bound $\sinh(\cdot) \leq \exp(\cdot)$, and, again, $\lambda_i/T = 1 + o_{T\to\infty}(1)$, we obtain

$$\mathbb{E}\left[||f^H||^2_{L^\infty(2\tilde{B})}\right]^2 \lesssim c(\tilde{x})^{-1}T^{-2}\exp\left(8\frac{T}{A}\right), \tag{6.14}$$

where, again, the constant implied in the '$\lesssim$'-notation may depend on (M, g) only. Since $c(\tilde{x})^{-1} \asymp T^{-2}$, (6.13) follows from (6.14).

Using (6.13) together with Chebyshev's inequality, (6.10), (6.11) and (6.12), we obtain

$$\mathbb{P}\left(\mathcal{N}(\tilde{x}, A^{-1}) > \frac{Q \cdot T}{A}\right) \lesssim \tau + \exp\left(8\frac{T}{A} - \frac{2Q \cdot T}{A}\right)\tau^{-2} + E_A(x).$$

Hence, Lemma 6.4 follows by taking $\tau = \exp(-QT/(50A))$ and $Q \geq 100$ (say). □

6.3. Sogge's bound and the decay of the doubling index

The aim of this section is to prove the following lemma, which shows that outside an event of small probability, the doubling index decreases uniformly for all $x \in M$. In accordance with the results in the previous section except Proposition 6.1 for $n \leq 4$, the following lemma is stated for $\rho(T) \geq \rho_0$ without the growth assumption of Theorem 1.1.

Lemma 6.7. *Let f^H be as in (5.3), and ρ_0 as in Corollary 4.4. If $\rho(T) \geq \rho_0$, then there exists some constant $C = C(M, g) > 1$ such that*

$$\mathbb{P}\left(\sup_{x\in M} \mathcal{N}_{f^H}\left((x,0), A^{-1}\right) \geq C\frac{T}{A}\right) \lesssim (\log T)^{-1},$$

where

$$A = A(T) = \begin{cases} T^{\frac{n-1}{4n}} \rho(T)^{\frac{1}{2n}} \cdot \frac{1}{(\log T)^{\frac{1}{n}}} & n \leq 4 \\ T^{\frac{1}{n}} \rho(T)^{\frac{1}{2n}} \cdot \frac{1}{(\log T)^{\frac{1}{n}}} & n \geq 5. \end{cases}.$$

To prove Lemma 6.7, we use Lemma 6.4 together with the following bound on the L^p-norm of eigenfunctions due to Sogge [54]. Let ϕ_i be an eigenfunction with eigenvalue λ_i^2. Then we have the following estimate on the L^p norms of ϕ_i (see also [60, Theorem 10.1]):

$$||\phi_i||_{L^p(M)} \lesssim \lambda_i^{\sigma(p)} ||\phi_i||_{L^2(M)}, \tag{6.15}$$

where

$$\sigma(p) = \begin{cases} \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{p}\right) & 2 < p \leq \frac{2(n+1)}{n-1} \\ n\left(\frac{1}{2} - \frac{1}{p}\right) - 1/2 & p \geq \frac{2(n+1)}{n-1}. \end{cases}$$

We are now in a position to prove Lemma 6.7.

Proof of Lemma 6.7. Let $A = A(T)$ be some parameter to be chosen later and apply Lemma 6.4 on the ball

$$\tilde{B}_A = B_g(x, A^{-1}) \times [-1/A, 1/A]$$

with $Q = 100$ (say) to yield

$$\mathbb{P}\left(\mathcal{N}_{f^H}(\tilde{x}, A^{-1}) \geq \frac{100T}{A}\right) \lesssim \exp\left(-\frac{2T}{A}\right) + E_A(x),$$

where $\tilde{x} := (x, 0)$. Using the monotonicity of the doubling index of Lemma 5.4 with $r = A$, we deduce that there exists some (large) constant $C_1 = C_1(M, g) \geq 1$ such that

$$\mathbb{P}\left(\sup_{y\in B_g(x,(10A)^{-1})} \mathcal{N}_{f^H}(y, (4A)^{-1}) \geq \frac{C_1 T}{A}\right) \lesssim \exp\left(-\frac{2T}{A}\right) + E_A(x),$$

where we have tacitly assumed that T/A is sufficiently large depending on M, g only. Taking the union bound over at most $O(A^n)$ balls $B(x_j, (10A)^{-1})$, we obtain

$$\mathbb{P}\left(\sup_{x\in M} \mathcal{N}_{f^H}(\tilde{x}, (4A)^{-1}) \geq \frac{C_1 T}{A}\right) \lesssim A^n \exp\left(-\frac{2T}{A}\right) + \sum_j E_A(x_j). \tag{6.16}$$

Assuming that A is sufficiently large so that A^{-1} is smaller than the injectivity radius, each ball $B(x_j, (10A)^{-1})$ intersects finitely many (depending on n only) other balls in the collection; therefore,

$$v(T)^{3/2} \sum_j E_A(x_j) = A^n \sum_{\lambda_i\in[T,T-\rho]} \sum_j \int_{B(x_j,(10A)^{-1})} |\phi_i|^3 d\,\mathrm{Vol} \lesssim A^n \sum_{\lambda_i\in[T,T-\rho]} ||\phi_i||^3_{L^3(M)}.$$

Using Sogge's bound (6.15), we conclude that

$$\sum_j E_A(x_j) \lesssim \sum_{\lambda_i \in [T, T-\rho]} \frac{A^n T^{3\sigma(3)}}{v(T)^{3/2}}, \tag{6.17}$$

with $\sigma(3)$ as in (6.15).

Finally, inserting (6.17) into (6.16) and summing over i, which gives a contribution of $v(T)$, we obtain

$$\mathbb{P}\left(\sup_{x\in M} \mathcal{N}_{f^H}(\tilde{x}, 4^{-1}A^{-1}) \geq \frac{C_1 T}{A}\right) \lesssim A^n \exp\left(-\frac{2T}{A}\right) + A^n T^{3\sigma(3)} v(T)^{-1/2}.$$

Hence, Lemma 6.7 follows by observing that

$$3\sigma(3) = \begin{cases} \frac{n-1}{4} & n \leq 4 \\ \frac{n}{2} - \frac{3}{2} & n \geq 5 \end{cases},$$

$$v(T)^{1/2} \asymp T^{\frac{n-1}{2}} \rho(T)^{1/2},$$

and taking for $n \leq 4$,

$$A = T^{\frac{n-1}{4n}} \rho(T)^{\frac{1}{2n}} \cdot \frac{1}{4(\log T)^{\frac{1}{n}}}$$

and for $n > 5$,

$$A = T^{\frac{1}{n}} \rho(T)^{\frac{1}{2n}} \cdot \frac{1}{4(\log T)^{\frac{1}{n}}}. \qquad \square$$

6.4. *Concluding the proof of Proposition 6.1*

Proof of Proposition 6.1. Since the proof of the Proposition 6.1 is somewhat long, we break it up into a series of steps:

Step 1: Controlling the distribution of $\mathcal{V}(F_x)$.

Recall that $d\sigma = \frac{d\,\mathrm{Vol}_g}{\mathrm{Vol}(M)} \otimes d\mathbb{P}$. The aim of this step is to obtain some bounds on $\sigma(\mathcal{V}(F_x) > t)$ for all $t \geq C_0$ for some $C_0 = C_0(M, g) \geq 1$. First, by Proposition 5.1, bearing in mind the rescaling factor, we have

$$\mathcal{V}(F_x) \leq C_1 \mathcal{N}_{f^H}((x,0), 8T^{-1}) \quad \text{and} \quad \mathcal{N}_{f^H}((x,0), 8T^{-1}) := \mathcal{N}_T(x) = \mathcal{N}(x)$$

for some $C_1 = C(M, g) \geq 1$. Therefore, Lemma 6.4, applied with $A = 4T$ and $Q = c_0 t := C_1^{-1} t$ (which is larger than 100 taking C_0 sufficiently large in terms of C_1), gives

$$\mathbb{P}(\mathcal{V}(F_x) \geq t) \leq \mathbb{P}(\mathcal{N}(x) \geq c_0 t) \lesssim \exp\left(-\frac{c_0 t}{10}\right) + E_T(x),$$

where E_T is as in Lemma 6.4 (and we write E_T in place of $E_{T/8}$ as shorthand). Thus, we have

$$\begin{aligned} \sigma(\mathcal{V}(F_x) > t) &= \frac{1}{\mathrm{Vol}(M)} \int_M \mathbb{P}(\mathcal{V}(F_x) \geq t)\, d\,\mathrm{Vol}_g \\ &\lesssim \exp\left(-\frac{c_0 t}{10}\right) + \int_M E_T(x)\, d\,\mathrm{Vol}_g . \end{aligned} \tag{6.18}$$

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We are now going to bound the second term on the r.h.s. of (6.18). By Sogge's bound, we have

$$\int_M |\phi_i(x)|^3 d\,\mathrm{Vol}_g \lesssim T^{3\sigma(3)},$$

with $\sigma(3)$ as in (6.15). Therefore, in light of the fact that the sum over i in the definition of $E_T(x)$ in Lemma 6.5 has $\nu(T)$-terms, $\nu(T) \asymp \rho(T)T^{n-1}$ and exchanging the integrals, we have

$$\int_M E_T(x) d\,\mathrm{Vol} \lesssim T^{\alpha(n)} \rho(T)^{-1/2}, \tag{6.19}$$

with

$$\alpha(n) := 3\sigma(3) - \frac{n-1}{2} = \begin{cases} -\frac{n-1}{4} & n \leq 4 \\ -1 & n \geq 5 \end{cases}.$$

Inserting (6.19) into (6.18), we see that

$$\sigma(\mathcal{V}(F_x) > t) \lesssim \exp\left(-\frac{c_0 t}{10}\right) + T^{\alpha(n)} \rho(T)^{-1/2}. \tag{6.20}$$

Step 2: Sharpening the upper bound

By Lemma 5.9, we have

$$\sup_{x \in M} \mathcal{V}(F_x) \lesssim T.$$

Our task in this step is to obtain a better upper bound, outside an event of small probability, using Lemma 6.7. Let $A = A(T)$ be as in Lemma 6.7. Since the monotonicity of the doubling index of Lemma 5.4 implies that

$$\mathcal{N}(x) \lesssim \mathcal{N}_{f^H}\left((x, 0), A^{-1}\right) + C_3$$

for some $C_3 = C_3(M, g) > 1$, an application of Lemma 6.7 gives

$$\sup_{x \in M} \mathcal{V}(F_x) \leq C_4 \frac{T}{A} =: p(T) \tag{6.21}$$

for some $C_4 = C_4(M, g) > 0$, outside an event Ω_1 with $\mathbb{P}(\Omega_1) \lesssim (\log T)^{-1}$.

We now show that the event Ω_1 does not positively contribute to the integral of Proposition 6.1. Indeed, we write

$$\begin{aligned} \int_{M \times \Omega} h(\mathcal{V}(F_x)) d\sigma &= \int_{M \times (\Omega \backslash \Omega_1)} h(\mathcal{V}(F_x)) d\sigma + \int_{M \times \Omega_1} h(\mathcal{V}(F_x)) d\sigma \\ &\leq \int_{M \times \Omega} h(\mathcal{V}(F_x)) \cdot \mathbb{1}_{\mathcal{V}(F_x) \leq p(T)} d\sigma + O\left((\log T)^{-1} \sup_{\omega \in \Omega} \int_M h(\mathcal{V}(F_x)) d\,\mathrm{Vol}(x) \right). \end{aligned} \tag{6.22}$$

Since $h(t) = t \log t$ and $\mathcal{V}(F_x) \lesssim T$, the second term on the r.h.s of (6.22) can be bounded by

$$(\log T)^{-1} \sup_{\omega \in \Omega} \int_M h(\mathcal{V}(F_x)) d\,\mathrm{Vol}(x) \lesssim \sup_{\omega \in \Omega} \int_M \mathcal{V}(F_x) d\,\mathrm{Vol}(x) = O(1),$$

where, in the last inequality, we have used Lemma 6.2, which is deterministic, in the form

$$\sup_{\omega\in\Omega}\int_M \mathcal{V}(F_x)d\,\mathrm{Vol}(x)=O(1).$$

Thus, we have shown that

$$\int_{M\times\Omega} h(\mathcal{V}(F_x))d\sigma \lesssim \int_{M\times\Omega} h(\mathcal{V}(F_x))\mathbb{1}_{\mathcal{V}(F_x)\le p(T)}d\sigma + O(1), \tag{6.23}$$

with $p(T)$ as in (6.21). This concludes step 2.

Step 3: Collecting the estimates.

We begin by rewriting the integral on the r.h.s of (6.23) as

$$\int_{M\times\Omega} h(\mathcal{V}(F_x))\mathbb{1}_{\mathcal{V}(F_x)\le p(T)}d\sigma = \int_0^{p(T)} h(t)d\sigma(\mathcal{V}(F_x)>t).$$

Integrating by parts, we obtain

$$\int_{M\times\Omega} h(\mathcal{V}(F_x))\mathbb{1}_{\mathcal{V}(F_x)\le p(T)}d\sigma \lesssim \int_{C_0}^{2p(T)} h'(t)\sigma(\mathcal{V}(F_x)>t)dt + O(1), \tag{6.24}$$

with C_0 as in Step 1. Now, recall that in Step 2, we had

$$A=A(T)=\begin{cases} T^{\frac{n-1}{4n}}\rho(T)^{\frac{1}{2n}}\cdot\frac{1}{(\log T)^{\frac{1}{n}}} & n\le 4\\ T^{\frac{1}{n}}\rho(T)^{\frac{1}{2n}}\cdot\frac{1}{(\log T)^{\frac{1}{n}}} & n\ge 5,\end{cases}$$

and in Step 1, we had

$$\alpha(n):=3\sigma(3)-\frac{n-1}{2}=\begin{cases}-\frac{n-1}{4} & n\le 4\\ -1 & n\ge 5\end{cases}.$$

Since $h(t)=t\log t$, we have $h'(t)=\log t+1\le 2\log t$. Thus, using Step 1, namely (6.20), Step 2, (6.24) and the definition of $A(T)$ above, we have

$$\begin{aligned}\int_{M\times\Omega} h(\mathcal{V}(F_x))d\sigma &\lesssim \int_{M\times\Omega} h(\mathcal{V}(F_x))\mathbb{1}_{\mathcal{V}(F_x)\le p(T)}d\sigma + O(1)\\ &\lesssim \int_{C_0}^{2p(T)} h'(t)\sigma(\mathcal{V}(F_x)>t)dt+O(1)\lesssim p(T)T^{\alpha(n)}\rho(T)^{-1/2}(\log T)+O(1)\\ &\lesssim \frac{T^{1+\alpha(n)}}{A}\rho(T)^{-1/2}(\log T)+O(1)\lesssim q(T)\rho(T)^{-\frac{n+1}{2n}}(\log T)^{\frac{n+1}{n}}+O(1),\end{aligned} \tag{6.25}$$

where the constant implied in the '$\lesssim$'-notation depends only on (M,g) and

$$q(T):=\begin{cases} T\cdot T^{-\frac{n-1}{4}\left(1+\frac{1}{n}\right)} & n\le 4\\ T^{-\frac{1}{n}} & n\ge 5.\end{cases}$$

Hence, taking

$$\rho(T)\ge\begin{cases} T^{\frac{-n^2+4n+1}{2(n+1)}}(\log T)^2 & n\le 4\\ 1 & n\ge 5\end{cases},$$

we see that the r.h.s. of (6.25) is bounded, as required. □

7. Proof of Theorem 1.1

Before concluding the proof of Theorem 1.1, we state a result, whose proof is a straightforward application of the Kac-Rice formula, performed below for the reader's convenience.

Lemma 7.1. *Let F_μ be as in (4.2) and ω_n be the volume of the unit ball in $\mathbb{R}^n$. Then we have*

$$\mathbb{E}[\mathcal{V}(F_\mu)] = 2^{-n}\omega_n\left(\frac{1}{\pi n}\right)^{1/2}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}.$$

Proof. Since the support of μ, being the unit sphere, is not contained in an hyperplane, the distribution of $(F_\mu, \nabla F_\mu)$ is nondegenerate. Thus, we may apply the Kac-Rice formula [3, Theorem 6.1] to see that

$$\mathbb{E}[\mathcal{V}(F_\mu)] = \int_{2^{-1}B_0} \mathbb{E}\left[\|\nabla F_\mu(y)\| \middle| F_\mu(y) = 0\right] \cdot \varphi_{F_\mu(y)}(0)dy, \tag{7.1}$$

where $\varphi_{F_\mu(y)}(0)$ is the density of $F_\mu(y)$ at the point 0. Since $\mathbb{E}[|F_\mu(y)|^2] = 1$, ∇F_μ and F_μ are independent, and bearing in mind that F_μ is stationary, we have

$$\mathbb{E}\left[\|\nabla F_\mu(y)\| \middle| F_\mu(y) = 0\right] \cdot \varphi_{F_\mu(y)}(0) = \mathbb{E}\left[\|\nabla F_\mu(0)\|\right] \cdot \varphi_{F_\mu(0)}(0). \tag{7.2}$$

The latter can be computed explicitly (see, for example, [50, Proposition 4.1]) to be

$$\mathbb{E}\left[\|\nabla F_\mu(0)\|\right] \cdot \varphi_{F_\mu(0)}(0) = \left(\frac{1}{\pi n}\right)^{1/2}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}. \tag{7.3}$$

Hence, Lemma 7.1 follows upon inserting (7.3) into (7.1) via (7.2). □

Proof of Theorem 1.1. Thanks to Lemma 6.2 and Fubini's Theorem, we have

$$\begin{aligned}\mathbb{E}[\mathcal{V}(f)] &= \frac{2^n T}{\omega_n}(1 + o_{T\to\infty}(1)) \cdot \int_M \mathbb{E}[\mathcal{V}(F_x)]d\,\mathrm{Vol}(x)\\ &= \frac{2^n\,\mathrm{Vol}(M)T}{\omega_n}(1 + o_{T\to\infty}(1)) \cdot \int_{M\times\Omega} \mathcal{V}(F_x)d\sigma.\end{aligned} \tag{7.4}$$

Thanks to Proposition 4.1, and since Proposition 6.1, valid under the hypotheses of Theorem 1.1, implies the uniform integrability hypothesis [11, (3.15)] of [11, Theorem 3.5], we have

$$\int_{M\times\Omega} \mathcal{V}(F_x)d\sigma = \mathbb{E}[\mathcal{V}(F_\mu)] \cdot (1 + o_{T\to\infty}(1)). \tag{7.5}$$

Combining (7.4), (7.5) and Lemma 7.1, we obtain

$$\begin{aligned}\mathbb{E}[\mathcal{V}(f)] &= \frac{2^n}{\omega_n}\mathrm{Vol}(M)\mathbb{E}[\mathcal{V}(F_\mu)] \cdot (T + o_{T\to\infty}(T))\\ &= \mathrm{Vol}(M)\left(\frac{1}{\pi n}\right)^{1/2}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}T + o_{T\to\infty}(T),\end{aligned}$$

as required. □

8. Asymptotic of the spectral projector for constant energy windows

The purpose of this section is to prove a substitute for Proposition 4.3 under the 'less restrictive' assumption on the energy window width $\rho(T) \equiv \rho_0(M)$ (with arguments working verbatim for

$\rho(T) \geq \rho_0$). Our result holds for all dimensions provided that the following assumption on M holds (cf. Section 1.3 and Theorem 1.3):

Definition 8.1 (Assumption $\mathscr{A}_0$). Let $n \geq 2$ and let (M, g) be a real analytic, compact Riemannian n-manifold. We say that (M, g) satisfies assumption $\mathscr{A}_0$ if either the geodesic flow on M is periodic or the geodesic flow on M is aperiodic and the set of self-focal points of M is of measure 0.

Since, in the case of constant energy windows, the local Weyl's law may fail around some 'bad' points $x \in M$, we will show that the set of such points is small; that is, we prove the following result:

Proposition 8.2. *Let $F_x(\cdot)$ be as in (4.1) (with $\rho \equiv \rho_0$) and μ the normalized Lebesgue measure on the $n-1$ dimensional sphere $\mathbb{S}^{n-1}$. Then, if that M satisfies assumption $\mathscr{A}_0$ as in (8.1), there exists a (measurable) subset $\mathcal{A}_1 = \mathcal{A}_1(T) \subseteq M$ of volume $\mathrm{Vol}(\mathcal{A}_1) = o_{T\to\infty}(1)$ such that*

$$\sup_{\substack{x\in M\setminus\mathcal{A}_1 \\ y,y'\in B_0}} \left| \mathbb{E}[F_x(y)\cdot F_x(y')] - (2\pi)^{\Lambda}\frac{J_\Lambda(|y-y'|)}{|y-y'|^{\Lambda}} \right| \to 0 \qquad T\to\infty,$$

with $\Lambda = (n-2)/2$ and $J_\Lambda(\cdot)$ the Λ-th Bessel function. Moreover, we can also differentiate both sides any arbitrary finite number of times; that is,

$$\mathbb{E}[D^{\alpha}F_x(y)\cdot D^{\alpha'}F_x(y')] = (-1)^{|\alpha'|} i^{|\alpha|+|\alpha'|}\int_{|\xi|=1} \xi^{\alpha+\alpha'}\exp(i\langle y-y',\xi\rangle)d\mu(\xi) + o_{T\to\infty}(1),$$

valid on $x \in M\setminus\mathcal{A}_1$, $y, y' \in B_0$, where α, α' are multi-indices, and $\xi^{\alpha} = (\xi_1^{\alpha_1}, \ldots, \xi_n^{\alpha_n})$.

8.1. Preliminaries: Geodesic flow and the spectrum of $\sqrt{-\Delta}$

For a reference to the facts contained in this section, we suggest the exposition in [60]. Let T^*M and S^*M be the co-tangent and the co-sphere bundle on M, respectively. The geodesic flow

$$G^t : T^*M \to T^*M$$

is the Hamiltonian flow of the metric norm function

$$H : T^*M \to \mathbb{R} \qquad\qquad H(x,\xi) = \sum_{i,j=1}^{n} g^{ij}\xi_i\xi_j,$$

where $g = g_{ij}$ is the metric on M and g^{ij} is its inverse. Since G^t is homogeneous, from now on, we will consider only its restriction to S^*M. We will need the following simple lemma (see also [51, Lemma 1.3.8]):

Lemma 8.3. *If (M, g) is a real analytic manifold, then the set of closed geodesics, on the co-sphere bundle equipped with the Liouville measure, has either full measure or measure zero.*

Proof. Since (M, g) is real analytic, the geodesic flow $G^t(\cdot,\cdot)$ is a real analytic function on $S^\star M$. Therefore, for fixed $t > 0$, solutions to

$$G^t(x,\xi) = (x,\xi)$$

consist of the zero set of an analytic function. This must have codimension at least 1 or be trivial. □

Lemma 8.3 implies that the geodesic flow on a real analytic manifold is either *aperiodic* if the set of closed geodesics has measure zero or *periodic* with (minimal) period $H > 0$ if $G^H = id$. For the former

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case, the two-term Weyl's law of Duistermaat-Guilleimin(-Ivrii) states

$$|\{i > 0 : \lambda_i \le T\}| = c_M T^n + o(T^{n-1}).$$

For the latter case, the spectrum of $\sqrt{\Delta}$ is a union of clusters of the form

$$C_k := \left\{ \frac{2\pi}{H}\left(k + \frac{\beta}{4}\right) + \mu_{k_i} \quad \text{for} \quad i = 1, ..., d_k \right\} \qquad k = 1, 2...,$$

where $\mu_{k_i} = O(k^{-1})$ uniformly for all i, d_k is a polynomial in k of degree $n-1$ and β is the common Morse index of the closed geodesics of M.

8.2. *Local Weyl's law revisited*

The aim of this section is to prove Proposition 8.2. As we will see below, Proposition 8.2 is a direct consequence of Egorov's Theorem and the following:

Proposition 8.4. *Let (M, g) be a compact, real analytic manifold with empty boundary, ρ_0 as in Corollary 4.4, and suppose that either the geodesic flow on M is periodic or $x \in M$ is not a self-focal point (Definition 1.2). Then*

$$\sup_{y,y' \in B_g(x,10/T)} \left| \sum_{\lambda_i \in [T-\rho_0, T]} \phi_i(y)\phi_i(y') - c_M T^n \mathcal{J}_{\Upsilon(T)}(T d_g(y, y')) \right| = o_x(T^{n-1}), \tag{8.1}$$

where $d_g(y, y')$ is the geodesic distance between y, y', $c_M > 0$ is given in (1.3), $\Upsilon(T) = 1 - \frac{\rho_0}{T}$ and

$$\mathcal{J}_{\Upsilon(T)}(w) = \int_{\Upsilon(T) \le |\xi| \le 1} \exp(i\langle w, \xi\rangle) d\xi.$$

Moreover, we can also differentiate both sides of (8.1) an arbitrary finite number of times; that is,

$$\sup_{y,y' \in B_g(x,10/T)} \frac{\left| \sum_{\lambda_i \in [T-\rho_0, T]} D_y^{\alpha}\phi_i(y) D_{y'}^{\alpha'}\phi_i(y') - \frac{c_M T^n D_y^{\alpha} D_{y'}^{\alpha'} \mathcal{J}_{\Upsilon(T)}(T d_g(y,y'))}{(2\pi)^n} \right|}{T^{|\alpha|+|\alpha'|}} = o_x(T^{n-1}),$$

where α, α' are multi-indices, and $\xi^{\alpha} = (\xi_1^{\alpha_1}, ..., \xi_n^{\alpha_n})$ and the derivatives are understood after taking normal coordinates around the point x.

The proof of Proposition 8.4 follows directly from the following two lemmas. In the periodic case, we have a full asymptotic expansion for the spectral projector kernel [58, Theorem 2]; see also [59]. In particular, we have the following:

Lemma 8.5 (Zelditch). *Let (M, g) be a compact, real analytic manifold with empty boundary. Suppose that the geodesic flow on M is periodic (i.e., M is a Zoll manifold). Then the conclusions of Proposition 8.4 hold.*

The second lemma is borrowed from Canzani-Hanin [16, 17]; see also the preceding work of Safarov [52]:

Lemma 8.6. *Let (M, g) be a compact, real analytic manifold with empty boundary, and suppose that $x \in M$ is not self-focal. Then the conclusions of Proposition 8.4 hold.*

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We are finally ready to prove Proposition 8.2:

Proof of Proposition 8.2. First we observe that, under the assumptions of Theorem 1.3, (8.1) and its term-wise differentiation hold for almost all $x \in M$, that is outside a set of measure zero. Indeed, thanks to Lemma 8.3, the geodesic flow on M is either aperiodic or periodic. In the latter case, the conclusion of Proposition 8.4 holds for all $x \in M$. In the former case, Proposition 8.4 holds for almost all $x \in M$. Thus, it remains to show that (8.1) and its term-wise differentiation, holding for almost all $x \in M$ implies the conclusion of Proposition 8.2.

Following along identical lines to the proof of Proposition 4.3 (which we do not reproduce here for the sake of brevity) shows that the function

$$h(x) := \sup_{y,y' \in B_0} \left| \mathbb{E}[F_x(y) \cdot F_x(y')] - (2\pi)^{\Lambda} \frac{J_{\Lambda}(|y-y'|)}{|y-y'|^{\Lambda}} \right|$$

converges point-wise to 0 for almost all $x \in M$. Therefore, Egorov's Theorem implies that there exists a (measurable) set $\mathcal{A}_1 = \mathcal{A}_1(T) \subseteq M$ of volume $\mathrm{Vol}(\mathcal{A}_1) = o(1)$ such that h converges to zero uniformly for all $x \in M \backslash \mathcal{A}_1$. This concludes the proof of the first claim of Proposition 8.2 (recall that the set of relevant T is a discrete subset of $\mathbb{R}$). The proof of the second claim is similar and therefore omitted. □

9. Proof of Theorem 1.3

In order to conclude the proof of Theorem 1.3, we need the following weaker, averaged w.r.t. position, version of Proposition 4.1, valid in all dimensions. The proof of Theorem 1.3 is verbatim the proof of Theorem 1.1, with Proposition 9.1 in place of Proposition 4.1 (see the discussion immediately after Proposition 4.1 and (4.3), in particular).

Proposition 9.1. *Let F_x be as in (4.1), ρ_0 as in Corollary 4.4, and F_μ be as above. Suppose that M satisfies assumption $\mathscr{A}_0$ as in Definition 8.1 and $\rho \equiv \rho_0$. Then one has*

$$\mathcal{V}(F_x) \xrightarrow{d} \mathcal{V}(F_\mu) \qquad T \to \infty, \tag{9.1}$$

where the convergence is in distribution as a random variable on $(M \times \Omega, d\sigma)$.

We stress that the convergence (9.1) is in the product space $(M \times \Omega, d\sigma)$ rather than for an individual $x \in M$ w.r.t. $d\mathbb{P}$. To the best of our knowledge, it is not known whether there exist counter-examples for the latter, stronger convergence; that is, whether, for some M (that might or might not satisfy the assumptions of Theorem 1.3), there exist $x \in M$ with the convergence (9.1) failing as a random function on $(\Omega, \mathbb{P})$.

Assuming Proposition 9.1, we can conclude the proof of Theorem 1.3 along identical lines to the proof of Theorem 1.1.

Proof of Theorem 1.3. Thanks to Lemma 6.2 and Fubini's Theorem, we have

$$\mathbb{E}[\mathcal{V}(f)] = \frac{2^n \,\mathrm{Vol}(M) T}{\omega_n}(1 + o_{T\to\infty}(1)) \cdot \int_{M\times\Omega} \mathcal{V}(F_x) d\sigma.$$

Thanks to Proposition 9.1, and since Proposition 6.1, which remains valid under the assumptions of Theorem 1.3, implies the uniform integrability hypothesis [11, (3.15)] of [11, Theorem 3.5], we have

$$\int_{M\times\Omega} \mathcal{V}(F_x) d\sigma = \mathbb{E}[\mathcal{V}(F_\mu)] \cdot (1 + o_{T\to\infty}(1)).$$

Hence, Lemma 7.1 gives

$$\begin{aligned}\mathbb{E}[\mathcal{V}(f)] &= \frac{2^n}{\omega_n}\mathrm{Vol}(M)\mathbb{E}[\mathcal{V}(F_\mu)]\cdot(T+o_{T\to\infty}(T))\\ &= \mathrm{Vol}(M)\left(\frac{1}{\pi n}\right)^{1/2}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}T+o_{T\to\infty}(T),\end{aligned}$$

as required. □

The rest of the script is dedicated to the proof of Proposition 9.1.

9.1. *Sogge's bound and large values of the eigenfunctions*

In addition to the possible failure of the local Weyl's law around self-focal points, another difficulty of the constant energy window regime is the possibility of

$$\sup_x |\phi_i(x)| \asymp v(T)^{1/2}$$

for some ϕ_i in the summation (1.2) (for example, it might occur for the sphere $\mathbb{S}^n$). This problem was not present in the growing energy window case thanks to Claim 4.7. Thus, we would not be able to apply Lindeberg's CLT as in the proof of Lemma 4.6. In order to circumvent this difficulty, we show that $|\phi_i(x)| = o(T)$ for all $\lambda_i \in [T-\rho_0, T]$ and for all $x \in M$ outside a set of small measure. That is, the main result of this section is to prove the following consequence of (6.15):

Lemma 9.2. *Let $T > 0$ be given, $v(T)$ as in (1.3), ρ_0 as in Corollary 4.4, and let $K = K \geq 1$ be some parameter (that may depend on T). Then there exists a subset $\mathcal{A}_2 = \mathcal{A}_2(T,K) \subseteq M$ of volume at most $O(K^{2\frac{n+1}{n-1}}T^{-1})$ with the following properties:*

(1) *We have*

$$\sup_{x\in M\setminus\mathcal{A}_2}\max_{\lambda_i\in[T-\rho_0,T]}||\phi_i||_{L^\infty(B(x,2/T))} \lesssim K^{-1}v(T)^{1/2}.$$

(2) *Uniformly for all multi-indices $|\alpha| \leq 2$, one has*

$$\sup_{x\in M\setminus\mathcal{A}_2}\max_{\lambda_i\in[T-\rho_0,T]}||T^{-\alpha}D^\alpha\phi_i||_{L^\infty(B(x,2/T))} \lesssim K^{-1}v(T)^{1/2}.$$

In order to state a preliminary result towards the proof of Lemma 9.2, we recall some notation. Given a Laplace eigenfunction ϕ_i, we denote by $\phi_{i,x}$ the scaled restriction of ϕ_i to $B_g(x, 4/T)$ via the exponential map; that is,

$$\phi_{i,x}(y) = \phi_i(\exp_x(y/T))$$

for $y \in B(0,4)$ (here we tacitly assume that T is sufficiently large so that $4/T$ is less than the injectivity radius). With this notation in mind, we prove the following consequence of elliptic regularity for harmonic functions:

Lemma 9.3. *Let $T \geq 1$, ρ_0 as in Corollary 4.4, and let ϕ_i be a Laplace eigenfunction with eigenvalue $\lambda_i \in [T-\rho_0, T]$. Then*

(1) *Uniformly for all $x \in M$, we have*

$$\sup_{B_g(x,2/T)}|\phi_i|^2 \lesssim \int_{B(0,4)}|\phi_{i,x}(y)|^2dy.$$

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(2) *Uniformly for all $x \in M$, we have*

$$\sup_{B_g(x,2/T)} |T^{-\alpha} D^{\alpha} \phi_i|^2 \lesssim \int_{B(0,4)} |\phi_{i,x}(y)|^2 dy,$$

uniformly for all multi-indices $|\alpha| \leq 2$.

Before embarking on the proof of Lemmas 9.2 and 9.3, we would like to briefly discuss their statements. Lemmas 9.2 and 9.3 are stated in the precise form that will be used in Section 4.3. However, in the literature, the conclusions of Lemmas 9.2 and 9.3 are often stated as

$$\sup_{B_g(x,c/\lambda_i)} |\phi_i| \lesssim K^{-1} v(T)^{1/2} \qquad \sup_{B_g(x,c/\lambda_i)} |\phi_i|^2 \lesssim \int_{B(0,2c)} |\phi_{i,x}(y)|^2 dy \tag{9.2}$$

for some small $c = c(M)$ with an analogous statement for the bounds on the derivatives. Since $\rho_0 = O_M(1)$, and for $\lambda_i \in [T - \rho_0, T]$,

$$\lambda_i^{-1} = T^{-1}(1 + o(1)),$$

(9.2) is equivalent to Lemmas 9.2 and 9.3 up to the constant 2 and a simple covering argument.

Proof of Lemma 9.3. Given ϕ_i, let us consider the function $h(x,t) = \phi_i(x)e^{\lambda_i t}$ defined on $M \times [-2, 2]$ and let us write $h_T(\cdot) = h(T^{-1}\cdot)$ (where the rescaling is to be understood in normal coordinates). Then, since the supremum norm is scale invariant, we have

$$\sup_{B_g(x,2/T)} |\phi_i| \lesssim \sup_{B_g(x,2/T)\times[-2/T,2/T]} |h| \lesssim ||h_T||_{L^\infty(\tilde{B})} \tag{9.3}$$

$$\sup_{B_g(x,2/T)} |D^{\alpha}\phi_i| \lesssim \sup_{B_g(x,2/T)\times[-2/T,2/T]} |D^{\alpha} h| \lesssim T^{\alpha} ||h_T||_{C^1(\tilde{B})}, \tag{9.4}$$

where $\tilde{B} = B_g(x, 2) \times [-2, 2]$ and $|\alpha| \leq 2$ is a multi-index. Since h is a harmonic function ($\Delta h = 0$) and $\tilde{B}$ has radius 4, for any $k \geq 0$, elliptic regularity [23, Page 330] gives

$$||h_T||_{C^k(\tilde{B})} \lesssim_k ||h_T||_{L^2(2\tilde{B})}. \tag{9.5}$$

The constant implied in the notation is independent of $x \in M$. Thus, Lemma 9.3 follows by inserting (9.5) into (9.3) and (9.4) and noticing that $||h_T||_{L^2(2\tilde{B})} \lesssim ||\phi_{i,x}||_{L^2(B(0,4))}$. □

Proof of Lemma 9.2. First, we observe that given $p \geq 2$, the function $x \to x^{p/2}$ is convex for $x \geq 0$. Therefore, applying Jensen's inequality to part (1) of Lemma 9.3, we obtain

$$\left(\sup_{B_g(x,2/T)} |\phi_i|\right)^p \lesssim_p \left(\int_{B(0,4)} |\phi_{i,x}(y)|^2 dy\right)^{p/2} \lesssim_p \int_{B(0,4)} |\phi_{i,x}(y)|^p dy \tag{9.6}$$

and similarly,

$$\left(\sup_{B_g(x,2/T)} |T^{-\alpha} D^{\alpha} \phi_i|\right)^p \lesssim_p \int_{B(0,4)} |\phi_{i,x}(y)|^p dy, \tag{9.7}$$

where $|\alpha| \leq 2$ is a multi-index. We are now going to prove part (1) of Lemma 9.2. By Sogge's bound (6.15) with $p \leq 2(n+1)/(n-1)$, bearing in mind that $||\phi_i||_{L^2} = 1$, we have

$$\left(\int_M |\phi_i(x)|^p d\,\mathrm{Vol}_g(x)\right)^{1/p} \lesssim T^{\frac{n-1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)} =: \tilde{T}$$

for all $\lambda_i \leq T$. Thus, integrating both sides of (9.6) with respect to $x \in M$ and exchanging the order of the integrals, we obtain

$$\int_M \left(\sup_{B_g(x,2/T)} |\phi_i|\right)^p d\,\mathrm{Vol}_g(x) \lesssim \int_{B(0,4)} \int_M |\phi_{i,x}(y)|^p d\,\mathrm{Vol}_g(x) dy \lesssim \tilde{T}^p.$$

Therefore, by Chebyshev's bound, for any $K_1 > 0$, we have

$$\mathrm{Vol}_g\left(\left\{x \in M : \sup_{B_g(x,2/T)} |\phi_i| \geq K_1\right\}\right) \lesssim K_1^{-p}\tilde{T}^p,$$

and taking the union bound over the $O(v(T))$ choices for i, we deduce

$$\mathrm{Vol}_g\left(\left\{x \in M : \max_{\lambda_i \in [T-\rho_0, T]} \sup_{B_g(x,2/T)} |\phi_i| \geq K_1\right\}\right) \lesssim K_1^{-p} v(T) \cdot \tilde{T}^p. \tag{9.8}$$

Thus, taking $K_1 = K^{-1}v(T)^{1/2} \gtrsim K^{-1}(T^{n-1})^{1/2}$ in (9.8) and recalling that $\tilde{T} = T^{\frac{n-1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)}$, we have

$$\mathrm{Vol}_g\left(\left\{x \in M : \max_{\lambda_i \in [T-\rho_0, T]} \sup_{B_g(x,2/T)} |\phi_i| \geq K^{-1}v(T)^{1/2}\right\}\right) \lesssim_p K^p T^{\nu(n,p)}, \tag{9.9}$$

where

$$\begin{aligned}\nu(n,p) : &= -p\frac{n-1}{2} + n - 1 + \frac{n-1}{2}\left(\frac{p}{2} - 1\right)\\ &= \frac{n-1}{2}\left(1 - \frac{p}{2}\right).\end{aligned}$$

Hence, taking $p = 2(n+1)/(n-1)$ in (9.9), we have

$$\mathrm{Vol}_g\left(\left\{x \in M : \max_{\lambda_i \in [T-\rho_0, T]} \sup_{B_g(x,2/T)} |\phi_i| \geq K^{-1}v(T)^{1/2}\right\}\right) \lesssim K^p T^{-1},$$

as required. Thanks to (9.7), the proof of part (2) of Lemma 9.2 follows along the lines of the proof of its part (1). □

9.2. Concluding the proof of Proposition 9.1

As we saw in the course of the proof of Proposition 4.1, to prove Proposition 9.1, it is sufficient to consider the convergence of finite-dimensional distributions. Indeed, Lemma 4.10 implies that the measure induced by F_x (as a random variable on $(M \times \Omega, d\sigma)$) onto $C^2(B_0)$ is tight. Thus, to conclude the proof of Proposition 9.1, it is sufficient to prove the following result:

Lemma 9.4 (Convergence of finite-dimensional distributions). *Let m be some positive integer, $B_0 = B(0,1)$, ρ_0 as in Corollary 4.4, F_x be as in 4.1 and F_μ be the random monochromatic wave as in (4.2).*

Then, assuming that M satisfies the assumption $\mathscr{A}_0$ *as in* (8.1) *and* $\rho \equiv \rho_0$*, for every* $y_1, \dots y_m \in B_0 \subseteq \mathbb{R}^n$*, we have*

$$(F_x(y_1), \dots, F_x(y_m)) \xrightarrow{d} (F_\mu(y_1), \dots, F_\mu(y_m)) \qquad T \to \infty,$$

where the convergence is in distribution as a random vector defined on $(M \times \Omega, d\sigma)$*. Moreover, for every* $\alpha = (\alpha_1, \dots, \alpha_n)$*, with* $|\alpha| \le 2$*, one has*

$$(D^\alpha F_x(y_1), \dots, D^\alpha F_x(y_m)) \xrightarrow{d} (D^\alpha F_\mu(y_1), \dots, D^\alpha F_\mu(y_m)) \qquad T \to \infty.$$

Using some classical probability language [11, Theorem 2.6], we may reformulate Lemma 9.4 as follows:

Lemma 9.5. *Let* $B_0 = B(0,1)$*,* ρ_0 *as in Corollary 4.4,* F_x *be as in* (4.1)*, and* F_μ *be as in* (4.2)*. Assuming that M satisfies assumption* $\mathscr{A}_0$ *of Definition 8.1 and* $\rho \equiv \rho_0$*, there exists a set* $\mathcal{A}_3 = \mathcal{A}_3(T) \subseteq M$ *of volume* $\mathrm{Vol}(\mathcal{A}_3) = o_{T\to\infty}(1)$*, such that the following holds. Given a uniformly continuous and bounded function* $g : \mathbb{R}^m \to \mathbb{R}$*, as* $T \to \infty$*, one has*

$$\sup_{x\in M\setminus\mathcal{A}_3} \left| \int_\Omega g(F_x(y_1), \dots, F_x(y_m)) d\mathbb{P} - \int_\Omega g(F_\mu(y_1), \dots, F_\mu(y_m)) d\mathbb{P} \right| \to 0,$$

and, for all multi-index $|\alpha| \le 2$*, one also has*

$$\sup_{x\in M\setminus\mathcal{A}_3} \left| \int_\Omega g(D^\alpha F_x(y_1), \dots, D^\alpha F_x(y_m)) d\mathbb{P} - \int_\Omega g(D^\alpha F_\mu(y_1), \dots, D^\alpha F_\mu(y_m)) d\mathbb{P} \right| \to 0.$$

For the reader's convenience, we provide a proof that Lemma 9.5 implies Lemma 9.4:

Proof of Lemma 9.4 assuming Lemma 9.5. By Portmanteau Theorem [11, Theorem 2.1], Lemma 9.4 is equivalent to the following:

$$\int_{M\times\Omega} g(F_x(y_1), \dots, (F_x(y_m)) d\sigma \to \int_\Omega g(F_\mu(y_1), \dots, (F_\mu(y_m)) d\mathbb{P} \qquad T \to \infty \tag{9.10}$$

for all bounded and uniformly continuous functions $g : \mathbb{R}^m \to \mathbb{R}$. Suppose that (9.10) fails for some g. Then there exists some $\varepsilon = \varepsilon(g) > 0$ such that

$$\left| \int_{M\times\Omega} g(F_x(y_1), \dots, (F_x(y_m)) d\sigma - \int_\Omega g(F_\mu(y_1), \dots, (F_\mu(y_m)) d\mathbb{P} \right| \ge \varepsilon$$

along a subsequence $T_i \to \infty$. Now, let $\mathcal{A}_3 \subseteq M$ be as in Lemma 9.5. Then

$$\begin{aligned} \int_{M\times\Omega} g(F_x(y_1), \dots, F_x(y_m)) d\sigma &= \int_{(M\setminus\mathcal{A}_3)\times\Omega} g(F_x(y_1), \dots, F_x(y_m)) d\sigma + o_g(1) \\ &= \int_{(M\setminus\mathcal{A}_3)\times\Omega} g(F_\mu(y_1), \dots, F_\mu(y_m)) d\sigma + o_g(1), \end{aligned}$$

making use of g being bounded. Therefore, we have

$$\left| \int_{M\times\Omega} g(F_x(y_1), \dots, (F_x(y_m)) d\sigma - \int_\Omega g(F_\mu(y_1), \dots, (F_\mu(y_m)) d\mathbb{P} \right| < \varepsilon$$

for all sufficiently large $T \ge T_0$. This contradiction concludes the proof of Lemma 9.4 □

We are now going to prove Lemma 9.5. The proof is similar to the proof of Lemma 4.6, but we reproduce it for completeness:

Proof of Lemma 9.5. Let $\phi_{i,x}$ be the restriction of ϕ_i to $B_g(x, 1/T)$ and let $\mathcal{A}_3 = \mathcal{A}_1 \cup \mathcal{A}_2$, where $\mathcal{A}_1$ is the exceptional set prescribed by Proposition 8.2 and $\mathcal{A}_2$ is the set constructed within Lemma 9.2 applied with $K = (\log T)^{\frac{n-1}{2(n+1)}} = (\log T)^c$. By Lemma 9.2, for all $x \in M \backslash \mathcal{A}_3$, we have

$$\max_{\lambda_i \in [T-\rho,T]} \sup_{B_g(x,2/T)} |\phi_i| \lesssim \frac{v(T)^{1/2}}{(\log T)^c},$$

where the constant implied in the '$\lesssim$' notation is absolute. Moreover, given and multi-index $|\alpha| \leq 2$, bearing in mind that

$$\sup_{B_0} |D^\alpha \phi_{i,x}| \lesssim \sup_{B_g(x,2/T)} |T^{-\alpha} D^\alpha \phi_i|,$$

we also have

$$\max_{\lambda_i \in [T-\rho,T]} \sup_{B_0} |D^\alpha \phi_{i,x}| \lesssim \frac{v(T)^{1/2}}{(\log T)^c}. \tag{9.11}$$

We are going to first consider the distribution of the vector $(F_x(y_1), \ldots, F_x(y_m))$ for $x \in M \backslash \mathcal{A}_3$. Thanks to Proposition 8.2, we have

$$\sup_{\substack{i,j \in \{1,\ldots,m\} \\ x \in M \backslash \mathcal{A}_3}} \left| \mathbb{E}[F_x(y_i) \cdot F_x(y_j)] - \mathbb{E}[F_\mu(y_i) \cdot F_\mu(y_j)] \right| \to 0 \qquad T \to \infty. \tag{9.12}$$

Therefore, by the multidimensional version of Lindeberg's Central Limit Theorem (Lemma 4.8), and upon using (9.12), it suffices to prove that for every $\varepsilon > 0$, we have

$$\sup_{\substack{y \in B_0 \\ x \in M \backslash \mathcal{A}_3}} \frac{1}{v(T)} \sum_{\lambda_i} \mathbb{E}[|a_i \phi_{i,x}(y)|^2 \mathbb{1}_{|a_i \phi_{i,x}(y)| > \varepsilon v(T)^{1/2}}] \to 0 \qquad T \to \infty \tag{9.13}$$

uniformly for all $y \in B_0$ and all $x \in M \backslash \mathcal{A}_3$, where $\mathbb{1}$ is the indicator function and $v(T) = c_M \rho T^{n-1}(1 + o(1))$. Mind that the convergence of $(F_x(y_1), \ldots, F_x(y_m))$ is not asserted for a single fixed $x \in M$, but rather for any sequence of 'good' x, and therefore, as a random variable on $M \times \Omega$, in accordance with the assertion of Lemma 9.5; see the explanation in Section 2. The calculation leading to (9.13) is identical to the calculation in Lemma 4.6 and is therefore omitted.

In order to prove the convergence of the derivative vector and upon recalling the second part of Proposition 8.2, again by the multidimensional version of Lindeberg's Central Limit Theorem, it is sufficient to prove that for any $\varepsilon > 0$ and $|\alpha| \leq 2$, we have

$$\sup \frac{1}{v(T)} \sum_{\lambda_i} \mathbb{E}[|a_i D^\alpha \phi_{i,x}(y)|^2 \mathbb{1}_{|a_i D^\alpha \phi_{i,x}(y)| > \varepsilon v(T)^{1/2}}] \to 0 \qquad T \to \infty. \tag{9.14}$$

Similar to the above argument, (9.11) implies (9.14) if $|\alpha| \leq 2$, thus concluding the proof of Lemma 9.5. $\square$

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References

[1] G. Alessandrini, L. Rondi, E. Rosset and S. Vessella, 'The stability for the Cauchy problem for elliptic equations', *Inverse Problems* **25**(12) (2009), 123004.
[2] J. Angst, V.-H. Pham and G. Poly, 'Universality of the nodal length of bivariate random trigonometric polynomials', *Trans. Amer. Math. Soc.* **370** (2018), 8331–8357.
[3] J. Azais and M. Wschebor, *Level Sets and Extrema of Random Processes and Fields* (Wiley, New York, 2009).
[4] V. Bally, L. Caramellino and G. Poly, 'Non universality for the variance of the number of real roots of random trigonometric polynomials', *Probab. Theory Related Fields* **174** (2019), 887–927.
[5] D. Beliaev, M. McAuley, and S. Muirhead, 'Fluctuations of the number of excursion sets of planar gaussian fields', Preprint, 2019, arXiv:1908.10708.
[6] D. Beliaev and I. Wigman, 'Volume distribution of nodal domains of random band-limited functions', *Probab. Theory Related Fields* **172** (2018), 453–492.
[7] P. Bérard, 'Volume des ensembles nodaux des fonctions propres du laplacien', in *Séminaire de Théorie Spectrale et Géométrie, Année 1984–1985*. (Univ Grenoble I, Saint-Martin-d'Hères, 1985), IV.1–IV.9.
[8] A. C. Berry, 'The accuracy of the Gaussian approximation to the sum of independent variates', *Trans. Amer. Math. Soc.* **49** (1941), 122–136.
[9] M. V. Berry, 'Regular and irregular semiclassical wavefunctions', *Journal of Physics A: Mathematical and General* **10** (1977), 2083.
[10] P. Billingsley, *Probability and Measure* (John Wiley & Sons, 2008).
[11] P. Billingsley, *Convergence of Probability Measures* (John Wiley & Sons, 2013).
[12] J. Bourgain, 'On toral eigenfunctions and the random wave model, *Israel J. Math.* **201** (2014), 611–630.
[13] J. Brüning, 'Über Knoten von Eigenfunktionen des Laplace-Beltrami-Operators', *Math. Z.* **158** (1978), 15–21.
[14] J. Brüning and D. Gromes, 'Über die Länge der Knotenlinien schwingender Membranen', *Math. Z.* **124** (1972), 79–82.
[15] J. Buckley and I. Wigman, 'On the number of nodal domains of toral eigenfunctions', **17** (2016), 3027–3062.
[16] Y. Canzani and B. Hanin, 'Scaling limit for the kernel of the spectral projector and remainder estimates in the pointwise Weyl law', *Anal. PDE* **8** (2015), 1707–1731.
[17] Y. Canzani and B. Hanin, 'C^∞ scaling asymptotics for the spectral projector of the Laplacian', *J. Geom. Anal*. **28** (2018), 111–122.
[18] Y. Canzani and B. Hanin, 'Local universality for zeros and critical points of monochromatic random waves', *Comm. Math. Phys.* **378** (2020), 1677–1712.
[19] M.-C. Chang, H. Nguyen, O. Nguyen and V. Vu, 'Random eigenfunctions on flat tori: universality for the number of intersections', *Int. Math. Res. Not. IMRN* (2020), 9933–9973.
[20] J. Cheeger, 'Finiteness theorems for Riemannian manifolds', *Amer. J. Math.* **92** (1970), 61–74.
[21] H. Donnelly and C. Fefferman, 'Nodal sets of eigenfunctions on reimannian manifolds', *Inventiones mathematicae* **93** (1988), 161–183.
[22] C.-G. Esseen, 'On the Liapounoff limit of error in the theory of probability', *Ark. Mat. Astr. Fys.* **28A** (1942), 19.
[23] L. C. Evans, *Partial Differential Equations*, vol. 19 of Graduate Studies in Mathematics (American Mathematical Society, Providence, RI, 1998).
[24] H. Flasche and Z. Kabluchko, 'Expected number of real zeroes of random Taylor series', *Commun. Contemp. Math.* **22** (2020), 1950059.
[25] N. Garofalo and F.-H. Lin, 'Monotonicity properties of variational integrals, a p weights and unique continuation', *Indiana Univ. Math. J.* **35** (1986), 245–268.
[26] L. Gass, 'Almost sure asymptotics for riemannian random waves', Preprint, 2020, arXiv:2005.06389.
[27] L. Hörmander, *Linear Partial Differential Operators* (Die Grundlehren der mathematischen Wissenschaften) Bd. 116 (Academic Press, Inc., Publishers, New York); (Springer-Verlag, Berlin- Göttingen -Heidelberg, 1963).
[28] L. Hörmander, 'The spectral function of an elliptic operator', *Acta Math.* **121** (1968), 193–218.
[29] I. A. Ibragimov and N. B. Maslova, 'The mean number of real zeros of random polynomials. I. Coefficients with zero mean', *Teor. Verojatnost. i Primenen.* **16** (1971), 229–248.
[30] I. A. Ibragimov and N. B. Maslova, 'The mean number of real zeros of random polynomials. II. Coefficients with a nonzero mean', *Teor. Verojatnost. i Primenen.* **16** (1971), 495–503.
[31] M. Krishnapur, P. Kurlberg and I. Wigman, 'Nodal length fluctuations for arithmetic random waves', *Ann. of Math.* **177** (2013), 699–737.
[32] I. Kukavica, 'Hausdorff measure of level sets for solutions of parabolic equations', *Internat. Math. Res. Notices* (1995), 671–682.

[33] I. Kukavica, 'Nodal volumes for eigenfunctions of analytic regular elliptic problems', *J. Anal. Math.* **67** (1995), 269–280.
[34] I. Kukavica, 'Quantitative uniqueness for second-order elliptic operators', *Duke Math. J.* **91** (1998), 225–240.
[35] F.-H. Lin, 'Nodal sets of solutions of elliptic and parabolic equations', *Comm. Pure Appl. Math.* **44** (1991), 287–308.
[36] A. Logunov, 'Nodal sets of Laplace eigenfunctions: polynomial upper estimates of the Hausdorff measure', *Ann. of Math. (2)* **187** (2018), 221–239.
[37] A. Logunov, 'Nodal sets of Laplace eigenfunctions: proof of Nadirashvili's conjecture and of the lower bound in Yau's conjecture', *Ann. of Math. (2)* **187** (2018), 241–262.
[38] A. Logunov and E. Malinnikova, 'Nodal sets of Laplace eigenfunctions: estimates of the Hausdorff measure in dimensions two and three', in *50 Years with Hardy Spaces* (Oper. Theory Adv. Appl.) vol. 261 (Birkhäuser /Springer, Cham, 2018), 333–344.
[39] D. Marinucci, G. Peccati, M. Rossi and I. Wigman, 'Non-universality of nodal length distribution for arithmetic random waves', *Geom. Funct. Anal.* **26** (2016), 926–960.
[40] D. Marinucci, M. Rossi and I. Wigman, 'The asymptotic equivalence of the sample trispectrum and the nodal length for random spherical harmonics', *Ann. Inst. Henri Poincaré Probab. Stat.* **56** (2020), 374–390.
[41] C. S. M. Christ and C. E. Kenig, *Harmonic Analysis and Partial Differential Equations: Essays in Honor of Alberto P. Calderon* (Chicago Lectures in Mathematics) (University of Chicago Press, 2001).
[42] F. Nazarov and M. Sodin, 'On the number of nodal domains of random spherical harmonics', *Amer. J. Math.* **131** (2009), 1337–1357.
[43] F. Nazarov and M. Sodin, 'Asymptotic laws for the spatial distribution and the number of connected components of zero sets of gaussian random functions', *J. Math. Phys. Anal. Geom.* **12** (2016), 205– 278.
[44] F. Nazarov and M. Sodin, 'Fluctuations in the number of nodal domains', *J. Math. Phys.* **61** (2020), 123302.
[45] O. Nguyen and V. Vu, 'Random polynomials: central limit theorems for the real roots', *Duke Math J.*, To appear, 2020, arXiv:1904.04347.
[46] F. Oravecz, Z. Rudnick and I. Wigman, 'The Leray measure of nodal sets for random eigenfunctions on the torus', *Ann. Inst. Fourier (Grenoble)* **58** (2008), 299–335.
[47] S. M. Prigarin, 'Weak convergence of probability measures in the spaces of continuously differentiable functions', *Sibirskii Matematicheskii Zhurnal* **34** (1993), 140–144.
[48] Á. Romaniega and A. Sartori, 'Nodal set of monochromatic waves satisfying the random wave model, Preprint, 2020, https://arxiv.org/abs/2011.03467.
[49] Y. Rozenshein, 'The number of nodal components of arithmetic random waves', *Int. Math. Res. Not. IMRN* 2017 (**2017**), 6990–7027.
[50] Z. Rudnick and I. Wigman, 'On the volume of nodal sets for eigenfunctions of the Laplacian on the torus', *Ann. Henri Poincaré* **9** (2008), 109–130.
[51] Y. Safarov and D. Vassiliev, *The Asymptotic Distribution of Eigenvalues of Partial Differential Operators* (Translations of Mathematical Monographs) vol. 155 (American Mathematical Society, Providence, RI, 1997). Translated from the Russian manuscript by the authors.
[52] Y. G. Safarov, 'Asymptotics of a spectral function of a positive elliptic operator without a nontrapping condition', *Funktsional. Anal. i Prilozhen.* **22** (1988), 53–65, 96.
[53] P. Sarnak and I. Wigman, 'Topologies of nodal sets of random band-limited functions', *Comm. Pure Appl. Math.* **72** (2019), 275–342.
[54] C. D. Sogge, 'Concerning the L^p norm of spectral clusters for second-order elliptic operators on compact manifolds', *J. Funct. Anal.* **77** (1988), 123–138.
[55] H. Whitney and F. Bruhat, 'Quelques propriétés fondamentales des ensembles analytiques-réels', *Comment. Math. Helv.* **33** (1959), 132–160.
[56] I. Wigman, 'Fluctuations of the nodal length of random spherical harmonics', *Comm. Math. Phys.* **298** (2010), 787–831.
[57] S. T. Yau, 'Survey on partial differential equations in differential geometry', in *Seminar on Differential Geometry* (Ann. of Math. Stud.) vol. 102 (Princeton Univ. Press, Princeton, N.J., 1982), 3–71.
[58] S. Zelditch, 'Fine structure of Zoll spectra', *J. Funct. Anal.* **143** (1997), 415–460.
[59] S. Zelditch, 'Real and complex zeros of Riemannian random waves', in *Spectral Analysis in Geometry and Number Theory* vol. 484 (Amer. Math. Soc. , Providence, RI, 2009), 321–342.
[60] S. Zelditch, *Eigenfunctions of the Laplacian on a Riemannian Manifold* (CBMS Regional Conference Series in Mathematics) vol. 125 (Published for the Conference Board of the Mathematical Sciences, Washington, DC by the American Mathematical Society, Providence, RI, 2017).