ON CAUCHY'S LEMMA CONCERNING CONVEX POLYGONS

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Offered to H. S. M. Coxeter on his sixtieth birthday

One of the authors having just presented Cauchy's theorem on convex polyhedra to a class on "Convexity" at the University of Wisconsin, a discussion on Cauchy's lemma on convex polygons led to an exchange of letters which are here reproduced. The three letters are independently readable. Two new proofs of Cauchy's lemma are given, of which the second (§3) is very short.

1. The case of plane convex polygons (Letter from S. K. Z. to I. J. S.)

In the plane, Cauchy's lemma seems to me quite simple. Let $P_1 P_2 \ldots P_n$ be an arbitrary convex polygon in which some, or all, the internal angles at $P_2, \ldots, P_{n-1}$ are increased, none of them being reduced, while the lengths, respectively $a_1, \ldots, a_{n-1}$, of the sides $P_1 P_2, \ldots, P_{n-1} P_n$ remain invariant; we want to show that, if the new polygon is still convex, then the length of the side $P_1 P_n$ is increased. Starting from a remark made in our conversation by Henry B. Mann, I concocted the following proof.

Let P_k be the vertex of the polygon which is farthest from the line carrying the side $P_1 P_n$. This vertex may not be unique, but the ensuing modifications of the argument are obvious. Let $P_1 P_n$ form the x axis directed from P_1 to P_n, the orthogonal y axis passing through P_k and being directed from the origin towards P_k. Let ϕ_i be the angle from the x axis to the ray $P_i P_{i+1}$ ($i = 1, \ldots, n - 1$). The given polygon can now be deformed by increasing its internal angles at $P_2, \ldots, P_{n-1}$ to their required values without altering the lengths of the sides $P_1 P_2, \ldots, P_{n-1} P_n$, leaving P_k in its initial position, and so that no vertex of the deformed polygon should have an ordinate exceeding that of P_k. Finally, we require that the new position of $\angle P_{k-1} P_k P_{k+1}$ should contain the old position of this angle.

The angles $\phi_k, \ldots, \phi_{n-1}$, initially between $-\pi$ and 0, increase, but not beyond 0, or remain unchanged. Now the abscissa of P_n was

$$a_k \cos \phi_k + \ldots + a_{n-1} \cos \phi_{n-1},$$

and since, in the interval $(-\pi, 0)$, the cosine is an increasing function of its argument, it follows that the abscissa of P_n increases, unless the angles in

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question are all unchanged. By a similar argument, or by symmetry, one sees that the abscissa of P_i decreases, unless all the relevant angles are unchanged. Hence, in any case, the difference between the abscissae of P_n and P_1 increases, and as, in their initial position, the two points had the same ordinate, it follows that their distance increases. Q.E.D.

2. The case of spherical convex polygons (Letter from I. J. S. to S. K. Z.)

Cauchy's lemma for the sphere can be stated as follows: Let $P_1 P_2 \ldots P_n$ be a convex polygon on the sphere whose sides $P_1 P_2, \ldots, P_{n-1} P_n$ are of fixed lengths but whose angles $\psi_i = \angle P_{i-1} P_i P_{i+1}$ $(i = 2, \ldots, n-1)$ can vary. Let

$$P_n P_1 = F(\psi_2, \ldots, \psi_{n-1}) = F(\psi)$$

express the dependence of $P_n P_1$ on these angles. Let $\Omega = \{(\psi)\}$ be the $(n-2)$-dimensional domain of variability of (ψ) corresponding to convex polygons in the non-strict sense. Cauchy's Lemma states that the assumptions

(ψ) and (ψ') are in the interior of Ω,

$\psi_i \leqslant \psi_i'$ $(i = 2, \ldots, n-1$, with the symbol $<$ for some $i)$,

imply that $F(\psi) < F(\psi')$.

Cauchy's incorrect proof (**1**, 27–28; also reproduced in **3**, **110**) increases one angle at a time but disregards the restriction that during these operations we should remain in Ω. The proofs by E. Steinitz (**4**, pp. 61–67) and A. D. Alexandrow (**1**, pp. 138–141) use induction on n and show that the first "point" (ψ) can be continuously moved into the final position (ψ'), *within* Ω, so that only one of the ψ_i is increased at one time, while $F(\psi)$ increases strictly during the entire process. The awkward feature of these proofs is this: As we bump on the way into the boundary of Ω the subscript i of the sole increasing angle ψ_i changes many times, perhaps returning to an old value. For this reason these proofs are aesthetically less than satisfying and I hope that this is also the opinion of our friends, Coxeter and Rademacher.

In the following pages I propose to show that your ingenious idea of proving Cauchy's lemma in the plane by dissecting the convex polygon appropriately into two parts works also for the spherical case.

Let

$$(\Pi) = (A_0 A_1 \ldots A_q)$$

be a convex polygon on the surface of the sphere (having center O and radius unity) subject to the restriction

$$A_0 A_1 + A_1 A_2 + \ldots + A_{q-1} A_q < \pi. \tag{1}$$

By saying that (Π) is convex I mean that the infinite rays $OA_0, OA_1, \ldots, OA_q$ are (essential) edges of a convex polyhedral angle having its vertex O as extreme point. By the symbol Π, without parentheses, I mean the spherical polygonal path $A_0 A_1 \ldots A_q$.

I define *the evolute* $\mathscr{E}(\Pi)$ *of* Π as follows: Draw the circular arc $A_q E_1$ with centre A_{q-1} so that A_{q-2}, A_{q-1}, E_1 are on a great circle, or "collinear." Draw next the circular arc $E_1 E_2$ with centre A_{q-2} terminating on the great circle carrying the side $A_{q-3} A_{q-2}$ a.s.f. until we draw the arc $E_{q-2} E_{q-1}$ with centre A_1 so that the points A_0, A_1, E_{q-1} are collinear. Call J the union of all circular arcs drawn. Let $P \in J$ and consider arc $A_1 P$. By $\mathscr{E} = \mathscr{E}(\Pi)$ I mean the set defined by

$$\mathscr{E}(\Pi) = \cup_{P \in J} \text{arc } A_1 P. \tag{2}$$

It is star-shaped with respect to the point A_1. Here we assumed that $q > 1$. I define

$$\mathscr{E}(\Pi) = \emptyset \qquad \text{if } q = 1. \tag{3}$$

We distinguish two cases: We shall say that $\mathscr{E}(\Pi)$ is of type I and write $\mathscr{E} \in T_1$ or of type II ($\mathscr{E} \in T_2$) depending on whether

$$\angle A_1 A_q A_{q-1} \leqslant \pi/2 \text{ or } > \pi/2.$$

The different aspects of $\mathscr{E}$ in each of these two cases are shown by Figs. 1a and 1b. In Fig. 1b the arc $A_1 T$ is tangent to J at the point T.

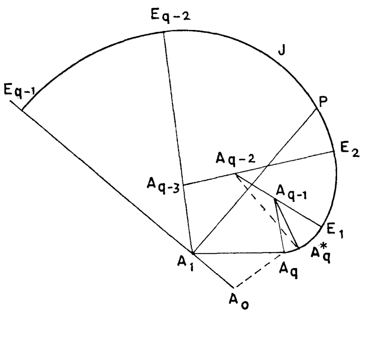

FIGURE 1*a*

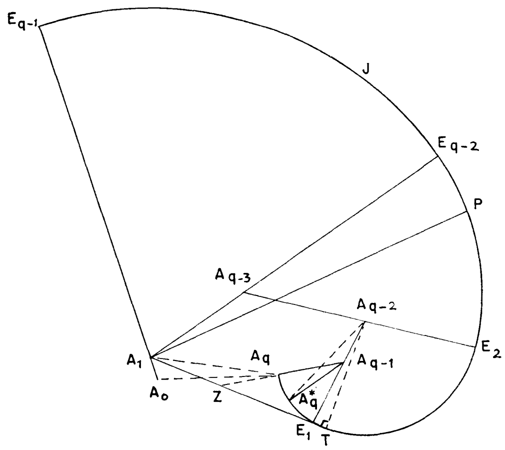

FIGURE 1*b*

The main difference between these two types is as follows: If $\mathscr{E} \in T_1$, then $A_0 \notin \mathscr{E}$. However, if $\mathscr{E} \in T_2$, then we may well have $A_0 \in \mathscr{E}$, in which case A_0 is anywhere within the triangle $A_1 A_q Z$, or even on its (open) side $A_1 Z$.

This last remark implies the following: Let us rotate the side $A_{q-1} A_q$ about A_{q-1} by an angle $\angle A_q A_{q-1} A_q^* < \angle A_q A_{q-1} E_1$. *If*

$$A_0 \notin \mathscr{E}(\Pi) \tag{4}$$

then

$$(A_0 A_1 \ldots A_{q-1} A_q^*) \textit{ is also a convex polygon.} \tag{5}$$

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Moreover, if we join A_{q-2} *and* A_q^* *by arc* $A_{q-2}A_q^*$, *then the new polygon*

(6) $$(\Pi^*) = (A_0A_1 \ldots A_{q-2}A_q^*) \text{ is convex.}$$

This follows from (5).

Our last remark concerning Π^* is as follows: From the triangle $A_{q-2}A_{q-1}A_q^*$ we conclude that

$$A_{q-2}A_q^* < A_{q-2}A_{q-1} + A_{q-1}A_q^* = A_{q-2}E_1 = A_{q-2}E_2.$$

This implies that

(7) $$\mathscr{E}(\Pi^*) \subset \mathscr{E}(\Pi).$$

Below I consider on the sphere a polygonal path $\Pi = A_0A_1 \ldots A_q$ and will denote by $(\Pi) = (A_0A_1 \ldots A_q)$ the closed polygon obtained by adding the last side arc A_qA_0.

LEMMA 1. *On a sphere on which we mark the North Pole N and South Pole S we consider a polygonal path*

(8) $$\Pi = A_0A_1 \ldots A_q$$

subject to the following assumptions:

(i) *The polygon* (Π) *is convex and such that the geographic longitude increases eastward as we proceed along* Π *from* A_0 *to* A_q *and returns decreasingly to its original value as we go from* A_q *to* A_0 *along* arc A_qA_0. *We also assume that on describing* (Π) *we move on the sphere clockwise as seen from outside the sphere.*

(ii)

(9) $$A_0A_1 + A_1A_2 + \ldots + A_{q-1}A_q < \pi.$$

(iii) *By* (i) *and* (ii) *we may consider the evolute* $\mathscr{E}(\Pi)$ *and shall assume that*

(10) $$S \notin \mathscr{E}(\Pi).$$

(iv) *We also consider a second polygonal path*

(11) $$\Pi' = A'_0A'_1 \ldots A'_q \qquad (A'_0 = A_0),$$

such that

(12) $$(\Pi') \text{ is convex.}$$

Moreover, we assume

(13) $$A'_iA'_{i+1} = A_iA_{i+1} \qquad (i = 0, \ldots, q-1),$$

and writing

$$\phi_0 = \angle NA_0A_1, \qquad \phi'_0 = \angle NA_0A'_1,$$
$$\phi_i = \pi - \angle A_{i-1}A_iA_{i+1}, \qquad \phi'_2 = \pi - \angle A'_{i-1}A'_iA'_{i+1}$$
$$(i = 1, \ldots, q-1),$$

we assume that

(14) $$\phi'_i \leqslant \phi_i \qquad (i = 0, \ldots, q-1).$$

Our last assumption is:

(v) *The inequality sign in* (14) *holds in the strict sense for at least one value of i. Then*

$$NA'_q < NA_q, \tag{15}$$

i.e. A'_q *is at a higher latitude than* A_q.

Proof. We use induction on q.

1. Let $q = 1$. By (14) and (v) we have $\phi'_0 < \phi_0$ and (9) implies that $A_0 A_1 = A_0 A'_1 < \pi$. The conclusion (15) follows from the cosine theorem of spherical trigonometry.

2. Let $q > 1$ and assume Lemma 1 for paths Π having $q - 1$ sides. We turn the side $A_{q-1} A_q$ about A_{q-1} to a new position $A_{q-1} A_q^*$ so that

$$\angle A_{q-2} A_{q-1} A_q^* = \angle A'_{q-2} A'_{q-1} A'_q.$$

Observe that $\angle NA_{q-1} A_q^* \leqslant \angle NA_{q-1} A_q$. Applying Lemma 1, for $q = 1$, to the 1-sided path $A_{q-1} A_q$ we conclude that

$$NA_q^* \leqslant NA_q. \tag{16}$$

Now we draw the arcs $A_{q-2} A_q^*$ and $A'_{q-2} A'_q$ (which are equal in length) and apply Lemma 1 to the two paths

$$\Pi^* = A_0 A_1 \ldots A_{q-2} A_q^*, \qquad \Pi^{*\prime} = A_0 A_1' \ldots A'_{q-2} A'_q \tag{17}$$

having $q - 1$ sides. Let us verify that the assumptions of Lemma 1 are satisfied. First of all we observe that our assumption (10) implies that

$$A_0 \notin \mathscr{E}(\Pi) \tag{18}$$

for the following reason. Let us assume that $A_0 \in \mathscr{E}(\Pi)$. We already know that this can happen only if $\mathscr{E}(\Pi) \in T_2$; also (Fig. 1b) that A_0 must then be in the interior of the triangle $A_1 A_q Z$ or on its side $A_1 Z$. In view of our assumption (i), S must then a fortiori be in this triangle and so $S \in \mathscr{E}(\Pi)$ in contradiction to our assumption (10).

We have just verified that (4) holds and we already know from (6) that the polygon (Π^*) is convex. Moreover, the longitude increases along the

$$\text{arc } A_{q-2} A_q^*$$

because S is outside the triangle $A_{q-2} A_{q-1} A_q^*$. Finally, the convexity of (Π') implies the convexity of $(\Pi^{*\prime})$. Therefore all required assumptions are satisfied and we may conclude that

$$NA'_q \leqslant NA_q^*. \tag{19}$$

Moreover, our assumption (v) implies that we have strict inequality in at least one of the relations (16) and (19). Therefore (15) holds and Lemma 1 is established.

Let now

$$(P) = (P_1 P_2 \ldots P_n) \tag{20}$$

be a convex polygon. Here $P_1 P_n$ is the "open" side in Cauchy's lemma. We wish to apply Lemma 1 to each of the two parts of (P) after an appropriate dissection. The dissection is performed as follows (Fig. 2). We may assume that $n \geqslant 4$ since Cauchy's lemma is evidently true for triangles. I claim that we can find a side $P_i P_{i+1}$, not adjacent to $P_n P_1$ (hence $i = 2, \ldots, n-2$) such that if we write

$$U = P_1 P_2 + P_2 P_3 + \ldots + P_{i-1} P_i, \tag{21}$$

$$L = P_{i+1} P_{i+2} + \ldots + P_{n-1} P_n, \tag{22}$$

then

$$U < \pi, \qquad L < \pi, \tag{23}$$

$$|U - L| < P_1 P_n + P_i P_{i+1}. \tag{24}$$

Indeed, let $x \in$ open arc $P_1 P_n$ and select on the polygon the point y such that the points x and y divide the perimeter of (P) into two equal parts. Let $y \in$ arc $P_i P_{i+1}$. By displacing x a little, if necessary, we may assume that y is between P_i and P_{i+1}. This does not exclude the possibility of $i = 1$ or $i = n - 1$. For instance, let us assume that $i = 1$; hence $y \in$ open arc $P_1 P_2$. This possibility can now be avoided as follows: since

$$P_1 P_2 < P_2 P_3 + \ldots + P_n P_1,$$

on moving x towards P_1 there comes a moment when y is forced to leave the side $P_1 P_2$ and enter the side $P_2 P_3$. Then $i = 2$ and our conditions are met. A verification of the inequalities (23) and (24) is now immediate: (23) are obvious because, by construction, U and L are each less than half the perimeter which is itself $<2\pi$ (see **3**, p. 37). Since x and y divide the perimeter into equal parts, we have

$$U < L + P_1 P_n + P_i P_{i+1}, \qquad L < U + P_1 P_n + P_i P_{i+1},$$

which imply (24).

Let now $(P) = (P_1, \ldots, P_n)$ and $(P') = (P'_1, \ldots, P'_n)$ be the two convex polygons of Cauchy's lemma and let us show that

$$P_1 P_n < P'_1 P'_n. \tag{25}$$

Let $P_i P_{i+1}$ be the side, not adjacent to $P_n P_1$, such that relations (21) to (24) hold. We produce the (directed) sides $P_n P_1$ and $P_{i+1} P_i$ beyond their end points until they meet in the point N. Likewise, let S be their intersection if we produce them in their opposite directions. The entire polygon (P) is within the lune shown in Fig. 2.

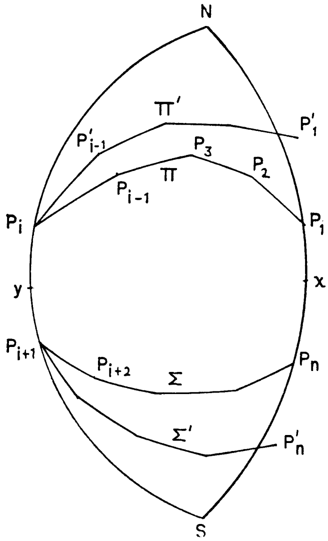

FIGURE 2

We now displace the polygon (P') such that its side $P'_i P'_{i+1}$ coincides with the side $P_i P_{i+1}$ of (P). If (P) and (P') should be differently oriented, then (P') must first be reflected in the great circle $NP_i S$ and then it will assume the aspect shown in Fig. 2. We now consider the two pairs of polygonal paths

(26) $$\Pi = P_i P_{i-1} \ldots P_2 P_1,$$

(27) $$\Pi' = P_i P'_{i-1} \ldots P'_2 P'_1,$$

and

(28) $$\Sigma = P_{i+1} P_{i+2} \ldots P_n,$$

(29) $$\Sigma' = P_{i+1} P'_{i+2} \ldots P'_n.$$

Let us first establish that

(30) $$S \notin \mathscr{E}(\Pi)$$

and

(31) $$N \notin \mathscr{E}(\Sigma).$$

It suffices, evidently, to establish (30), as (31) will follow by switching the role of the poles N and S. We prove (30) by assuming that

(32) $$S \in \mathscr{E}(\Pi)$$

and thereby reaching a contradiction.

Evidently (32) implies that $\mathscr{E}(\Pi)$ is of type II. Let Z be the intersection of arc $P_{i-1} T$ (tangent to J at the point T) with the extension beyond P_1 of the arc $P_2 P_1$. It follows that S is inside the triangle $P_{i-1} P_1 Z$ or perhaps on arc $P_{i-1} Z$, between P_{i-1} and Z. Whether P_i is in $\mathscr{E}(\Pi)$ or not, the convexity

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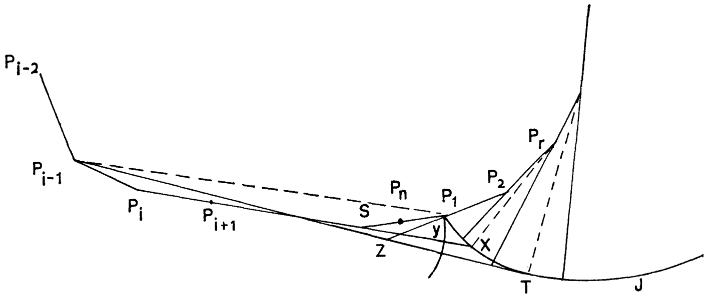

FIGURE 3

of (P) shows that the extension of arc $P_i S$, beyond S, must intersect the sectionally circular arc J at a point X between P_1 and T. Since $X \in J$, it follows that X is on one of the circular arcs which make up J. Let P_r be its centre. Finally, we draw the circular arc $P_1 Y$, with centre at S, and call Y its intersection with arc $P_i X$. We now argue as follows: Clearly

$$SP_1 = SY < SX$$

and therefore (see Fig. 2)

$$P_i P_{i+1} + L + P_n P_1 < P_i S + SP_1 < P_i S + SX = P_i X. \tag{33}$$

On the other hand, great-circle arcs being shortest paths, we have

$$P_i X < P_i P_{i-1} + P_{i-1} P_{i-2} + \ldots + P_{r+1} P_r + P_r X = U$$

and from (33) we obtain

$$P_i P_{i+1} + L + P_n P_i < U \quad \text{or} \quad P_i P_{i+1} + P_n P_i < U - L$$

which clearly contradicts our assumption (24). The relations (30) and (31) are therefore established.

In view of the assumptions of Cauchy's lemma we may therefore apply our Lemma 1 to the two pairs of paths (26), (27) and (28), (29) to conclude that

$$NP'_1 \leqslant NP_1, \qquad SP'_n \leqslant SP_n, \tag{34}$$

where the assumptions of Cauchy's lemma (that *some* angle actually increases) imply that in at least one of the inequalities (34) we have strict inequality. Adding them, we thus obtain

$$NP'_1 + P'_n S < NP_1 + P_n S. \tag{35}$$

However,

$$NP'_1 + P'_1 P'_n + P'_n S \geqslant NP_1 + P_1 P_n + P_n S,$$

because a meridian is the shortest path between the two poles. Subtracting the inequality (35) we obtain the desired inequality (25). Unfortunately, your simple idea became a bit complicated in its application to the sphere. Can this proof be simplified?

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3. Again the case of spherical convex polygons (Letter from S. K. Z. to I. J. S.)

I greatly enjoyed your proof which is very instructive and aesthetically preferable to that of Steinitz–Alexandrow (**4**, pp. 61–67 and **1**, pp. 138–141). However, you may be interested in the following simplification of the latter proof, which makes it quite short.

As in their original proof, we note that the proposition is trivial when we have three vertices. The inductive step from $n - 1$ to n vertices is facilitated by the fact that if one of the angles of the polygon remains unchanged, we can cut off the corresponding vertex, reducing the case of n vertices to that of $n - 1$.

Otherwise, let $P_1, P_2, \ldots, P_n$ be the vertices of the ploygon (Fig. 4), the angles at $P_2, \ldots, P_{n-1}$ being increased, the sides $P_1 P_2$, $P_2 P_3, \ldots, P_{n-1} P_n$

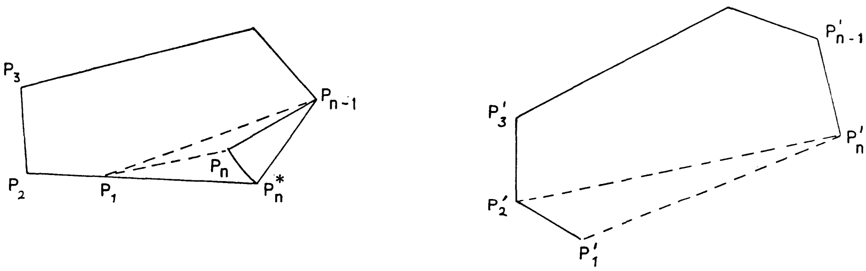

FIGURE 4

having invariant lengths, and $P_1 P_n$ being the "open" side. We assume the proposition to be true for $n - 1$ vertices and begin by moving P_n alone and increasing continuously the angle at P_{n-1} to its prescribed size. The side $P_1 P_n$ of the triangle $P_1 P_{n-1} P_n$ increases as the opposite angle increases. If, in the process, the convexity of the polygon is preserved, in view of the preceding remark, this brings us back to the case of $n - 1$ vertices.

If, on the other hand, the polygon ceases to be convex, let P_n^* be the last position of P_n for which the polygon is still convex. One of its angles must, then, become equal to π. It is not difficult to see that this angle is bound to be the one at P_1, but the argument which follows would also apply, with a change of notations, if it were at P_n^*.

Now P_2, P_1, and P_n^* are on a great circle, and

$$(*) \qquad P_1 P_n^* = P_2 P_n^* - P_2 P_1.$$

Moreover, the polygon $P_2 P_3 \ldots P_{n-1} P_n^*$ is convex. Deform the polygon $P_1 P_2 \ldots P_{n-1} P_n^*$ into $P'_1 P'_2 \ldots P'_n$ with the prescribed angles at $P'_2, \ldots, P'_{n-1}$ and $P'_1 P'_2 = P_1 P_2, \ldots, P'_{n-1} P'_n = P_{n-1} P_n = P_{n-1} P_n^*$. At the same time, the polygon $P_2 P_3 \ldots P_{n-1} P_n^*$ is deformed into $P'_2 P'_3 \ldots P'_n$, and, in view of our inductive assumption,

$$P'_2 P'_n > P_2 P_n^*.$$

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As a result of it, noting that the triangle $P'_1 P'_2 P'_n$ is bound to be convex, and bearing in mind the relation (*), we find

$$P'_1 P'_n \geqslant P'_2 P'_n - P'_1 P'_2 > P_2 P_n^* - P_1 P_2 = P_1 P_n^* \geqslant P_1 P_n.$$

Hence the proof is complete.

P.S. Here, as in the case of the plane, we do not really need to assume the convexity of the polygon $P'_1 P'_2 \ldots P'_n$. All that is required is that none of the internal angles of the polygon at $P'_2, \ldots, P'_{n-1}$ should exceed π, but, clearly, we need the convexity of the initial polygon $P_1 P_2 \ldots P_n$ to be sure that the triangle $P_1 P_{n-1} P_n$ is contained in it, so that an increase in the angle of the polygon at P_{n-1} entails an equal increase in the angle of the triangle at the same point.

Of course, the side $P'_2 P'_n$ considered in my proof is, by definition, the shortest arc of a great circle joining these two points. Although this plays no part in my proof, it may be worth noting that neither ambiguities nor discontinuities can arise in this connection when the polygon is subjected to deformation. This follows from the following, very simple proposition.

Let none of the angles of the simple spherical polygon $P_1 P_2 \ldots P_n$ at $P_2, \ldots, P_{n-1}$ exceed π, at least one of them being smaller than π; then the side $P_1 P_n$ cannot have a length equal to π. As you pointed out in our conversation, this corresponds to the fact, well known to any mariner, that one cannot reach one pole from the other by following a non-intersecting path composed of a finite number of arcs of great circles and turning at the vertices always right or always left, unless, of course, there is no turning and one follows one meridian all the way. A formal proof is easily obtained by observing that one has both to leave one pole and to arrive at the other by following an arc of a meridian.

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