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CONVERGENCE IN LOTKA-VOLTERRA TYPE DIFFUSIVE DELAY SYSTEMS WITHOUT DOMINATING INSTANTANEOUS NEGATIVE FEEDBACKS

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Abstract

This paper deals with the convergence aspect of diffusive delay Lotka-Volterra systems with infinite delays. It is well known that such a system has a globally asymptotically stable steady state if the negative feedbacks of the intraspecific competitions are dominant and instantaneous. It is shown here that such a globally asymptotically stable steady state continues to exist even if the instantaneous assumption is removed, provided that solutions of the system are eventually uniformly bounded and the delays involved in the intraspecific competitions are small. This work generalises several recent related ones.

1. Introduction

Ordinary differential equations have long played a central role in the modelling of various real systems and will no doubt continue to serve as important tools in future scientific investigations. However, frequently, more realistic models require the inclusion of the effects of both time delays and spatial variations. This is especially important in population dynamics, as pointed out by Okubo [43] " ... time and space are inseparable 'sister coordinates,' and only when population of organisms are considered in both time and space can the ecological situation be understood." Ideally, a real ecological system should be modeled by differential equations (ODEs for discrete diffusions in patchy environment, PDEs for continuous diffusions) with time delays.

Systematic study on abstract functional differential equations reaction-diffusion systems with time delays has been documented in the recent papers

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of Martin and Smith [31–32]. Recent and specific results on Lotka-Volterra type diffusive delay equations can be found in the work of Luckhaus [29], Friesecke [7, 8], and Yamada [40]. Results for systems can be found in Gopalsamy [10], Kuang and Smith [26], Martin and Smith [30, 32] and the references cited therein.

The results of [10, 26, 30] are all concerned with the global stability of a steady state of the considered diffusive delay Lotka-Volterra system. These results have a distinct common feature: in order to show that a globally asymptotically stable steady state exists in these systems, it is imperative to assume, mathematically, that the negative feedbacks of the intraspecific competitions are dominant and instantaneous. While the dominant requirement sounds reasonable ecologically, the instantaneous requirement is certainly rather artificial since some kinds of time delays are inevitable in any population interactions. This ecological consideration is the main motivation of the present paper.

Other motivations come from the recent works of Kuang [24] and Kuang and Smith [25]. In these two papers, a technique is developed to remove the instantaneous requirement for intraspecific competitions in order to establish global stability of steady state in delayed Lotka-Volterra systems. Roughly, we replace a delayed term, say $u(t-\tau)$, by $u(t)+u'(\xi)\tau$, where $\xi \in [t-\tau, t]$ is determined by the mean value theorem. This involves the estimation of $u'(t)$ in terms of $u(t)$. In [24] and [25], it is shown that the attractivity of a steady state remains unchanged as long as the delays involved in the intraspecific competitions are small and initial values are selected in a proper space. In this paper we attempt to do the same for diffusive delay systems. In order to do so, we need to adopt the approach of Hutson and Moran [21] to estimate Laplacians, and thus derivatives, with respect to time. Then we combine the methods developed in Kuang and Smith [25] and Martin and Smith [30] to establish the desired convergence. Our main result here indicates somehow that a globally asymptotically stable steady state indeed continues to exist even if the instanteous assumption is removed, provided that solutions of the diffusive delay system are eventually uniformly bounded and the delays involved in the intraspecific competitions are small enough. This in some sense provides reasonable support for the usual practice of ignoring time delay effects in models of population dynamics when the delay lengths are deemed to be small.

This paper is organised as follows. In the next section we describe our model in detail and introduce some proper notations and Banach spaces that will be used in subsequent sections. Section 3 deals with the estimation of partial derivatives with respect to the time coordinate. Section 4 contains our main results. The last section is devoted to discussion.

2. Preliminaries

In this paper we consider the following autonomous Lotka-Volterra type diffusive delay system with infinite delays

$$\partial_t u_i(x, t) = d_i \Delta u_i(x, t) + b_i u_i(x, t) G_i(u_t(x, \cdot)), \text{ on } \Omega \times (0, \infty); \quad (2.1a)$$

$$\partial_\nu u_i(x, t) = 0, \quad \text{on } \partial\Omega \times (0, \infty); \quad (2.1b)$$

$$u_i(x, \theta) = \xi_i(x, \theta), \quad \text{on } \overline{\Omega}(-\infty, 0]; \quad (2.1c)$$

where $i = 1, 2, \dots, n$, $u = (u_1, u_2, \dots, u_n)$, $x = (x_1, \dots, x_N)$ and Ω is an open, connected and bounded region in $\mathbf{R}^N$ with $\partial\Omega$ smooth enough, say $\partial\Omega \in C^{2+\alpha}$, $0 < \alpha < 1$. Furthermore, Δ denotes the Laplacian on Ω, ∂_ν denotes the outward normal derivative on $\partial\Omega$; d_i and b_i are positive constants. The initial functions $\xi_i : \overline{\Omega} \times (-\infty, 0] \to \mathbf{R}$ *are always assumed to be continuous and nonnegative, with* $\xi_i(x, 0) > 0$, $i = 1, 2, \dots, n$. In the following, additional conditions will be imposed on ξ_i. Throughout the rest of this paper, $C(-\infty, 0]^n$ denotes the space of continuous functions from $(-\infty, 0]$ into $\mathbf{R}^n$ and for each $x \in \overline{\Omega}$, $u_t(x, \cdot) = (u_{it}(x, \cdot))_1^n$ denotes the member of $C(-\infty, 0]^n$ defined by $u_t(x, \theta) \equiv u(x, t+\theta)$ for $\theta \in (-\infty, 0]$. The functionals G_i in (2.1a) map $C(-\infty, 0]^n$ into $\mathbf{R}$ and are assumed to have the form

$$g_i(u_t, (x, \cdot)) = r_i - a_i \int_{-\tau_i}^0 u_{it}(x, \theta)\, d\mu_i(\theta) + \sum_{j=1}^n \int_{-\infty}^0 u_{jt}(x, \theta)\, d\mu_{ij}(\theta), \quad (2.1d)$$

where $a_i > 0$, $\tau_i > 0$ and r_i are real numbers; $\mu_i(\theta)$ are nondecreasing, $\mu_i(0) - \mu_i(-\tau_i) = 1$; $\mu_{ij}(\theta)$ are bounded real-valued Borel measures on $(-\infty, 0]$ with total variation $|\mu_{ij}|$.

The system (2.1a–d) can be used to model the population dynamics of a closed ecological system containing n interacting species (which means there is no immigration and emigration); $u_i(x, t)$ may represent the population density of the ith species at time t and location x and $r = (r_1, r_2, \dots, r_n)$ may stand for the vector of intrinsic population growth rates.

It is natural from a biological point of view to seek a solution of (2.1a–d) corresponding to nonnegative initial data belonging to the Banach space $CB(\overline{\Omega} \times (-\infty, 0])$ of bounded and continuous functions that map $\overline{\Omega} \times (-\infty, 0]$ into $\mathbf{R}^n$, with the uniform norm

$$\|\xi(x, \theta)\| = \sup_{x \in \overline{\Omega}, \theta \le 0} |\xi(x, \theta)|, \quad (2.2)$$

where $\xi \in BC(\overline{\Omega} \times (-\infty, 0])$ and $|\cdot|$ is a chosen norm on $\mathbf{R}^n$. Observe that

$G_i(u_t(x, \cdot))$ can be rewritten as

$$G_i(u_t(x, \cdot)) = r_i(t) - a_i \int_{-\tau_i}^{0} u_{it}(x, \theta)\, d\mu_i(\theta) + \sum_{j=1}^{n} \int_{-\tau}^{0} u_{jt}(x, \theta)\, d\mu_{ij}(\theta), \tag{2.3}$$

where $\tau = \max\{\tau_i : i = 1, 2, \dots, n\}$, and

$$r_i(t) = r + \sum_{j=1}^{n} \int_{-\infty}^{-\tau} u_{jt}(x, \theta)\, d\mu_{ij}(\theta). \tag{2.4}$$

Since for $t \leq \tau$, $r_i(t)$ is known, we see that $u(x, t)$ is a solution of (2.1a–d) with infinite delays if and only if it is a solution of the following nonautonomous system with bounded delays

$$\partial_t u_i(x, t) = d_i \Delta u_i(x, t) + b_i u_i(x, t) \left[r_i(t) - a_i \int_{-\tau_i}^{0} u_{it}(x, \theta)\, d\mu_i(\theta) + \sum_{j=1}^{n} \int_{-r}^{0} u_{jt}(x, \theta)\, d\mu_{ij}(\theta) \right], \quad \text{on } \Omega \times (0, \infty]); \tag{2.5a}$$

$$\partial_\nu u_i(x, t) = 0, \qquad \text{on } \partial\Omega \times (0, \infty); \tag{2.5b}$$

$$u_i(x, \theta) = \xi_i(x, \theta), \qquad \text{on } \overline{\Omega} \times [-\tau, 0], \tag{2.5c}$$

where $r_i(t)$ are defined as in (2.4). Therefore, local existence, uniqueness and continuability of a mild solution of (2.1a–d) follow from results in [31, 32]. By a mild solution of (2.1a–d), we mean a continuous function $u: [0, t^*) \to C(\overline{\Omega}, \mathbf{R}^n)$ which satisfies a certain integral equation obtained by applying the variation of constant formula to the abstract formulation of (2.1a) as a delay differential equation in the space $C(\overline{\Omega}, \mathbf{R}^n)$ (see either of [31, 32]). It is shown in [31] that this mild solution $u(t)$ is a strong solution (continuously differentiable) of the abstract delay differential equation in $C(\overline{\Omega}, \mathbf{R}^n)$ for $t > \tau$ and thus is a classical solution for $t > \tau$. *In the following we shall simply assume that our solutions are classical ones.*

If $u = (u_i)_1^n$ is a solution of (2.1a–d) on $\Omega \times (-\infty, t^*]$, $t^* > 0$, and

$$M_i \equiv \sup\{|G_{it}(u(x, \cdot))| : (x, t) \in \overline{\Omega} \times [0, t^*]\},$$

then

$$d_i \Delta u_i - b_i u_i M_i \leq \partial_t u_i \leq d_i \Delta u_i + b_i u_i M_i$$

and it follows by the comparison and maximum principle (see Smoller [36]) that:

(i) $\xi_i(x, 0) \equiv 0$, $x \in \overline{\Omega}$, implies $u_i(x, t) \equiv 0$ for $x \in \overline{\Omega}$, $t > 0$;

(ii) $\xi_i(x_0, 0) > 0$ for some $x_0 \in \Omega$ implies that $u_i(x, t) > 0$ for all $x \in \overline{\Omega}$, $t > 0$.

In this paper we always assume that:

(2.1a–d) has a unique, spatially homogeneous steady state solution $u^* = (u_1^*, u_2^*, \dots, u_n^*)$ such that (H0)

$$u_i^* \geq 0, \quad G_i(u^*) \leq 0 \quad \text{and} \quad u_i^* G_i(u^*) = 0. \tag{2.6}$$

Such a steady state is called a *saturated equilibrium* by Hofbauer and Sigmund [20]. In the following we shall call u^* a *saturated steady state.*

It is easy to see that bounded solutions of system (2.1a–d) may not have precompact orbits in $BC(\overline{\Omega} \times (-\infty, 0])$. This makes the usual properties (nonempty, connected, compact and invariant) of positive limit sets invalid. In order to overcome this difficulty, we choose the following well-known more friendly spaces [1, 14, 15, 17]:

$$UC_g = \left\{ \phi(x, \theta) \in C(\overline{\Omega} \times (-\infty, 0], \mathbf{R}^n) : \|\phi\|_g = \sup_{x \in \overline{\Omega}, \theta \leq 0} \frac{|\phi(x, \theta)|}{g(\theta)} < \infty, \right.$$
$$\left. \phi(x, \theta)/g(\theta) \text{ is uniformly continuous on } \overline{\Omega} \times (-\infty, 0] \right\},$$

where $g: (-\infty, 0] \to [1, \infty)$ satisfies

$g: (-\infty, 0]$ is a continuous nonincreasing function on $(-\infty, 0]$ such that $g(0) = 1$; (g1)

$g(s+u)/g(s) \to 1$ uniformly on $(-\infty, 0]$ as $u \to 0^-$; (g2)

$g(s) \to \infty$ as $s \to -\infty$. (g3)

Clearly, UC_g is a Banach space with norm

$$\|\phi\|_g = \sup_{x \in \overline{\Omega}, \theta \leq 0} \frac{|\phi(x, \theta)|}{g(\theta)}.$$

UC_g is a strong fading memory space (in the sense of [14, 17] which implies that bounded solutions of an autonomous system (such as (2.1a–d)) corresponding to initial data $\phi \in BC(\overline{\Omega} \times (-\infty, 0])$ have precompact orbits in UC_g (rigorous proof of this statement can be obtained by modifying similar arguments presented in [17]). Thus positive limit sets are nonempty and have their usual properties. A further reason for choosing UC_g is that $BC(\overline{\Omega} \times (-\infty, 0])' \hookrightarrow UC_g$ with continuous inclusion for g satisfying (g1)–(g3).

It is known (Lemma 2.1 in [12] and [27]) that with the above assumptions made for system (2.1a–d), there is a $g(s)$, satisfying (g1)–(g3), such that

$$|\mu_{ijg}| = \int_{-\infty}^{0} g(s)|d\mu_{ij}(\theta)| < \varphi, \qquad i, j = 1, \dots, n.$$

We chose the norm $|\cdot|$ in $\mathbf{R}^n$ as

$$|\phi| = \max\{|\phi_i| : i = 1, \dots, n\}$$

where $\phi = (\phi_1, \dots, \phi_n) \in \mathbf{R}^n$. Thus, for $g(s)$ satisfying (g1)–(g3), $\phi(x, \theta) \in UC_g$, we have

$$\|\phi\|_g = \sup_{x\in\overline{\Omega}, \theta\le 0} \max\left\{\frac{|\phi_i(x, \theta)|}{g(\theta)} : i = 1, \dots, n\right\}.$$

As usual, $C(\overline{\Omega}, \mathbf{R}^n)$, respectively $C^k(\overline{\Omega}, \mathbf{R}^n)$, will denote the set of continuous, respectively k times continuously differentiable functions $\Omega \to \mathbf{R}^n$ with their standard norms that will be denoted by $\|\cdot\|$, respectively $\|\cdot\|_{C^k}$.

3. Estimation of Laplacian

The purpose of this section is to provide an estimate of $\Delta u_i(x, t)$ in terms of $\|u_t(x, \theta) - u^*\|_g$. We first estimate $\|u(\cdot, t)_{C^1}\|$ in terms of $\|u_t\|_g$. The following lemma is adapted from Lemma 2.1 in Hutson and Moran [21].

Lemma 3.1. *There is a polynomial $B_1(x, y) = \alpha_1 x + \alpha_2 y + \alpha_3 y^2$, where positive constants α_1, α_2 and α_3 are independent of $u_0(x, \theta)$ such that*

$$\|u(\cdot, t)\|_{C^1} \le B_1(\|u_{t-1}(x, \theta)\|_g, \|u_t(x, \theta)\|_g), \quad \text{for } t \ge 1. \tag{3.1}$$

Proof. With $v = u_i$, $d = d_i$ in turn, each equation may be written in the form

$$\partial_t v - (d\Delta - a)v = q(x, t), \tag{3.2}$$

where

$$q(x, t) = \alpha u_i(x, t) + b_i u_i(x, t) G_i(u_t(x, \cdot)), \tag{3.3}$$

and $v(x, 0) = v_0$.

By [41, page 88], for some positive constant a (depending only on Ω), under both homogeneous Neumann and Dirichlet conditions, $(\Delta - aI)$ generates an analytic semigroup in $E = L_p(\overline{\Omega})$ for $p > 1$, and with $-A$ the

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associated operator, there is a $\delta > 0$ such that Re $\sigma(A) > \delta$, where $\sigma(A)$ denotes the spectrum of A. Equation (3.2) may then be written as

$$\frac{dv}{dt} + Av = Q, \tag{3.4}$$

with $A = -d\Delta + a$, and $Q(s) = q(\cdot, s) \in E$, and with mild solution

$$v(t) = e^{-At}v_0 + \int_0^t e^{-A(t-s)}Q(s)\,ds. \tag{3.5}$$

Note that

$$|q(x, t)| \le \|u_{it}(x, \theta)\|_g \left\{a + b_i\left[|r_i| + \left(a_i + \sum_{j=1}^n \mu_{ijg}\right)\|u_t(x, \theta)\|_g\right]\right\} \tag{3.6}$$

where

$$|\mu_{ijg}| = \int_{-\infty}^0 g(\theta)\,d|\mu_{ij}(\theta)|.$$

Denote

$$\eta = \max_{1\le i\le n}\{a + b_i|r_i|\}, \qquad \sigma = \max_{1\le i\le n}\left\{b_i\left(a_i + \sum_{j=1}^n \mu_{ijg}\right)\right\}. \tag{3.7}$$

Thus,

$$|q(x, t)| \le \|u_t(x, \theta)\|_g(\eta + \sigma\|u_t(x, \theta)\|_g), \tag{3.8}$$

which implies that

$$\|Q(t)\|_{L^\infty(\overline{\Omega})} = \|q(\cdot, t)\|_{L^\infty(\overline{\Omega})} \le \|u_t\|_g(\eta + \sigma\|u_t\|_g).$$

From [19, page 26], we have for $\alpha > 0$,

$$\|A^\alpha e^{-At}\|_E \le C(\alpha)t^{-\alpha}e^{-\delta t}, \tag{3.9}$$

where $C(\alpha)$ is bounded for α in any compact interval of $(0, \infty)$ and bounded as $\alpha \to 0^+$. Here $\|\cdot\|_E$ is also used to denote the operator norm. Choosing $\alpha \in (0, 1)$, applying A^α to (3.5), and then taking norms, we obtain

$$\begin{aligned}\|A^\alpha v(t)\varepsilon\|_E &\le \|A^\alpha e^{-A}\|_E\|v_{t-1}\|_E \\ &\quad + \int_0^1 \|A^\alpha e^{-A(1-s)}\|_E\|Q(t-1+s)\|_E\,ds \\ &\le C(\alpha)\left[e^{-\delta}\|v_{t-1}\|_E \right.\\ &\quad \left. + \max_{0\le s\le 1}\|Q(t-1+s)\|_E\int_0^1(1-s)^{-\alpha}e^{-\delta(1-s)}\,ds\right],\end{aligned} \tag{3.10}$$

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by (3.9). Clearly, there is a positive constant $\beta = \beta(\Omega)$ depending only on the domain Ω such that

$$\|v_0\|_E \le \beta \|v_0(x, \theta)\|_g, \qquad \|Q(t)\|_E \le \beta \|q(\cdot, t)\|_{L^\infty(\overline{\Omega})}. \tag{3.11}$$

Also, there is a positive constant $\gamma = \gamma(\alpha, \delta)$, such that

$$\int_0^1 (1-s)^{-\alpha} e^{-\delta(1-s)}\, ds < \gamma. \tag{3.12}$$

Observe that $\max_{t-1 \le s \le t} \|u_s\|_g \le \overline{g}\|u_t\|_g$, where (cf.$_\nu$(g2))

$$\overline{g} = \sup_{s \le 0}\{g(s-1)/g(s)\}.$$

We denote that $\overline{\gamma} = \gamma\overline{q}$ and $\overline{\sigma} = \sigma\overline{q}$. Hence, we have

$$\|A^\alpha v(t)\|_E \le \beta C(\alpha)[\|u_{t-1}(x, \theta)\|_g + \overline{\gamma}\|u_t(x, \theta)\|_g(\eta + \overline{\sigma}\|u_t(x, \theta)\|_g]. \tag{3.13}$$

From the definition of the fractional space E^α [19, page 29], and a standard imbedding theorem [19, page 39], respectively, we obtain

$$\|A^\alpha v(t)\|_E = \|v(t)\|_{E^\alpha},$$
$$\|v(t)\|_{C^\nu} \le k\|v(t)\|_{E^\alpha}, \qquad 0 \le \nu \le 2\alpha - N/p,$$

where k is independent of v. Taking $\alpha = 3/4$, $p = 2N$, $\nu = 1$, we have

$$\|v(t)\|_{C^1} \le k\beta C(\alpha)[\|u_{t-1}(x, \theta)\|_g + \overline{\gamma}\|u_t(x, \theta)\|_g(\eta + \overline{\sigma}\|u_t(x, \theta)\|_g)]. \tag{3.14}$$

Now see that the lemma is proved by letting

$$\alpha_1 = k\beta C(\alpha), \quad \alpha_2 = k\beta C(\alpha)\overline{\gamma}\eta \quad \text{and} \quad \alpha_3 = k\beta C(\alpha)\overline{\gamma}\overline{\sigma}.$$

It is reasonable to assume that in realistic biological models, intraspecific competition operates. As a consequence, we assume that every solution of (2.1a–d) will enter and remain in a fixed bounded L^∞ neighborhood of the origin. For $Y \subset \mathbf{R}_+^N$, denote

$$X_0 = \{u \in C(\overline{\Omega}, \mathbf{R}^n) : u(x) \in Y,\ x \in \overline{\Omega}\}. \tag{3.15}$$

We assume the following throughout the rest of this paper:

> There is a compact neighborhood Y of the origin $\mathbf{R}_+^N$ such that for every
>
> $u_0(x, \theta) \in BC(\overline{\Omega} \times (-\infty, 0]) \cap C(\overline{\Omega} \times (-\infty, 0])$, (H1)
>
> there is a finite time $t_0(u_0)$ such that the corresponding solution $u(\cdot, t) \in X_0$ for $t \ge t_0(u_0)$.

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In the following, we denote

$$M_1 = \max\{|y| : y \in Y\}. \tag{3.16}$$

Clearly, (H1) implies that for every $u_0(x, \theta) \in BC(\overline{\Omega} \times (-\infty, 0]) \cap C^2(\overline{\Omega} \times (-\infty, 0])$, there is a $t_1(u_0) > t_0(u_0)$ such that

$$\|u_t(x, \theta)\|_g \leq M_1, \quad \text{for } t \geq t_1(u_0). \tag{3.17}$$

We thus have:

COROLLARY 3.1. *Assume* (H1) *holds and* $u_0(x, \theta) \in BC(\overline{\Omega} \times (-\infty, 0]) \cap C^2(\overline{\Omega} \times (-\infty, 0])$. *Then there exist positive constants* β_1 *and* β_2, *both independent of* $u_0(x, \theta)$, *such that*

$$\|u(\cdot, t)\|_{C^1} \leq \beta_1 M_1 + \beta_2 M_1^2, \qquad t \geq t_1(u_0) + 1. \tag{3.18}$$

PROOF. We note that $\beta_1 = \alpha_1 + \alpha_2$, $\beta_2 = \alpha_3$. The conclusion follows from Lemma 3.1.

For convenience, we would like to denote

$$M_2 = \beta_1 M_1 + \beta_2 M_1^2, \tag{3.19}$$

$$w_{ij}(x, t) = \frac{\partial}{\partial x_j} u_i(x, t), \quad i = 1, \dots, n, \; j = 1, \dots, N, \tag{3.20}$$

$$q_{ij}(x, t) = \frac{\partial}{\partial x_j}[b_i u_i(x, t) G_i(u_t, (x, \cdot))], \quad i = 1, \dots, n, \; j = 1, \dots, N. \tag{3.21}$$

It is easy to see that w_{ij} satisfies

$$\partial_t w_{ij}(x, t) = d_i \Delta w_{ij}(x, t) + q_{ij}(x, t), \quad \text{on } \Omega \times (0, \infty), \tag{3.22}$$

$$w_{ij}(x, \theta) = \frac{\partial}{\partial x_j} \xi_j(x, \theta), \quad \text{on } \overline{\Omega} \times (-\infty, 0], \tag{3.23}$$

$$w_{ij}(x, t) = \omega_{ij}(x, t), \quad \text{on } \partial\Omega \times (0, \infty), \tag{3.24}$$

where $w_{ij}(x, t)$ satisfies $|w_{ij}| \leq M_2$, $t \geq t_1(u_0) + 1$. Corollary 3.1 implies that there is a $t_2 = t_2(u_0) > t_1(u_0)$, such that for $t \geq t_2(u_0)$,

$$\|w_{ijt}(x, \theta)\|_g \leq M_2. \tag{3.25}$$

Denote $W_{ij}(x, t) = w_{ij}(x, t + t_2)$. Then $W_{ij}(x, t)$ satisfies

$$\partial_t W_{ij}(x, t) = d_i \Delta W_{ij}(x, t) + q_{ij}(x, t + t_2), \quad \text{on } \Omega \times (0, \infty), \tag{3.26}$$

$\|W_{ij0}(x, \theta)\|_g \leq M_2$, and $|W_{ij}(x, t)| \leq M_2$, $(x, t) \in \partial\Omega \times (0, \infty)$, by Corollary 3.1. Now we see that W_{ij} has almost the same properties as that of $u_i(x, t)$. We can therefore apply Lemma 3.1 to obtain

Lemma 3.2. *Assume that* $u_0(x, \theta) \in BC(\overline{\Omega} \times (-\infty, 0]) \cap C^3(\overline{\Omega} \times (-\infty, 0])$. *There is a polynomial* $B_2(x, y) = \gamma_1 x + \gamma_2 y + \gamma_3 y^2$, *where positive constants* γ_1, γ_2 *and* γ_3 *are independent of* $u_0(x, \theta)$, *and a* $\bar{t}(u_0) > 1$, *such that for* $t \geq \bar{t}(u_0)$,

$$\|w_{ij}(\cdot, t)\|_{C^1} \leq B_2(\|w_{ij\ t-1}(x, \theta)\|_g, \|w_{ijt}(x, \theta)\|_g). \tag{3.27}$$

Indeed, similar to Corollary 3.1, we have

$$\|w_{ij}(\cdot, t)\|_{C^1} \leq (\gamma_1 + \gamma_2)M_2 + \gamma_3 M_2^2 \quad \text{for } t \geq \bar{t}(u_0). \tag{3.28}$$

Note that $\Delta u_i(x, t) = \sum_{j=1}^N w_{ij}(x, t)$ and $M_2 = \beta_1 M_1 + \beta_2 M_1^2$. We thus have:

Theorem 3.1. *In* (2.1a–d), *assume that* (H1) *holds and* $u_0(x, \theta) \in BC(\overline{\Omega} \times (-\infty, 0]) \cap C^3(\overline{\Omega} \times (-\infty, 0])$. *Then there exist positive constants* δ_i, $i = 1, \ldots, 4$, *independent of* $u_0(x, \theta)$, *and a* $\bar{t}(u_0) > 1$, *such that for* $t \geq \bar{t}(u_0)$, $i = 1, \ldots, n$,

$$|\Delta u_i(x, t)| \leq \sum_{i=1}^{4} \delta_i M_1^i. \tag{3.29}$$

Now we are ready to state and prove the main result of this section.

Theorem 3.2. *Assume that assumptions of Theorem* 3.1 *are satisfied and* $u^* = (u_1^*, \ldots, u_n^*)$ *is the unique saturated equilibrium of* (2.1a). *There is a constant* $\delta = \delta(M_1)$, *independent of* $u_t(x, \theta)$, *and a* $\bar{t}(u_0) > 1$ *such that for* $t \geq \bar{t}(u_0)$,

$$|\partial_t u_i(x, t)| \leq \delta \|u_t(x, \theta) - u^*\|_g, \qquad i = 1, \ldots, n. \tag{3.30}$$

Proof. Assume first that $u_i^* = 0$. Then

$$|u_i(x, t)| = |u_i(x, t) - u_i^*| \leq \|u_t(x, \theta) - u^*\|_g. \tag{3.31}$$

We need only to adjust our estimates in the proof of Lemma 3.1. In the estimate of (3.8), we will have

$$|q(x, t)| \leq \|u_t(x, \theta) - u^*\|_g (\eta + \sigma M_1),$$

and obviously

$$\|v_0\| = \|u_i(\cdot, t-1)\| \leq \bar{g}\|u_t(x, \theta) - u^*\|_g.$$

The inequality (3.14) can thus be replaced by

$$\|v(t)\|_{C^1} \leq \bar{g} k \beta C(\alpha)[1 + \gamma(\eta + \bar{\sigma} M_1)]\|u_t(x, \theta) - u^*\|_g. \tag{3.32}$$

Assume now that $u_i^* \neq 0$. Then $G_i(u^*) = 0$. In (3.2) we let $v = u_i(x, t) - u_i^*$. We thus have

$$\|v_0\| = \|u_i(\cdot, t-1) - u_i^*\| \leq \bar{g}\|u_t(x, \theta) - u^*\|_g$$

and

$$|q(x, t)| \leq \left[a + b_i M_1\left(|a_i| + \sum_{j=1}^{n} |\mu_{ijg}|\right)\right] \|u_t(x, \theta) - u^*\|_g .$$

In this case, (3.14) should be replaced by

$$\|v(t)\|_{C^1} \leq \bar{g} k \beta C(\alpha)\left\{1 + \gamma\left[a + b_i M_1\left(|a_i| + \sum_{j=1}^{n} |\mu_{ijg}|\right)\right]\right\} \times \|u_t(x, \theta) - u^*\|_g . \quad (3.33)$$

Denote

$$\Delta_1 = \bar{g} k \beta C(\alpha) \max\left\{[1 + g(n + \sigma M_1)],\ 1 + \gamma\left[a + b_i M_1\left(|a_i| + \sum_{j=1}^{n} |\mu_{ijg}|\right)\right], i = 1, \dots, n\right\}. \quad (3.34)$$

Then we have shown that for $t \geq t_1(u_0) + 1$,

$$\|u_i(\cdot, t)\|_{C^1} \leq \Delta_1 \|u_t(x, \theta) - u^*\|_g, \qquad i = 1, \dots, n. \quad (3.35)$$

By modifying the arguments that lead to Lemma 3.2, we easily arrive at for $t \geq \bar{t}(u_0) > t_1(u_0) + 1$, where $\bar{t}(u_0)$ as defined in Lemma 3.2,

$$\|w_{ij}\|(\cdot, t)\|_{C^1} \leq [\gamma_1 + \gamma_2 + \gamma_3 \Delta_1 M_1]\Delta_1 \|u_t(x, \theta) - u^*\|_g, \quad i, j = 1, \dots, n, \quad (3.36)$$

and where γ_i, $i = 1, 2, 3$, are defined in Lemma 3.2. Let

$$\Delta_2 = N\Delta_1[\gamma_1 + \gamma_2 + \gamma_3 \Delta_1 M_1]; \quad (3.37)$$

then

$$|\Delta u_i(x, t)| \leq \Delta_2 \|u_t(x, \theta) - u^*\|_g . \quad (3.38)$$

Clearly,

$$|\partial_t u_i(x, t)| \leq d_i |\Delta u_i(x, t)| + b_i |u_i(x, t) G_i(u_t(x, \cdot))| . \quad (3.39)$$

Denote

$$\omega = \max_{1 \leq i \leq n}\left\{b_i\left[r_i + \left(a_i + \sum_{j=1}^{n} \mu_{ijg}\right) M_1\right], b_i M_i\left(a_i + \sum_{j=1}^{n} \mu_{ijg}\right)\right\}. \quad (3.40)$$

We have

$$b_i|u_i(x, t)G_i(u_t(x, \cdot))| \leq \omega\|u_t(x, \theta) - u^*\|_g .$$

(3.39) thus implies that

$$|\partial_t u_i(x, t)| \leq (d_i\Delta_2 + \omega)\|u_t(x, \theta) - u^*\|_g .$$

The proof of the theorem is completed by letting

$$\delta = \max\{\delta_i\Delta_2 + \omega : 1 \leq i \leq n\} .$$

4. Main results

We say that system (2.1a–d) is *diagonally dominant* if

$$a_i > \sum_{j=1}^{n} |\mu_{ij}| , \qquad i = 1, \dots, n . \tag{4.1}$$

Throughout this section, we assume that system (2.1a–d) is diagonally dominant and denote $g = g(s)$ as a function which satisfies (g1)–(g3), $a_i > \sum_{j=1}^{n} |\mu_{ijg}|$, $i = 1, \dots, n$, and $g(s) = 1$ for $s \in [-\tau, 0]$, where $\tau = \max\{\tau_i, i = 1, \dots, n\}$. Clearly, such a $g(s)$ always exists as long as (2.1a–d) is diagonally dominant. Also, we always denote $u^* = (u_1^*, \dots, u_n^*)$ as the unique saturated equilibrium of (2.1a–d).

For $\xi \in \mathbf{R}^n$, we define

$$V[\xi] = \max\{|\xi_i - u_i^*| : i = 1, \dots, n\} , \tag{4.2}$$

$$I(\xi) = \{i : V[\xi] = |\xi_i - u_i^*|\} . \tag{4.3}$$

Also, let $C(\overline{\Omega})^n$ be the space of continuous functions $u : \overline{\Omega} \to \mathbf{R}_+^n$ and define W and J on $C(\overline{\Omega})^n$ as

$$W[u] = \max\{V[u(x)] : x \in \overline{\Omega}\} , \tag{4.4}$$

$$J(u) = \{x : W[u] = V[u(x)]\} . \tag{4.5}$$

It is easy to check that if

$$D\_W[u]z \equiv \lim_{h \to 0^+} \frac{W[u - hz] - W[u]}{-h} \tag{4.6}$$

for all $u, z \in C(\overline{\Omega})^n$, then for $u(x) \not\equiv u^*$ on $\overline{\Omega}$,

$$D\_W[u](z) = \min\{\operatorname{sgn}(u_i(x) - u_i^*)z_i(x) : x \in J(u),\ i \in I(u(x))\} \tag{4.7}$$

where $\operatorname{sgn}(r)$ is 1 if $r > 0$ and -1 if $r < 0$.

Let A be the closure in $C(\overline{\Omega})^n$ of the linear operator B defined by

$$Bu = (d_i \Delta u_i)_1^n, \qquad \text{for all } u \in (u_i)_1^n \in D(B),$$
$$D(B) = \{u : u \in C^3(\overline{\Omega})^n \text{ and } \partial_\nu u = 0 \text{ on } \partial\Omega\}.$$

Also, define the map $F = (F_i)_1^n : C(\overline{\Omega} \times (-\infty, 0]) \to C(\overline{\Omega})^n$ by

$$[F_i(\phi)](x) = b_i \phi_i(x, 0) G_i(\phi(x, \cdot)), \qquad x \in \overline{\Omega}, \; i = 1, \dots, n. \tag{4.8}$$

Note that (2.1a) can be written in the abstract form

$$\partial_t u = Au + F(u_t). \tag{4.9}$$

The results in Rothe [42, page 15] indicate that A generates an analytic semigroup $T = \{T(t) : t \geq 0\}$ on $C(\overline{\Omega})^n$, and so by variation of constants, (4.9) can be integrated and written in the form

$$u(t) = T(t-s)u(s) + \int_s^t T(t-r)F(u_r)\,dr \quad \text{for } t \geq s \geq 0. \tag{4.10}$$

Noting that $T(t)u^* \equiv u^*$ for $t \geq 0$ and applying the maximum principle shows that

$$W[T(t)y] \leq W[y] \quad \text{for all } t \geq 0 \text{ and } y \in C(\overline{\Omega})^n. \tag{4.11}$$

If $t > 0$ and $0 < h < t$, we have from (4.10) and the continuity of T and F that

$$u(t) - hF(u_t) = T(h)u(t-h) + o(h),$$

where $h^{-1}|o(h)| \to 0$ as $h \to 0^+$. Hence,

$$\begin{aligned} W[u(t) - hF(u_t)] &\leq W[T(h)u(t-h)] + o(h) \\ &\leq W[u(t-h)] + o(h), \end{aligned}$$

and it follows that if d^-/dt denotes the lower Dini derivative, then

$$\frac{d^-}{dt} W[u(t)] \leq D_- W[u(t)](F(u_t)) \tag{4.12}$$

for $t > 0$ and solutions u to (2.1a–d).

Throughout the rest of this section, we assume that

$$a_i(1 - \delta\tau_i) > \sum_{j=1}^n |\mu_{ijg}|, \qquad i = 1, \dots, n, \tag{H2}$$

where δ is defined as in Theorem 3.2.

REMARK 4.1. It is easy to see that (H2) is equivalent to

$$\tau_i < \delta^{-1}\left[1 - a_i^{-1}\left(\sum_{j=1}^n |\mu_{ijg}|\right)\right], \qquad i = 1, \dots, n.$$

In other words, this is a typical "smallness assumptions delay."

We define further that

$$\mathscr{V}(\phi) = \sup_{\theta \le 0} W[\phi(x, \theta)]/g(\theta). \tag{4.13}$$

It is clear that $\mathscr{V}(u_t) = \|u_t - u^*\|_g$. The following lemma is crucial to the main result, Theorem 4.1, of this paper.

LEMMA 4.1. *If* $u_0(x, \theta) \in BC(\overline{\Omega} \times (-\infty, 0]) \cap C^3(\overline{\Omega} \times (-\infty, 0])$ *and for some* $t_0 \ge \bar{t}(u_0)$, $\mathscr{V}(u_t) = W(u(\cdot, t))$, *where* $u_t(x, \theta)$ *is the solution of* (2.1a–d), *then*

$$\left.\frac{d^-}{dt}\right|_{t=t_0} W(u(\cdot, t)) \le 0,$$

where $\bar{t}(u_0)$ *is as in Theorem* 3.2. *Moreover, equality holds only if* $u(x, t) \equiv u^*$, $x \in \overline{\Omega}$.

PROOF. We denote $\phi(x, \theta) = u_{t_0}(x, \theta)$ and assume that $t_0 \ge \bar{t}(u_0)$. If $\phi(x, \theta) = u^*$ for all $x \in \overline{\Omega}$, then the lemma is obvious, so we assume this is not the case. It follows that $\phi(x, 0) \not\equiv u^*$. Assume that $W(\phi(x, 0)) = |\phi_i(x_0, 0) - u_i^*|$. Formulas (4.7) and (4.12) indicate

$$\frac{d^-}{dt} W(\phi(x, 0)) \le \operatorname{sgn}(\phi_i(x_0, 0) - u_i^*) b_i \phi_i(x_0, 0) G_i(\phi(x_0, \cdot)).$$

It suffices to show that if $\phi_i(x_0, 0) > 0$ and $\phi_i(x_0, 0) \ne u_i^*$, then

$$\operatorname{sgn}(\phi_i(x_0, 0) - u_i^*) G_i(\phi(x_0, \cdot)) < 0.$$

If $u_i^* = 0$, we have $\operatorname{sgn}(\phi_i(x_0, 0) - u_i^*) = 1$ and $G_i(u^*) \le 0$, and if $u_i^* > 0$, we have $G_i(u^*) = 0$. We thus have

$$\begin{aligned}
&\operatorname{sgn}(\phi_i(x_0, 0) - u_i^*) G_i(\phi(x_0, \cdot)) \\
&\quad \le \operatorname{sgn}(\phi_i(x_0, 0) - u_i^*) \left\{ -a_i \int_{-\tau_i}^0 (\phi_i(x_0, \theta) - u_i^*)\, d\mu_i(\theta) \right. \\
&\qquad\qquad \left. + \sum_{j=1}^n \int_{-\infty}^0 (\phi_j(x_0, \theta) - \mu_j^*)\, d\mu_{ij}(\theta) \right\}.
\end{aligned} \tag{4.14}$$

Note that

$$\int_{-\tau_i}^0 (\phi_i(x_0, \theta) - u_i^*)\, d\mu_i(\theta) = \phi_i(x_0, 0) - u_i^* + \int_{-\tau_i}^0 \theta \partial_t \phi_i(x_0, \xi(\theta))\, d\mu_i(\theta),$$

where $\xi(\theta) \in (\theta, 0)$. By Theorem 3.2, we have

$$|\theta \partial_t \phi_i(x_0, \xi(\theta))| \le \delta |\theta| \, \|u_t - u^*\|_g \le \delta \tau_i |\phi_i(x_0, 0) - u_i^*|.$$

This implies that

$$\begin{aligned} &\operatorname{sgn}(\phi_i(x_0, 0) - u_i^*) \left[-a_i \int_{-\tau_i}^{0} (\phi_i(x_0, \theta) - u_i^*) \, d\mu_i(\theta) \right] \\ &\quad \le -a_i(1 - \delta\tau_i)|\phi_i(x_0, 0) - u_i^*|. \end{aligned} \tag{4.15}$$

It is easy to verify that

$$\begin{aligned} &\operatorname{sgn}(\phi_i(x_0, 0) - u_i^*) \sum_{j=1}^{n} \int_{-\infty}^{0} (\phi_j(x_0, \theta) - u_j^*) \, d\mu_{ij}(\theta) \\ &\quad \le \sum_{j=1}^{n} \int_{-\infty}^{0} (|\phi_j(x_0, \theta) - u_j^*|/g(\theta)) |\, d\mu_{ijg}(\theta)| \\ &\quad \le \left(\sum_{j=1}^{n} |\mu_{ijg}| \right) |\phi_i(x_0, 0) - u_i^*|. \end{aligned} \tag{4.16}$$

Inequalities (4.15) and (4.16) clearly imply that

$$\begin{aligned} &\operatorname{sgn}(\phi_i(x_0, 0) - u_i^*) G_i(\phi(x_0, \cdot)) \\ &\quad \le \left(-a_i(1 - \delta\tau_i) + \sum_{j=1}^{n} |\mu_{ijg}| \right) |\phi_i(x_0, 0) - u_i^*| < 0, \end{aligned}$$

proving the lemma.

Lemma 4.2. *Let* $u_t(x, \theta)$ *be the solution of* (2.1a–d) *with* $u_0(x, \theta) \in BC(\overline{\Omega} \times (-\infty, 0]) \cap C^3(\overline{\Omega} \times (-\infty, 0])$. *Then for* $t \ge \bar{t}(u_0)$, *the map* $t \to \mathscr{V}(u_t)$ *is nonincreasing on* $[0, \infty)$.

Proof. That Lemma 4.1 implies the above lemma is fairly standard (see, e.g., [13, 28]). Nevertheless, we prove it here for completeness.

Suppose the lemma is false. Then for some $t_0 > \bar{t}(u_0)$,

$$\liminf_{h \to 0^+} \frac{\mathscr{V}(u_{t_0 - h}) - \mathscr{V}(u_{t_0})}{-h} > 0, \tag{4.17}$$

and hence there exist $h_n \to 0^+$ as $n \to \infty$ and $\alpha > 0$ such that

$$\frac{\mathscr{V}(u_{t_0 - h_n}) - \mathscr{V}(u_{t_0})}{-h_n} \ge \alpha, \qquad n \ge 1. \tag{4.18}$$

In particular, we must have $\mathscr{V}(u_{t_0}) > \mathscr{V}(u_{t_0-h_n})$. We claim that

$$\mathscr{V}(u_{t_0}(x, \theta)) = W(u(\cdot, t_0)).$$

Otherwise, $\mathscr{V}(u_{t_0}) > W(u(\cdot, t_0))$ implies that there exists $s_0 < 0$, $x_0 \in \overline{\Omega}$ such that

$$\mathscr{V}(u_{t_0}) = V(u(x_0, t_0 + s_0))/g(s_0). \tag{4.19}$$

If n is chosen such that $t_0 - h_n > t_0 + s_0$, then

$$\begin{aligned}\mathscr{V}(u_{t_0-h_n}) &\geq V(u(x_0, t_0 + s_0))/g(s_0 + h_n)\\ &= \frac{g(s_0)}{g(s_0+h_n)} \cdot \frac{V(u(x_0, t_0+s_0))}{g(s_0)} = \frac{g(s_0)}{g(s_0+h_n)}\mathscr{V}(u_{t_0})\\ &\geq \mathscr{V}(u_{t_0}),\end{aligned}$$

which contradicts (4.18) and hence $\mathscr{V}(u_{t_0}) = W(u(\cdot, t_0))$ as claimed. By Lemma 4.1, we have

$$\lim_{h\to 0^+} \frac{W(u(\cdot, t_0 - h)) - W(u(\cdot, t_0))}{-h} \leq 0. \tag{4.20}$$

But

$$\begin{aligned}&W(u(\cdot, t_0 - h_n)) - W(u(\cdot, t_0))\\ &\quad = W(u(\cdot, t_0 - h_n)) - \mathscr{V}(u_{t_0}) \leq \mathscr{V}(u_{t_0-h_n}) - \mathscr{V}(u_{t_0}),\end{aligned}$$

which clearly implies that

$$\frac{W(u(\cdot, t_0 - h_n)) - W(u(\cdot, t_0))}{-h_n} \geq \frac{\mathscr{V}(u_{t_0-h_n}) - \mathscr{V}(u_{t_0})}{-h_n} \geq \alpha,$$

contradicting (4.20). This completes our proof.

Now we are ready to state and prove our main result of this paper.

Theorem 4.1. *Assume that* (H1)–(H2) *hold for the system* (2.1a–d) *and* $u_0(x, \theta) \in BC(\overline{\Omega} \times (-\infty, 0]) \cap C^3(\overline{\Omega} \times (-\infty, 0])$. *Then the solution* $u_t(x, \theta)$ *tends to* u^* *as* $t \to +\infty$ *(in* UC_g *norm).*

Proof. By Lemma 4.2, we know that there is a $c \geq 0$, such that

$$\lim_{t\to\infty} \mathscr{V}(u_t) = c.$$

It suffices to show that $c = 0$; so assume for contradiction that $c > 0$. Since the semigroup T in (4.10) is compact, the ω-limit set $\omega(u_0(x, \theta))$ of u_t is nonempty, compact and invariant (in the UC_g space). In particular, if

$\phi \in \omega(u_0(x, \theta))$ and v is the mild solution to (4.19) with $u_0(x, \theta)$ replaced by ϕ, then

$$0 < c = \mathscr{V}(v_t) \quad \text{for all } t \geq 0. \tag{4.21}$$

Now for each $t > 0$, select $x_t \in \overline{\Omega}$ and $i \in \{1, \dots, n\}$ so that

$$W[v(\cdot, t)] = |v_i(x_t, t) - u_i^*|.$$

If $t > 0$ is such that $V[v(x_t, t)] = c$ and $v_i(x_t, t) > 0$, then Lemma 4.1 implies that

$$\liminf_{h \to 0^+} \frac{W[v(\cdot, t-h)] - W[v(\cdot, t)]}{-h} < 0.$$

But this implies that for some small h,

$$\mathscr{V}(v_{t-h}) \geq W[v(\cdot, t-h)] > W[v(\cdot, t)] = c, \tag{4.22}$$

a contradiction to (4.21). We thus have two cases to consider

(i) $W[v(x, t)] < c$, for all $t > 0$;

(ii) For some $i_0 \in \{1, \dots, n\}$, $v_{i_0}(x, t) \equiv 0$ and hence $c = v_{i_0}^*$.

Consider first case (i). Let $s_1 > 0$ such that $g(s_1) > 1$. For $t > -s_1$, denote

$$\bar{c} = \max\{W[v(x, s)], \ t + s_1 \leq s \leq t\}. \tag{4.23}$$

Then for such a t, we must have

$$\mathscr{V}(v_t) \leq \max\{\bar{c}, c/g(s_1)\} < c, \tag{4.24}$$

a contradiction.

Consider now case (ii). Without loss of generality, we may assume that $u_i^* \neq u_j^*$, $i \neq j$, since otherwise we can choose $e_i > 0$ such that $e_i u_i^* \neq e_j u_j^*$, $i \neq j$, and make the change of variable $\bar{u}_i = e_i u_i$. Since $\lim_{s \to -\infty} g(s) = +\infty$, there is a $\sigma > 0$, such that

$$M_1/g(-\sigma) < c/2. \tag{4.25}$$

Clearly, we must have

$$|v_j(x, t) - u_j^*| < u_{i_0}^*, \qquad (x, t) \in \overline{\Omega} \times [0, \infty), \ i_0 \neq j, \tag{4.26}$$

since otherwise we will again have $v_j(x, t) \equiv 0$ and thus obtain $u_j^* = u_{i_0}^*$, $i_0 \neq j$, contradicting our previous assumption.

By continuity and compactness, we have

$$\max\{|v_j(x, t) - u_j^*| : (x, t) \in \overline{\Omega} \times [0, \sigma], \ j \neq i_0\} < u_{i_0}^*. \tag{4.27}$$

Since $v_\sigma \in \omega(u_0, (x, \theta))$, there is a sequence $t_k \to \infty$ such that $v_{t_k} \to v_\sigma$ in UC_g norm. Therefore, there is a $K > 0$ such that for $k \geq K$,

$$\max_{j \neq i_0}\{|u_j(x, t_k + \theta) - u_j^*| : x \in \overline{\Omega}, \ -\sigma \leq \theta \leq 0\} < u_{i_0}^*. \tag{4.28}$$

By Lemma 4.2, we know $\mathscr{V}(u_{i_k}) \downarrow u^*_{i_0} = c$. Hence for $k \geq K$, we must have

$$u^*_{i_0} \leq \mathscr{V}(u_{t_k}) = \max\{|u_{i_0}(x, t_k + \theta) - u^*_{i_0}| : x \in \overline{\Omega},\ -\sigma \leq \theta \leq 0\}. \quad (4.29)$$

Since $u_{i_0 t_k} \to u_{i_0} \equiv 0$ as $k \to +\infty$, we may assume that K is sufficiently large so that $0 \leq u_{i_0}(x, t_k + \theta) \leq u^*_{i_0}$ for all $(x, \theta) \in \overline{\Omega} \times [-\tau, 0]$ and $k > K$. Furthermore, we have $u_{i_0}(x, t) > 0$ and, therefore, for $k \geq K$,

$$\max\{|u_{i_0}(x, t_k + \theta) - u^*_{i_0}| : x \in \overline{\Omega},\ -\sigma \leq \theta \leq 0 < u^*_{i_0},$$

an obvious contradiction to (4.29). This contradiction shows that c must equal 0 and the proof is thus complete.

A more general version of Theorem 4.1 takes the form:

Theorem 4.2. *Assume that* (H1) *holds for system* (2.1a–d) *and* $u_0(x, \theta) \in BC(\overline{\Omega} \times (-\infty, 0]) \cap C^3(\overline{\Omega} \times (-\infty, 0])$. *Assume further that there exist* $c_i > 0$, $i = 1, \ldots, n$, *such that*

$$a_i(1 - r\delta\tau_i) > c_i \sum_{j=1}^{n} c_j^{-1} |\mu_{ijg}|, \qquad r = \max\{c_i/c_j : i, j = 1, \ldots, n\}, \quad \text{(H3)}$$

where δ *is defined as in Theorem* 3.2. *Then the solution* $u_t(x, \theta)$ *tends to* u^* *as* $t \to +\infty$ *(in* UC_g *norm).*

Proof. We may denote $U_i(x, t) = c_i u_i(x, t)$, then $U(t) = (U_1(t), \ldots, U_n(t))$ satisfies

$$\partial_t U_i(x, t) = d_i \Delta U_i(x, t) + b_i U_i(x, t) \overline{G}_i(U_t(x, \cdot)), \qquad i = 1, \ldots, n,$$

where

$$\overline{G}_i(U_t(x, \cdot)) = r_i - a_i c_i^{-1} \int_{-\tau_i}^{0} U_i(x, t + \theta)\, d\mu_i(\theta) + \sum_{j=1}^{n} \int_{-\infty}^{0} U_j(x, t + \theta)\, d(c_j^{-1} \mu_{ij}(\theta)).$$

Let $c \equiv \max\{c_i : i = 1, \ldots, n\}$. Then Theorem 3.2 gives

$$|\partial_t U_i(x, t)| \leq c_i \delta \|u(x, t) - u^*\|_g = \delta \frac{c_i}{c} \cdot c \|u(x, t) - u^*\|_g \leq r\delta \|U(x, t) - U^*\|_g,$$

where $U^* = (c_1 u_1^*, \ldots, c_n u_n^*)$. The theorem now follows immediately from Theorem 4.1 by letting $\overline{a}_i = a_i c_i^{-1}$, $\overline{u}_{ij}(\theta) = c_j u_{ij}(\theta)$.

By virtue of the proof of Theorem 4.1, we have obtained the following result that generalises Theorem 1 (the result that partially motivated this work) in Martin and Smith [30].

Theorem 4.3. *In system* (2.1a–d), *assume that* $\int_{-\tau_i}^{0} u_{it}(x\theta)\, d\mu_i(\theta) = u_i(x, t)$ (*e.g.,* $\mu_i(\theta) = 1$, $\mu_i(\theta) = 0$, *for* $\theta < 0$) *and* $u_0(x, \theta) \in BC(\overline{\Omega} \times (-\infty, 0]) \cap C^3(\overline{\Omega} \times (-\infty, 0])$. *Assume further that there exist* $c_i > 0$, $i = 1, \dots, n$, *such that*

$$a_i > c_i \sum_{j=1}^{n} c_j^{-1} |\mu_{ijg}|. \tag{H4}$$

Then its solution $u_t(x, \theta)$ *tends to* u^* *as* $t \to +\infty$ (*in* UC_g *norm*).

Proof. We observe that the boundedness of $u_t(x, \theta)$ follows from the proofs of Lemmas 4.1 and 4.2. The rest of the proof is similar to that of Theorem 4.2 (in fact, simpler than it).

5. Discussion

To some extent, this work can be viewed as continuations and generalisations of that of Martin and Smith [30] and Kuang and Smith [27], and is closely related to the work of Kuang and Smith [28]. In [30], Lotka-Volterra type diffusive delay systems with bounded delays and dominating negative feedbacks are considered. Reference [27] deals with systems similar to (2.1a) but without diffusion. Reference [28] discusses delayed Lotka-Volterra systems with infinite delays but with discrete diffusions. Roughly, our results in this paper suggest that if a Lotka-Volterra system has globally asymptotically stable steady state when the negative feedback of the intraspecific competition is instantaneous, then it remains to have such a globally stable steady state, provided that the delay involved in the intraspecific competition is sufficiently small. This, indeed, in some sense, confirms our intuition that small delays are negligible in these dynamical systems.

Our results here are also closely related to the recent ones of Friesecke [7], in which he considers the scalar equation

$$\partial_t u - \Delta u = f(u(t), u(t-\tau)) \tag{5.1}$$

with either homogeneous Neumann or Dirichlet boundary conditions. Here $f(u(t), u(t-\tau)) = u(t)(1 - u(t-\tau))$ is its typical example. The main result in [7] states that *for a given* $K > 0$, *there is a* $\tau_0 = \tau_0(f, \Omega, K)$ *such that for* $\tau < \tau_0$, *all trajectories* u *of* (5.1) *with* $\overline{\lim} \int_\Omega u^2(t)\, dx < K$ *tend to steady states of* (5.1). For $f(u, v)$ satisfying the negative feedback condition

$$\overline{\lim_{|u|, |v| \to +\infty}} \frac{f(u, v)}{uv} < 0, \qquad uv > 0; \tag{5.2}$$

it is shown in Luckhaus [29] that solutions of (5.1) are indeed bounded as long as τ is small enough. Surprisingly, in [8], Friesecke succeeded in showing that for large τ, small μ, solutions of

$$\partial_t u - \mu \Delta u = u(t)(1 - u(t - \tau))$$

can grow exponentially to infinity as $t \to +\infty$. This somehow indicates that our assumption (H1) is both biologically and mathematically reasonable. It should be pointed out here that the results stated in this paper are independent of that of [7].

Generally, boundedness of solutions in diffusive delay systems is difficult to prove mathematically. This can be seen from the work of Luckhaus [29]. However, for some diffusive delay population interaction models, this may be easy to show if some (no matter how weak) instantaneous intraspecific competitions are present. The following are two examples of this kind.

EXAMPLE 1. Diffusive delay Lotka-Volterra competition model [32]:

$$\partial_t u_i = d_i \Delta u_i + b_i u_i \left[1 - a_{ii} u_i(x, t) - \sum_{j=1}^{n} c_{ij} \int_{-\tau}^{0} u_j(x, t+\theta)\, d\eta_{ij}(\theta) \right], \quad x \in \overline{\Omega} \subset \mathbf{R}^n,\ t > 0,\ 1 \le i \le n, \tag{5.3}$$

$$\partial_\nu u_i = 0, \qquad x \in \partial\Omega,\ t > 0, \tag{5.4}$$

$$u_i(x, t) = \phi_i(x, t) \ge 0, \qquad x \in \overline{\Omega},\ t \in [-\tau, 0], \tag{5.5}$$

d_i, b_i and a_{ii} are positive, c_{ij} and τ are nonnegative; η_{ij} is nondecreasing on $[-\tau, 0]$, and $\eta_{ij}(0^+) - \eta_{ij}(-\tau^-) = 1$. Let $\overline{u}_1(t)$ be the solution of the initial value problem

$$\overline{u}_1' = b_1 \overline{u}_1 (1 - a_{11} \overline{u}_1), \qquad \overline{u}_0 = \max_{x \in \overline{\Omega}} u_i(x, 0). \tag{5.6}$$

Define $\overline{u}_1(x, t) = \overline{u}_1(t)$, $x \in \overline{\Omega}$. Clearly, $\overline{u}_1$ satisfies (5.4) and $\Delta \overline{u}_i = 0$. Also,

$$\partial_t \overline{u}_1 - d_1 \Delta \overline{u}_1 - b_1 \overline{u}_1 \left[1 - a_{11} \overline{u}_1 - \sum_{j=1}^{n} c_{ij} \int_{-\tau}^{0} \overline{u}_j(x, t+\theta)\, d\eta_{ij}(\theta) \right]$$
$$\ge 0 = \partial_t u_1 - d_1 \Delta u_1 - b_1 u_1 \left[1 - a_{11} u_1 - \sum_{j=1}^{n} c_{ij} \int_{-\tau}^{0} u_j(x, t+\theta)\, d\eta_{ij}(\theta) \right], \quad \text{for } (x, t) \in \overline{\Omega} \times [0, +\infty). \tag{5.7}$$

Thus, by Theorem 10.1 in Smoller [36], we conclude that

$$u_1(x, t) \le \overline{u}_1(t), \qquad t \ge 0.$$

Clearly, $\lim_{t\to+\infty} \overline{u}_1(t) = a_{11}^{-1} < +\infty$. Similarly, we can show that

$$\varlimsup_{t\to+\infty} u_i(x, t) \le a_{ii}^{-1}, \qquad i = 1, \dots, n.$$

EXAMPLE 2. Diffusive delay Lotka-Volterra food chain [23]:

$$\partial_t u_i = d_i \Delta u_i + f_i(u), \qquad (x, t) \in \overline{\Omega} \times (0, +\infty), \tag{5.8}$$

with (5.4) and (5.5), and

$$f_i(u) = \begin{cases} b_1 u_1 \left[\delta_1 - a_{11} u_1 - \sum_{j=1}^{n} p_{j1} \int_{-\tau_{j1}}^{0} u_j(x, t+\theta)\, d\eta_{j1}(\theta)\right], & i = 1, \\ b_i u_i \left[-\delta_i - a_{ii} u_i + \sum_{j=1}^{i-1} q_{ij} \int_{-\tau_{ij}}^{0} u_j(x, t+\theta)\, d\eta_{ij}(\theta) - \sum_{j=1}^{n} p_{ji} \int_{-\tau_{ji}}^{0} u_j(x, t+\theta)\, d\eta_{ji}(\theta)\right], & i \ne 1, n, \\ b_n u_n \left[-\delta_n - a_{nn} u_n + \sum_{j=1}^{n-1} q_{nj} \int_{-\tau_{nj}}^{0} u_j(x, t+\theta)\, d\eta_{nj}(\theta) - p_{nn} \int_{-\tau_{nn}}^{0} u_n(x, t+\theta)\, d\eta_{nn}(\theta)\right], & i = n, \end{cases}$$

where b_i, δ_i and a_{ii} are positive, p_{ij}, q_{ij} and τ_{ij} are nonnegative. By an argument similar to that presented in the previous example, we can first show that

$$\varlimsup_{t\to+\infty} u_1(x, t) \le \delta_1 a_{11}^{-1}.$$

Next one can show that

$$\varlimsup_{t\to+\infty} u_2(x, t) \le \max\{0, (q_{21}\delta_1 a_{11}^{-1} - \delta_2) a_{22}^{-1}\}.$$

By repeating this argument, one can easily show that solutions of (5.6) are eventually uniformly bounded.

For a related account of boundedness of solutions, invariance and comparison in diffusive delay systems, the readers are referred to [32].

For more details about estimates of the constant δ that first appeared in Theorem 3.2, see Henry [19, pages 26–39].

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