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BOUNDARY VALUE PROBLEMS FOR HARMONIC FUNCTIONS ON THE HEISENBERG GROUP

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Analysis on the Heisenberg group has become an important area with strong connections to Fourier analysis, group representations, and partial differential operators. We propose to show in this work that special functions methods can also play a significant part in this theory. There is a one-parameter family of second-order hypoelliptic operators L_γ, ($\gamma \in \mathbf{C}$), associated to the Laplacian L_0 (also called the subelliptic or Kohn Laplacian). These operators are closely related to the unit ball for reasons of homogeneity and unitary group invariance. The associated Dirichlet problem is to find functions with specified boundary values and annihilated by L_γ inside the ball (that is, L_γ-harmonic). This is the topic of this paper.

Gaveau [**9**] proved the first positive result, showing that continuous functions on the boundary can be extended to L_0-harmonic functions in the ball, by use of diffusion-theoretic methods. Jerison [**15**] later gave another proof of the L_0-result. Hueber [**14**] has recently obtained some results dealing with special values of the Poisson kernel for L_0. Greiner [**10**] suggested that the L_γ-problem be treated by means of decomposition according to the unitary group action. This is the approach used here. The Poisson kernel associated to each irreducible unitary group module will be found, and an L^2-type convergence theorem will be proved, for all values of γ satisfying $-N < \gamma < N$ (on the Heisenberg group H_N, $N = 1, 2, \dots$). It remains, as yet, to establish the convergence of the summation over all the modules, but the Poisson kernel is at least formally known.

The operator L_γ is a prototype of homogeneous differential operators on nilpotent Lie groups, and there is a general theorem of Helffer and Nourrigat [**12**] that such an operator is hypoelliptic if and only if all of its images under continuous irreducible representations (of the group) are injective. This occurs for L_γ for all $\gamma \in \mathbf{C}$ with the non-hypoelliptic exceptions $\gamma = \pm N, \pm(N + 2), \pm(N + 4), \dots$. This was first shown by Folland and Stein [**6**] who also constructed the fundamental solution. It is striking that the Dirichlet problem can not be solved in general on the ball, for L_γ when $\gamma \leqq -N$ or $\gamma \geqq N$, a much larger set than the exceptional values for hypoellipticity, as will be shown in this paper.

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The unitary group decomposition of the Dirichlet problem leads to a boundary value problem on the upper half of the unit disk in the complex plane. Our method is to conformally map the disk to a strip and to transform the associated differential operator into one which commutes with translation along the strip. This allows the use of Fourier transforms and it becomes possible to determine the Fourier transform of the Poisson kernel; indeed it is a ratio of entire functions both of which are representable as Laplace transforms of compactly supported functions. We will also discuss the relationships between the Poisson kernel and the harmonic polynomials introduced by Greiner, and finally we will mention areas for further investigation, especially complex values of γ, and the problem of continuous boundary values. Certain hypergeometric functions and a family of Meixner-Pollaczek orthogonal polynomials form a fundamental part of the analysis.

1. The differential operator and its Fourier transform. The Heisenberg group H_N is the space $\mathbf{C}^N \times \mathbf{R}$ furnished with the group operation

$$(z, t) \cdot (w, s) := (z + w, t + s + 2 \operatorname{Im} \langle z, w \rangle),$$

where $N = 1, 2, \dots$ and

$$\langle z, w \rangle := \sum_{j=1}^{N} z_j \bar{w}_j, \ (z, w \in \mathbf{C}^N).$$

The left-invariant tangent fields are spanned by

$$Z_j := \frac{\partial}{\partial z_j} + i\bar{z}_j \frac{\partial}{\partial t},$$

$$\bar{Z}_j := \frac{\partial}{\partial \bar{z}_j} - iz_j \frac{\partial}{\partial t}, \quad 1 \leqq j \leqq N,$$

and

$$T := \frac{\partial}{\partial t},$$

and the subelliptic Laplacian is

$$L := -\frac{1}{2} \sum_{j=1}^{N} (Z_j\bar{Z}_j + \bar{Z}_jZ_j)$$

(see **[6]**). For $\gamma \in \mathbf{C}$ define the operator

$$L_\gamma := L + i\gamma T,$$

(this arises for $\gamma = -N + 2, -N + 4, \dots, N - 2$ when applying $\Box_b$ to forms [6]). We say a twice-differentiable function f on an open subset of H_N is L_γ-harmonic if $L_\gamma f = 0$.

The unitary group $U(N)$ acts on H_N by $u(z, t) := (uz, t)$, $u \in U(N)$, $(z, t) \in H_N$, and the action induced on functions commutes with L_γ. The $U(N)$-orbits of H_N can be indexed by

$$c(z, t) := t + i|z|^2.$$

Important $U(N)$-invariant sets are the ball

$$B := \{(z, t) \in H_N : |c(z, t)| < 1\}$$

and its boundary

$$\partial B := \{(z, t) : t^2 + |z|^4 = 1\}.$$

The Dirichlet problem for L_γ on B consists of extending functions on ∂B to L_γ-harmonic functions on B. This problem can be decomposed into $U(N)$-modules and the trivial $U(N)$ component is actually typical. We will first present the solution to the Dirichlet problem for $U(N)$-invariant functions (that is, depending only on $c(z, t)$), and then show how any other $U(N)$-module can be treated.

In terms of $c(z, t)$ we are considering the upper half of the complex plane, which we will conformally map onto a strip, with the half-disk being mapped onto a narrower strip. We will find a transformed version of the original differential equation which is invariant under the action of translation parallel to the axis of the strip. We begin with the half-disk.

1.1 PROPOSITION. *Let g be a twice differentiable function on an open subset of* $\{\zeta \in \mathbf{C} : \operatorname{Im} \zeta \geqq 0\}$,

$$L_\gamma g(c(z, t)) = 2i\left((\zeta - \bar{\zeta})\frac{\partial^2 g(\zeta)}{\partial\zeta\partial\bar{\zeta}} - \left(\frac{N - \gamma}{2}\right)\frac{\partial g(\zeta)}{\partial\zeta} + \left(\frac{N + \gamma}{2}\right)\frac{\partial g(\zeta)}{\partial\bar{\zeta}}\right),$$

where $\zeta = t + i|z|^2$, $(z, t) \in$ (*an open subset of*) H_N.

1.2 *Definition*. Let $\alpha, \beta \in \mathbf{C}$ and define a differential operator on functions on $\mathbf{C}$ by

$$D_{\alpha\beta}g(\zeta) = \left((\zeta - \bar{\zeta})\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}} - \alpha\frac{\partial}{\partial\zeta} + \beta\frac{\partial}{\partial\bar{\zeta}}\right)g(\zeta), \quad \zeta \in \mathbf{C}.$$

Thus our Dirichlet problem reduces to finding functions on

$$\{\zeta \in \mathbf{C} : |\zeta| \leqq 1, \operatorname{Im} \zeta \geqq 0\}$$

annihilated by $D_{\alpha\beta}$ in $\{|\zeta| < 1\}$ with specified boundary values on $\{|\zeta| = 1\}$, where

$$\alpha := (N - \gamma)/2, \ \beta := (N + \gamma)/2.$$

The homogeneous polynomials annihilated by $D_{\alpha\beta}$ are already known: namely the Heisenberg polynomials

$$C_n^{(\alpha,\beta)}(\zeta) := \sum_{j=0}^{n} \frac{(\alpha)_j(\beta)_{n-j}}{j!(n-j)!} \overline{\zeta}^j \zeta^{n-j}, \quad n = 0, 1, 2, \ldots.$$

(introduced by Greiner [**10**] with a different notation; Gasper [**8**] found a complex orthogonality on the entire circle). Greiner and Koornwinder [**11**] pointed out that the hypoellipticity of L_γ implies that any function L_γ-harmonic in a neighborhood of $(0, 0) \in H_N$ must be real-analytic, and if it is also $U(N)$-invariant, then it has an expansion

$$\sum_{n=0}^{\infty} a_n C_n^{(\alpha,\beta)}(t + i|z|^2)$$

at least locally at $(0, 0)$.

From our work [**4**] on the limiting case $\alpha = (1 - \mu)\nu$, $\beta = (1 + \mu)\nu$, μ fixed, $\nu \to 0_+$, we are led to study the effect on $D_{\alpha\beta}$ of the Möbius group fixing the upper half disk. This group consists of the transformations

$$\mathscr{F}_t(\zeta) := \frac{\zeta + \operatorname{th} t}{1 + \zeta(\operatorname{th} t)}, \quad t \in \mathbf{R};$$

note that

$$\mathscr{F}_{t_1} \circ \mathscr{F}_{t_2} = \mathscr{F}_{t_1+t_2}.$$

Define the conformal map

$$\rho(\zeta) := \frac{1}{2} \log \frac{1 + \zeta}{1 - \zeta},$$

which maps the open unit disk to the strip

$$S := \left\{ \sigma + i\tau : \sigma \in \mathbf{R}, -\frac{\pi}{4} < \tau < \frac{\pi}{4} \right\}$$

Then

$$\rho(\mathscr{F}_t(\zeta)) = \rho(\zeta) + t, \quad (\zeta \neq \pm 1, t \in \mathbf{R}).$$

In the sequel we only require $\alpha > 0$ and $\beta > 0$ and let $\nu := (\alpha + \beta)/2$ (that is, it is not necessary for $N = 2\nu$ to be an integer).

Applying the map ρ to $D_{\alpha\beta}$ we get:

$$D_{\alpha\beta} g(\rho(\zeta)) = i \sin 2\tau \operatorname{ch} \rho \operatorname{ch} \overline{\rho} \frac{\partial^2 g}{\partial \rho \partial \overline{\rho}} - \alpha \operatorname{ch}^2 \rho \frac{\partial g}{\partial \rho} + \beta \operatorname{ch}^2 \overline{\rho} \frac{\partial g}{\partial \overline{\rho}},$$

where $\tau = \operatorname{Im} \rho$. We try an integrating factor of the form

$$(\operatorname{ch} \bar{\rho})^A (\operatorname{ch} \rho)^B$$

to get a differential operator commuting with translation, that is, with coefficients independent of $\sigma = \operatorname{Re} \rho$; indeed $A = \alpha$, $B = \beta$ works.

1.3 PROPOSITION.

$$D_{\alpha\beta}((\operatorname{ch} \bar{\rho})^{\alpha}(\operatorname{ch} \rho)^{\beta} g(\rho)) = \frac{i}{4}(\operatorname{ch} \bar{\rho})^{\alpha+1}(\operatorname{ch} \rho)^{\beta+1} D'_{\alpha\beta} g(\rho),$$

where

$$D'_{\alpha\beta} := (\sin 2\tau)\left(\frac{\partial^2}{\partial\sigma^2} + \frac{\partial^2}{\partial\tau^2} - 4\alpha\beta\right)$$
$$+ 2 \cos 2\tau\left(2\nu \frac{\partial}{\partial\tau} - i(\beta - \alpha)\frac{\partial}{\partial\sigma}\right),$$

$\rho = \sigma + i\tau$; *and* $D_{\alpha\beta} g = 0$ *if and only if*

$$D'_{\alpha\beta}((\operatorname{ch} \bar{\rho})^{-\alpha}(\operatorname{ch} \rho)^{-\beta} g) = 0$$

for functions on the strip.

Since $D'_{\alpha\beta}$ commutes with translation (in σ) we will solve the associated Dirichlet problem by a convolution integral, on the upper edge of the strip S.

We reduce the equation to an ordinary differential equation by taking Fourier transforms in σ.

For smooth functions g on the strip let

$$\hat{g}(y, \tau) := \int_{\mathbf{R}} g(\sigma, \tau) e^{-iy\sigma} d\sigma$$

($y \in \mathbf{R}$, for each τ for which $\int_{\mathbf{R}} |g(\sigma, \tau)| d\sigma < \infty$). Then $D'_{\alpha\beta} g = 0$ implies

$$(\sin 2\tau)\frac{\partial^2}{\partial\tau^2}\hat{g} + 4\nu(\cos 2\tau)\frac{\partial}{\partial\tau}\hat{g}$$
$$- ((\sin 2\tau)(4\alpha\beta + y^2) - 2(\beta - \alpha)y \cos 2\tau)\hat{g} = 0.$$

Under the change of variable $t = 1 - e^{-4i\tau}$ the equation becomes

$$8i(1 - t)^{-1/2}\left(t(1 - t)^2 \frac{\partial^2}{\partial t^2}\hat{g} - (1 - t)(t - \nu(2 - t))\frac{\partial}{\partial t}\hat{g}\right.$$
$$\left. + (t\left(\frac{\alpha\beta}{4} + \frac{y^2}{16}\right) - \left(\frac{\beta - \alpha}{8}\right)iy)\hat{g}\right) = 0.$$

This turns into the hypergeometric equation by use of the integrating factor $(1 - t)^{h(y)}$ where

$$h(y) = \frac{\alpha}{2} - \frac{iy}{4} \text{ or } \frac{\beta}{2} + \frac{iy}{4}.$$

The unique solution of $D'_{\alpha\beta}k = 0$, up to multiplication by functions in y, which is regular at $\tau = 0$ ($t = 0$) is

$$(1.4) \quad k_{\alpha\beta}(y, \tau) := e^{-2i\tau\alpha - y\tau} {}_2F_1\left(\begin{matrix} \nu - iy/2, \alpha \\ 2\nu \end{matrix}; 1 - e^{-4i\tau} \right),$$

normalized by $k_{\alpha\beta}(y, 0) = 1$; and the hypergeometric function is given by the series in

$$|1 - e^{-4i\tau}| < 1.$$

Since this function is fundamental in all that follows, especially at $\tau = \pi/4$, we need a better representation; indeed for real τ the series only converges for $|\tau| < \pi/12$.

1.5 THEOREM. *For* $\alpha, \beta > 0$, $y \in \mathbf{C}$, $0 < \tau \leqq \pi/4$,

$$k_{\alpha\beta}(y, \tau) = \frac{1}{B(\alpha, \beta)} (\sin 2\tau)^{1-\alpha-\beta} \times \int_{-\tau}^{\tau} e^{yt} (\sin(\tau + t))^{\alpha-1} (\sin(\tau - t))^{\beta-1} dt;$$

also

$$k_{\alpha\beta}(y, -\tau) = k_{\alpha\beta}(-y, \tau) = k_{\beta\alpha}(y, \tau).$$

(*Here B denotes the beta-function.*)

Proof. By the Euler integral formula

$$k_{\alpha\beta}(y, \tau) = e^{-2i\tau\alpha - y\tau} B(\alpha, \beta)^{-1} \times \int_0^1 t^{\alpha-1} (1 - t)^{\beta-1} (1 - t(1 - e^{-4i\tau}))^{-\nu+iy/2} dt,$$

(note this integral becomes singular at $t = 1/2$ as $\tau \to (\pi/4)-$). Make the substitution $t = \frac{1}{2}(1 - \operatorname{th} x)$ to obtain

$$k_{\alpha\beta}(y, \tau) = \frac{2^{1-2\nu}}{B(\alpha, \beta)} \int_{\mathbf{R}} \exp((\beta - \alpha)(x + i\tau))(\operatorname{ch} x)^{-\nu-iy/2} \times (\operatorname{ch}(x + 2i\tau))^{-\nu+iy/2} dx.$$

This is an integral of an analytic function whose singularities nearest to $\mathbf{R}$ are

$$x = \pm\frac{\pi}{2}i \text{ and } x = \pm i\frac{\pi}{2} - 2i\tau,$$

and so the path of integration can be deformed to $\mathbf{R} - i\tau$ (by the usual large rectangular contour method). This gives

$$k_{\alpha\beta}(y, \tau) = \frac{2^{1-2\nu}}{B(\alpha, \beta)} \int_{\mathbf{R}} e^{(\beta-\alpha)x}(\operatorname{ch}(x - i\tau))^{-\nu-iy/2} \times (\operatorname{ch}(x + i\tau))^{-\nu+iy/2}dx.$$

This formula shows that

$$k_{\alpha\beta}(y, -\tau) = k_{\alpha\beta}(-y, \tau) = k_{\beta\alpha}(y, \tau).$$

Now make the substitution

$$e^{2x} = \sin(\tau - t)/\sin(\tau + t), \ -\tau < t < \tau$$

to get the required form $k_{\alpha\beta}$.

Note that in the proof, as in the sequel, we always use the principal branch of complex power functions.

1.6. COROLLARY. *For* $\alpha, \beta > 0$, $-\pi/4 \leqq \tau \leqq \pi/4$, $y \in \mathbf{R}$, $k_{\alpha\beta}(y, \tau) > 0$, *indeed,*

$$0 < e^{-|y\tau|}k_{\alpha\beta}(0, \tau) \leqq k_{\alpha\beta}(y, \tau) \leqq e^{|y\tau|}k_{\alpha\beta}(0, \tau).$$

Further for each $\tau \neq 0$, $k_{\alpha\beta}$ *is a convex function of* y, *that is*

$$\left(\frac{\partial}{\partial y}\right)^2 k_{\alpha\beta}(y, \tau) > 0.$$

The relevance of $k_{\alpha\beta}$ to the Dirichlet problem is as follows: suppose g is a function on the strip

$$\left\{\sigma + i\tau : \sigma \in \mathbf{R}, -\frac{\pi}{4} < \tau \leqq \frac{\pi}{4}\right\}$$

which is reasonably well-behaved and such that $D'_{\alpha\beta}g = 0$, then

$$\hat{g}(y, \tau) = h(y)k_{\alpha\beta}(y, \tau) \quad \text{for some } h;$$

further

$$\hat{g}(y, \pi/4) = h(y)k_{\alpha\beta}(y, \pi/4)$$

implying that

$$\hat{g}(y, \tau) = \hat{g}(y, \pi/4)(k_{\alpha\beta}(y, \tau)/k_{\alpha\beta}(y, \pi/4)),$$

and so $g(\sigma, \tau)$ is the convolution of $g(\sigma, \pi/4)$ with the kernel $K_{\alpha\beta}(\sigma, \tau)$ where

$$\hat{K}_{\alpha\beta}(y, \tau) = k_{\alpha\beta}(y, \tau)/k_{\alpha\beta}(y, \pi/4)\,).$$

To establish the existence and necessary properties of $K_{\alpha\beta}$ we will develop some bounds and asymptotic expressions for $k_{\alpha\beta}$. However, first we consider some immediate consequences of the theorem.

The special case $\alpha = \beta = \nu$ allows a more explicit formula.

1.7 THEOREM.

$$k_{\nu\nu}(y, \tau) = {}_2F_1\left(\begin{matrix}(\nu/2) - iy/4, (\nu/2) + iy/4 \\ \nu + 1/2\end{matrix}; \sin^2(2\tau)\right)$$

$$-\pi/4 \leqq \tau \leqq \pi/4, y \in \mathbf{C}, \textit{and}$$

$$k_{\nu\nu}(y, \pi/4) = \frac{\Gamma\left(\nu + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\,(\nu + 1)/2 + iy/4)\Gamma(\,(\nu + 1)/2 - iy/4)},$$

an entire function with zeros at

$$y = \pm 2i(\nu + 2n + 1), n = 0, 1, 2, \dots .$$

Proof. The expression for $k_{\nu\nu}(y, \tau)$ comes from applying a quadratic transformation (**[5]**, vol. 1, p. 112, #26) to

$${}_2F_1\left(\begin{matrix}\nu - iy/2, \nu \\ 2\nu\end{matrix}; 1 - e^{-4i\tau}\right).$$

The Gauss sum can be used for $k_{\nu\nu}\left(y, \frac{\pi}{4}\right)$.

Later we will state the explicit form of $K_{\nu\nu}(\sigma, \tau)$.

1.8 PROPOSITION. For $\alpha, \beta > 0, -\pi/4 \leqq \tau \leqq \pi/4$,

$$k_{\alpha\beta}(0, \tau) = {}_2F_1\left(\begin{matrix}\alpha/2, \beta/2 \\ \nu + 1/2\end{matrix}; \sin^2 2\tau\right).$$

Proof. Use the same quadratic transformation as in 1.7.

We can find an upper bound for $1/k_{\alpha\beta}(y, \pi/4)$ which appeared to be fairly sharp when tested in several numerical experiments.

1.9 THEOREM *For* $\alpha, \beta > 0, -\pi/4 \leqq \tau \leqq \pi/4$,

$$k_{\alpha\beta}(y, \tau) \geqq (\Gamma(\nu)^2/(\Gamma(\alpha)\Gamma(\beta)\,)\,)k_{\nu\nu}(0, \tau);$$

in particular

$$k_{\alpha\beta}(y, \pi/4) \geqq \left(\Gamma\left(\nu + \frac{1}{2}\right)\Gamma(\nu)^2\pi^{1/2}\right)/(\Gamma(\alpha)\Gamma(\beta)\Gamma(\,(\nu + 1)/2)^2),$$

$$(y \in \mathbf{R}).$$

Proof. Fix ν, τ and let

$$F(y, s) := \int_{-\tau}^{\tau} e^{yt}(\sin(\tau + t))^{\nu-s-1}(\sin(\tau - t))^{\nu+s-1}dt$$
$$= \int_{-\tau}^{\tau} e^{yt+sg(t)}w(t)dt$$

where

$$w(t) = (\sin(\tau + t)\sin(\tau - t))^{\nu-1} \quad \text{and}$$
$$g(t) = \log(\sin(\tau + t)/\sin(\tau - t)).$$

We will show $F(y, s) \geqq F(0, 0)$, for $y \in \mathbf{R}$ and $-\nu < s < \nu$. Indeed g is an odd function, and w is even implying that $y = 0$, $s = 0$ is a critical point of F. Then by the Cauchy-Schwarz inequality

$$\left|\frac{\partial^2 F}{\partial y \partial s}\right| < \left(\frac{\partial^2 F}{\partial y^2}\right)^{1/2}\left(\frac{\partial^2 F}{\partial s^2}\right)^{1/2}.$$

Thus F has a global minimum at $(0, 0)$. Finally we note that

$$K_{\alpha\beta}(y, \tau) = B(\alpha, \beta)^{-1}F(y, (\beta - \alpha)/2).$$

We get $F(0, 0)$ from Theorem 1.7.

By use of distributions we can extract a family of solutions of $D'_{\alpha\beta}f = 0$ from $k_{\alpha\beta}$.

1.10 *Definition*. For $j = 0, 1, 2, \dots$ let

$$k_{\alpha\beta,j}(\tau) := B(\alpha, \beta)^{-1}(\sin 2\tau)^{1-2\nu}$$
$$\times \int_{-\tau}^{\tau} t^j(\sin(\tau + t))^{\alpha-1}(\sin(\tau - t))^{\beta-1}dt$$

for $0 < \tau \leqq \dfrac{\pi}{4}$, and

$$k_{\alpha\beta,j}(-\tau) = (-1)^j k_{\alpha\beta,j}(\tau),\ k_{\alpha\beta,0}(0) = 1,\ k_{\alpha\beta,j}(0) = 0$$

for $j > 0$. That is,

$$k_{\alpha\beta,j}(\tau) = \left(\frac{\partial}{\partial y}\right)^j k_{\alpha\beta}(y, \tau)\,|_{y=0}.$$

1.11 THEOREM. *For* $n = 0, 1, 2, \dots$

$$D'_{\alpha\beta}\left(\sum_{j=0}^{n}\binom{n}{j}\sigma^{n-j}(-i)^j k_{\alpha\beta,j}(\tau)\right) = 0.$$

Proof. We must show

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$$D'_{\alpha\beta}\left(\left(\sigma - i\frac{\partial}{\partial y}\right)^n k_{\alpha\beta}(y, \tau)\,|_{y=0}\right) = 0$$

(as a function of σ, τ). Because $k_{\alpha\beta}$ is annihilated by the Fourier transform of $D'_{\alpha\beta}$, this is a consequence of the identity

$$\left[\left(\frac{\partial}{\partial\sigma}\right)^m\left(\sigma - i\frac{\partial}{\partial y}\right)^n - \left(\sigma - i\frac{\partial}{\partial y}\right)^n (iy)^m\right]F(y) = 0$$

when $y = 0$, where F is a function of y. But

$$\left(\sigma - i\frac{\partial}{\partial y}\right)^n (iy)^m F(y)$$
$$= \sum_{j=\max(0,m-n)}^{m} \frac{m!}{j!}\binom{n}{m-j}(iy)^j\left(\sigma - i\frac{\partial}{\partial y}\right)^{n-m+j} F(y),$$

and at $y = 0$ the sum reduces to

$$\frac{n!}{(n-m)!}\left(\sigma - i\frac{\partial}{\partial y}\right)^{n-m} F(0), \quad (0 \text{ if } n < m),$$

which is the same as

$$\left(\frac{\partial}{\partial\sigma}\right)^m\left(\sigma - i\frac{\partial}{\partial y}\right)^n F(0).$$

Thus we have found a family of $D'_{\alpha\beta}$-harmonic functions which are polynomial in σ. They are unbounded at $\sigma = \pm\infty$ but are of relatively slow growth.

2. Bounds for the Poisson kernel. The next task is to show that

$$k_{\alpha\beta}(y, \tau)/k_{\alpha\beta}(y, \pi/4)$$

is a Fourier transform for each τ, $0 \leqq \tau < \pi/4$, and to establish uniform upper bounds for this function in $y \in \mathbf{R}$, $0 \leqq \tau \leqq \pi/4$.

2.1 PROPOSITION. *There is a constant $C_{\alpha\beta}$ depending only on α, β such that*

$$k_{\alpha\beta}(y, \tau) \leqq C_{\alpha\beta} \min\left(\frac{1}{\beta}, \frac{\Gamma(\beta)}{(2\tau y)^\beta}\right)e^{\tau y}$$

for $y \geqq 0$, and

$$k_{\alpha\beta}(y, \tau) \leqq C_{\alpha\beta} \min\left(\frac{1}{\alpha}, \frac{\Gamma(\alpha)}{(2\tau|y|)^\alpha}\right)e^{-\tau y}$$

for $y \leqq 0$, $0 \leqq \tau \leqq \pi/4$.

Proof. It suffices to consider $y \geqq 0$. Assume first $\alpha \geqq 1$ and $\beta \geqq 1$. In the formula of 1.5 use the bounds

$$(\sin(\tau + t))^{\alpha-1} \leqq (\sin 2\tau)^{\alpha-1} \quad \text{and}$$
$$(\sin(\tau - t))^{\beta-1} \leqq (\tau - t)^{\beta-1}$$

to obtain

$$k_{\alpha\beta}(y, \tau) \leqq B(\alpha, \beta)^{-1}(\sin 2\tau)^{-\beta} \int_{-\tau}^{\tau} e^{yt}(\tau - t)^{\beta-1}dt.$$

The integral $\leqq e^{y\tau}(2\tau)^{\beta}/\beta$ and also equals

$$e^{y\tau}y^{-\beta}\int_0^{2\tau y} e^{-s}s^{\beta-1}ds \leqq e^{y\tau}y^{-\beta}\Gamma(\beta).$$

Thus we require

$$C_{\alpha\beta} \geqq \sup_{0\leqq\tau\leqq 1} B(\alpha, \beta)^{-1}(2\tau/(\sin 2\tau))^{\beta} = B(\alpha, \beta)^{-1}(\pi/2)^{\beta}.$$

If $0 < \beta < 1$ then

$$(\sin(\tau - t))^{\beta-1} \leqq ((\tau - t)(\sin 2\tau)/(2\tau))^{\beta-1}$$

and we proceed similarly with a slightly different bound for $C_{\alpha\beta}$. If $0 < \alpha < 1$ then split up the integral into two parts to get

$$\begin{aligned} k_{\alpha\beta}(y, \tau) \leqq{} & B(\alpha, \beta)^{-1}(\sin 2\tau)^{1-2\nu} \\ & \times \left[\int_{-\tau}^{0} (\sin(\tau + t))^{\alpha-1}(\sin(\tau - t))^{\beta-1}dt \right. \\ & \left. + (\sin 2\tau)^{\alpha-1} \int_0^{\tau} e^{yt}(\sin(\tau - t))^{\beta-1}dt\right]. \end{aligned}$$

The first integral is less than some constant (depending on α and β) times $\tau^{2\nu-1}$, and the second integral is treated similarly to the above.

2.2 Proposition.

$$k_{\alpha\beta}(y, \tau)k_{\alpha\beta}(-y, \tau) = {}_4F_3\left(\begin{matrix}\alpha, \beta, \nu - iy/2, \nu + iy/2 \\ 2\nu, \nu, \nu + 1/2\end{matrix}; \sin^2 2\tau\right),$$

for $\alpha, \beta > 0$, $-\pi/4 \leqq \tau \leqq \pi/4$, $y \in \mathbf{C}$.

Proof. For τ near 0 ($|\tau| < \pi/12$) we use the ${}_2F_1$ form of $k_{\alpha\beta}$ given in (1.4). By the identity (see **[18]**, p. 80, #2.5.32)

$$\begin{aligned} & {}_2F_1\left(\begin{matrix}a, b \\ c\end{matrix}; z\right){}_2F_1\left(\begin{matrix}a, c - b \\ c\end{matrix}; z\right) \\ & = (1 - z)^{-a}{}_4F_3\left(\begin{matrix}a, b, c - a, c - b \\ c, c/2, (c + 1)/2\end{matrix}; -\frac{z^2}{4(1 - z)}\right) \end{aligned}$$

(with $a = \alpha$, $b = \nu - iy/2$, $c = 2\nu$, $z = 1 - e^{-4i\tau}$), the stated formula holds. Both sides are analytic in τ and entire in y.

2.3 COROLLARY. *For* $-\pi/4 \leqq \tau \leqq \pi/4, y \in \mathbf{R}$,

$$k_{\alpha\beta}(y, \tau)k_{\alpha\beta}(-y, \tau) \geqq (\Gamma(\nu)^2/(\Gamma(\alpha)\Gamma(\beta)))k_{\nu\nu}(y, \tau)^2.$$

Proof. For $y \in \mathbf{R}$ each term of the ${}_4F_3$-series is positive (for $\tau \neq 0$), and it suffices to show

$$(\alpha)_n(\beta)_n \geqq (\Gamma(\nu)^2/(\Gamma(\alpha)\Gamma(\beta)))(\nu)_n(\nu)_n$$

(using the formula with $\alpha = \beta = \nu$). But if

$$t_n := \frac{(\alpha)_n(\beta)_n}{(\nu)_n(\nu)_n}$$

then

$$\frac{t_{n+1}}{t_n} = 1 - \frac{((\beta - \alpha)/2)^2}{(\nu + n)^2} < 1$$

(if $\alpha, \beta > 0$), so $\{t_n\}$ is a decreasing sequence; further

$$t_n = \frac{\Gamma(\nu)^2}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha + n)\Gamma(\beta + n)}{\Gamma(\nu + n)\Gamma(\nu + n)}$$

which is asymptotic to $\Gamma(\nu)^2/(\Gamma(\alpha)\Gamma(\beta))$.

We need a lemma to establish the key bound on $k_{\alpha\beta}(y, \tau)$.

2.4 LEMMA. *For* $\nu > 0$, *there is a constant* C'_ν *depending on* ν *such that*

$$|\Gamma((\nu + 1)/2 + iy/4)|^2 \leqq C'_\nu (4(\nu + 1)^2 + y^2)^{\nu/2}e^{-|y|\pi/4}$$

for all $y \in \mathbf{R}$.

Proof. We use the asymptotic expression for $\log \Gamma$, to get

$$\begin{aligned}
&\log(\Gamma((\nu + 1)/2 + iy/4)\Gamma((\nu + 1)/2 - iy/4)) \\
&= \log 2\pi + (\nu/2)\log((\nu + 1)^2/4 + y^2/16) \\
&- (\nu + 1) - \frac{y}{2} \arctan(y/(2(\nu + 1))) \\
&+ J((\nu + 1)/2 + iy/4) + J((\nu + 1)/2 - iy/4)
\end{aligned}$$

where J is the Binet function (for example, see [**13**], p. 457). The Binet function has an asymptotic development and using one term we obtain

$$J(z) = \frac{1}{12z} + \frac{1}{3}\int_0^\infty \frac{B_3^*(t)}{(t + z)^3}dt$$

(valid for $z \in \mathbf{C}\backslash[-\infty, -1]$), where B_3^* is the Bernoulli function of order 3, a function of period 1 equal to

$$t^3 - \frac{3}{2}t^2 + \frac{1}{2}t \quad \text{on } 0 \leqq t \leqq 1,$$

thus $|B_3^*(t)| \leqq \sqrt{3}/36$. From this bound,

$$J((\nu + 1/2 + iy/4) + J((\nu + 1)/2 - iy/4)$$
$$\leqq (\nu + 1)/\left(12\left(\left(\frac{\nu + 1}{2}\right)^2 + \left(\frac{y}{4}\right)^2\right)\right)$$
$$+ \frac{\sqrt{3}}{36}\left(\frac{2}{\nu + 1}\right)^2,$$

by use of

$$|t + z|^{-3} \leqq (t + \operatorname{Re} z)^{-3}.$$

Next we use the elementary inequality

$$x \arctan x \geqq \frac{\pi}{2}|x| - 1 \quad \text{for } x \in \mathbf{R}$$

and obtain

$$-\frac{y}{2}\arctan\frac{y}{2(\nu + 1)} \leqq (\nu + 1) - \frac{\pi}{4}|y|.$$

These bounds suffice to establish the lemma.

2.5 THEOREM. *For α, $\beta > 0$ there is a constant $C'_{\alpha\beta}$ such that*

$$\left(k_{\alpha\beta}(y, \tau)/k_{\alpha\beta}\left(y, \frac{\pi}{4}\right)\right) \leqq \tau^{-\delta}C'_{\alpha\beta}e^{-|y|(\pi/4 - \tau)}$$

for $0 < \tau \leqq \pi/4$, and $|y| \geqq 2(\nu + 1)$, and

$$\left(1/k_{\alpha\beta}\left(y, \frac{\pi}{4}\right)\right) \leqq C'_{\alpha\beta}(1 + |y|^{\delta})e^{-|y|\pi/4},$$

where $\delta = \beta$ for $y > 0$ and $\delta = \alpha$ for $y < 0$. Further

$$\sup\left\{k_{\alpha\beta}(y, \tau)/k_{\alpha\beta}\left(y, \frac{\pi}{4}\right): y \in \mathbf{R}, 0 \leqq \tau \leqq \pi/4\right\} < \infty.$$

Proof. It suffices to consider $y > 0$. By Proposition 2.1 and Corollary 2.3,

$$\frac{k_{\alpha\beta}(y, \tau)}{k_{\alpha\beta}(y, \pi/4)} = \frac{k_{\alpha\beta}(y, \tau)k_{\alpha\beta}(-y, \pi/4)k_{\nu\nu}(y, \pi/4)^2}{k_{\alpha\beta}(y, \pi/4)k_{\alpha\beta}(-y, \pi/4)k_{\nu\nu}(y, \pi/4)^2}$$
$$\leqq \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\nu)^2}\frac{k_{\alpha\beta}(y, \tau)k_{\alpha\beta}(-y, \pi/4)}{k_{\nu\nu}(y, \pi/4)^2}$$

$$\leqq \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\nu)^2} C_{\alpha\beta}^2 \min\left(\frac{1}{\beta}, \frac{\Gamma(\beta)}{(2\tau y)^\beta}\right) \min\left(\frac{1}{\alpha}, \frac{\Gamma(\alpha)}{(2y)^\alpha}\right)$$

$$\times\ e^{y(\tau+\pi/4)} k_{\nu\nu}(y, \pi/4)^{-2}.$$

By Lemma 2.4 and Theorem 1.7

$$k_{\nu\nu}(y, \pi/4)^{-2} \leqq C_\nu''(4(\nu + 1)^2 + y^2)^\nu e^{-\pi|y|/2}$$

for some constant C_ν''. Thus if $y \geqq 2(\nu + 1)$ then

$$(k_{\alpha\beta}(y, \tau)/k_{\alpha\beta}(y, \pi/4))$$
$$\leqq C_{\alpha\beta}''\tau^{-\beta}(1 + (2(\nu + 1)/y)^2)^\nu e^{-y(\pi/4-\tau)}, \ (0 < \tau \leqq \pi/4);$$

when $\tau = 0$

$$(1/k_{\alpha\beta}(y, \pi/4)) \leqq C_{\alpha\beta}'' y^\beta((2(\nu + 1)/y)^2 + 1)^\nu e^{-y\pi/4}.$$

The stated inequalities follow from these bounds.

To show $k_{\alpha\beta}(y, \tau)/k_{\alpha\beta}(y, \pi/4)$ is uniformly bounded we argue separately for $\tau < \pi/8$ and $\tau \geqq \pi/8$.

For $0 \leqq \tau \leqq \pi/8$, the function is bounded by

$$C_{\alpha\beta}'' \min\left(\frac{1}{\alpha}, \frac{\Gamma(\alpha)}{(2y)^\alpha}\right)(4(\nu + 1)^2 + y^2)^\nu e^{-\pi y/8}$$

which is bounded on $y \geqq 0$. For $\pi/8 \leqq \tau \leqq \pi/4$, the function is bounded by

$$C_{\alpha\beta}''(4(\nu + 1)^2 + y^2)^\nu$$

on $0 \leqq y \leqq 2(\nu + 1)$ and by

$$C_{\alpha\beta}''\tau^{-\beta}\left(\left(\frac{2(\nu + 1)}{y}\right)^2 + 1\right)^\nu \leqq C_{\alpha\beta}''\left(\frac{8}{\pi}\right)^\beta\left(\left(\frac{2(\nu + 1)}{y}\right)^2 + 1\right)^\nu$$

on $y \geqq 2(\nu + 1)$. ($C_{\alpha\beta}''$ denotes constants depending only on α and β which may be different in different contexts.)

It is relatively easy to obtain asymptotic results for $y \to \pm\infty$.

2.6 PROPOSITION. *For* $0 < \tau \leqq \pi/4$,

$$k_{\alpha\beta}(y, \tau) \sim (\sin 2\tau)^{-\beta} \frac{\Gamma(2\nu)}{\Gamma(\alpha)} y^{-\beta} e^{\tau y} \quad \textit{as } y \to +\infty,$$

$$k_{\alpha\beta}(y, \tau) \sim (\sin 2\tau)^{-\alpha} \frac{\Gamma(2\nu)}{\Gamma(\beta)} |y|^{-\alpha} e^{\tau|y|} \quad \textit{as } y \to -\infty.$$

Further

$$(k_{\alpha\beta}(y, \tau)/k_{\alpha\beta}(y, \pi/4)) \sim (\sin 2\tau)^{-\delta} e^{-|y|(\pi/4-\tau)} \quad as\ y \to \pm\infty$$

with $\delta = \beta$ *for* $y > 0$, $\delta = \alpha$ *for* $y < 0$.

Proof. Consider the case $y > 0$ so that the integrand in the expression 1.5 for $k_{\alpha\beta}$, in a neighborhood of $t = \tau$ determines the asymptotic behavior. Indeed $k_{\alpha\beta}(y, \tau)$ is asymptotic to

$$B(\alpha, \beta)^{-1}(\sin 2\tau)^{1-2\nu}(\sin 2\tau)^{\alpha-1} \cdot \int_{-\tau}^{\tau} e^{yt}(\tau - t)^{\beta-1}dt$$

(as $y \to +\infty$), and the integral is asymptotic to

$$y^{-\beta}e^{y\tau}\Gamma(\beta),$$

by Watson's lemma. A similar argument applies to $y \to -\infty$.

2.7 COROLLARY. *For* α, $\beta > 0$, *and* $-\pi/4 \leqq \tau < 0$,

$$(k_{\alpha\beta}(y, \tau)/k_{\alpha\beta}(y, \pi/4)) \sim \left(\frac{\Gamma(\beta)}{\Gamma(\alpha)}(|y| \sin 2\tau)^{\beta-\alpha}\right)^{\operatorname{sgn} y} e^{-|y|(\pi/4+\tau)}.$$

Proof. Use the relation

$$k_{\alpha\beta}(y, -\tau) = k_{\alpha\beta}(-y, \tau).$$

Note that $k_{\alpha\beta}(y, -\pi/4)/k_{\alpha\beta}(y, \pi/4)$ is unbounded for $\alpha \neq \beta$.

2.8 COROLLARY. *For* $-\pi/4 < \tau < \pi/4$, $k_{\alpha\beta}(y, \tau)/k_{\alpha\beta}(y, \pi/4)$ *is a rapidly decreasing function of* y, *that is,*

$$\sup_{y\in\mathbf{R}} \left| (1 + y^2)^m \left(\frac{\partial}{\partial y}\right)^n \left(\frac{k_{\alpha\beta}(y, \tau)}{k_{\alpha\beta}(y, \pi/4)}\right) \right| < \infty$$

for each m, $n = 0, 1, 2, \ldots$.

Proof. For $0 < \tau \leqq \pi/4$, $\left(\frac{\partial}{\partial y}\right)^n k_{\alpha\beta}(y, \tau)$ is a multiple of

$$\int_{-\tau}^{\tau} t^n e^{yt}(\sin(\tau + t))^{\alpha-1}(\sin(\tau - t))^{\beta-1}dt$$

so the same asymptotic relations as in 2.6 hold for each derivative, with a factor of $(-1)^n$ for $y \to -\infty$. This shows that

$$\left(\frac{\partial}{\partial y}\right)^n (k_{\alpha\beta}(y, \tau)/k_{\alpha\beta}(y, \pi/4))$$

has exponential decay as $y \to \pm\infty$, for each n.

2.9 THEOREM. *There exists a smooth function* $K_{\alpha\beta}(\sigma, \tau)$ *on the strip*

$$S = \{\sigma + i\tau : \sigma \in \mathbf{R}, -\pi/4 < \tau < \pi/4\}$$

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such that

1) $\hat{K}_{\alpha\beta}(y, \tau) = k_{\alpha\beta}(y, \tau)/k_{\alpha\beta}(y, \pi/4)$;

2) $K_{\alpha\beta}(\sigma, \tau)$ *is a rapidly decreasing function of* $\sigma \in \mathbf{R}$ *and is analytic in* $\sigma \in \mathbf{C}$ *with*

$$|\mathrm{Im}\ \sigma| < \pi/4 - |\tau|,$$

for each τ, $-\pi/4 < \tau < \pi/4$;

3) $K_{\alpha\beta}(\sigma, \tau)$ *is positive definite in* $\sigma \in \mathbf{R}$.

Proof. The existence of a rapidly decreasing function $K_{\alpha\beta}(\sigma, \tau)$ satisfying condition (1) comes from Corollary 2.8. From the exponential decay established in Proposition 2.6 we see that

$$K_{\alpha\beta}(\sigma, \tau) = (1/2\pi)\int_{\mathbf{R}} e^{iy\sigma}(k_{\alpha\beta}(y, \tau)/k_{\alpha\beta}(y, \pi/4))dy$$

is analytic in

$$|\mathrm{Im}\ \sigma| < \pi/4 - |\tau|.$$

Property (3) of course is a consequence of $k_{\alpha\beta}(y, \tau) > 0$.

Next we have to show

$$\left(\frac{\partial}{\partial\tau}\right)^n (k_{\alpha\beta}(y, \tau)/k_{\alpha\beta}(y, \pi/4))$$

is a Fourier transform for each n. This follows from the expression

$$k_{\alpha\beta}(y, \tau) = \int_{-1}^{1} e^{yt\tau}g(t, \tau)(1 + t)^{\alpha-1}(1 - t)^{\beta-1}dt,$$

where $g(t, \tau)$ is infinitely differentiable in τ for $-1 \leqq t \leqq 1$; indeed

$$g(t, \tau) = B(\alpha, \beta)^{-1}(\tau/\sin 2\tau) \times \left(\frac{\sin(\tau(1 + t))}{(1 + t)\sin 2\tau}\right)^{\alpha-1}\left(\frac{\sin(\tau(1 - t))}{(1 - t)\sin 2\tau}\right)^{\beta-1}$$

Thus $\left(\frac{\partial}{\partial\tau}\right)^n K_{\alpha\beta}(\sigma, \tau)$ exists for all n.

3. The Poisson integral. We have enough information about $K_{\alpha\beta}$ to define the Poisson integral for the differential operator $D'_{\alpha\beta}$. We use $L^p(\mathbf{R} + i\pi/4)$ to denote the space of measurable functions on $\mathbf{R} + i\pi/4$ (the upper edge of S) such that

$$\int_{\mathbf{R}} |f(\sigma + i\pi/4)|^p d\sigma < \infty, \quad 1 \leqq p < \infty.$$

3.1 *Definition.* For $f \in L^p(\mathbf{R} + i\pi/4)$, $1 \leqq p < \infty$, let

$$P'_{\alpha\beta}[f](\sigma + i\tau) := \int_{\mathbf{R}} f(w + i\pi/4)K_{\alpha\beta}(\sigma - w, \tau)dw, \quad (\sigma + i\tau \in S).$$

3.2 THEOREM. *For* $f \in L^p(\mathbf{R} + i\pi/4)$, $1 \leqq p < \infty$, $P'_{\alpha\beta}[f]$ *is a smooth function on S such that*

$$D'_{\alpha\beta}P'_{\alpha\beta}[f] = 0.$$

Further if $\|f\|_2 < \infty$ *then*

$$\int |f(\sigma + i\pi/4) - P'_{\alpha\beta}[f](\sigma + i\tau)|^2 d\sigma \to 0 \quad \text{as } \tau \to (\pi/4)_-.$$

Proof. The fact that $P'_{\alpha\beta}[f]$ is smooth follows from $K_{\alpha\beta}(\sigma, \tau)$ being rapidly decreasing in σ, and smooth in (σ, τ). Similarly we can apply $D'_{\alpha\beta}$ to $P'_{\alpha\beta}[f]$ and interchange the integration with $D'_{\alpha\beta}$. The function

$$\sigma + i\tau \mapsto K_{\alpha\beta}(\sigma - w, \tau)$$

is annihilated by $D'_{\alpha\beta}$ for each $w \in \mathbf{R}$, because we can map $D'_{\alpha\beta}$ to its Fourier transform acting on

$$k_{\alpha\beta}(y, \tau)/k_{\alpha\beta}(y, \pi/4).$$

If $f \in L^2(\mathbf{R} + i\pi/4)$, let f be its Plancherel transform; then

$$\int_{\mathbf{R}} |f(\sigma + i\pi/4) - P'_{\alpha\beta}[f](\sigma + i\tau)|^2 d\sigma$$
$$= \frac{1}{2\pi} \int |\hat{f}(y)|^2 \left(1 - \frac{k_{\alpha\beta}(y, \tau)}{k_{\alpha\beta}(y, \pi/4)}\right)^2 dy \to 0 \text{ as } \tau \to (\pi/4)_-,$$

by the dominated convergence theorem. Here we use the uniform boundedness of $k_{\alpha\beta}(y, \tau)/k_{\alpha\beta}(y, \pi/4)$ from Theorem 2.5.

We note the uniqueness and reproducing properties for $P'_{\alpha\beta}$. If f is smooth on the open strip and continuous on the half-closed strip (union with upper edge), annihilated by $D'_{\alpha\beta}$, and if f has a Fourier transform on every line $\sigma + i\tau$, $-\pi/4 < \tau \leqq \pi/4$, then

$$f = P'_{\alpha\beta}[f_b],$$

where $f_b = f|(\mathbf{R} + i\pi/4)$, the boundary value. To see this, note that $\hat{f}(y, \tau)$ must be a multiple (depending on y) of $k_{\alpha\beta}(y, \tau)$.

We apply these results to the original differential operator $D_{\alpha\beta}$, still considered on the strip.

3.3 *Definition.* For a measurable function f on $\mathbf{R} + i\pi/4$ with

$$\int_{\mathbf{R}} |f(\sigma + i\pi/4)|^p (\text{ch } 2\sigma)^{-p\nu} d\sigma < \infty$$

for some p, $1 \leqq p < \infty$, let

$$\begin{aligned} P_{\alpha\beta}[f](\sigma + i\tau) := {} & (\text{ch}(\sigma - i\tau))^\alpha (\text{ch}(\sigma + i\tau))^\beta \\ & \times P'_{\alpha\beta}[f(w + i\pi/4)(\text{ch}(w - i\pi/4))^{-\alpha} \\ & \times (\text{ch}(w + i\pi/4))^{-\beta}](\sigma + i\tau), \end{aligned}$$

$\sigma + i\tau \in S$.

3.4 THEOREM. *For a function f as in* 3.3, $P_{\alpha\beta}[f]$ *is smooth on* S *and is annihilated by* $D_{\alpha\beta}$. *Further if*

$$\int_{\mathbf{R}} |f(\sigma + i\pi/4)|^2 (\text{ch } 2\sigma)^{-2\nu} d\sigma < \infty$$

then

$$\int_{\mathbf{R}} |(f(\sigma + i\pi/4) - P_{\alpha\beta}[f](\sigma + i\tau)|^2 (\text{ch } 2\sigma)^{-2\nu} d\sigma \to 0$$

$$\text{as } \tau \to (\pi/4)_{-}.$$

Proof. This follows from Theorem 3.2 and the relationship between $D_{\alpha\beta}$ and $D'_{\alpha\beta}$ established in Proposition 1.3. Further

$$|\text{ch}(w - i\pi/4)^{-\alpha} \text{ch}(w + i\pi/4)^{-\beta}| = \left(\frac{1}{2} \text{ ch } 2w\right)^{-\nu}.$$

The L^2-convergence is a consequence of Theorem 3.2 and the uniform convergence of

$$\text{ch}(\sigma - i\tau)^{-\alpha} \text{ch}(\sigma + i\tau)^{-\beta}$$

to

$$(\text{ch}(\sigma - i\pi/4))^{-\alpha}(\text{ch}(\sigma + i\pi/4))^{-\beta}$$

as $\tau \to \pi/4$, $(\alpha, \beta > 0)$.

We can find $K_{\alpha\beta}$ explicitly when $\alpha = \beta = \nu$.

3.5 THEOREM. *For* $\nu > 0$,

$$K_{\nu\nu}(\sigma, \tau) = 2^{-\nu+1} B(\nu + 1/2, 1/2)^{-1}$$
$$\times \frac{\cos 2\tau}{(\text{ch } 2\sigma + \sin 2\tau)^{\nu}(\text{ch } 2\sigma - \sin 2\tau)}$$
$$\times {}_2F_1\left(\begin{matrix}\nu, \nu - 1\\ 2\nu\end{matrix}; \frac{2 \sin 2\tau}{\text{ch } 2\sigma + \sin 2\tau}\right).$$

Also $K_{\nu\nu}(\sigma, \tau) > 0$ *and the* ${}_2F_1$*-function assumes only values between* 1 *and*

$$\Gamma(2\nu)/(\Gamma(\nu)\Gamma(\nu + 1)), \quad \sigma \in \mathbf{R}, 0 \leqq \tau < \pi/4.$$

Proof. When $\alpha = \beta$ the operator $D_{\alpha\beta}$ becomes the operator associated to the ultraspherical polynomials, and the Poisson integral for the half-disk is known, namely

$$re^{i\varphi} \mapsto B(\nu + 1/2, 1/2)^{-1} \int_0^{\pi} f(e^{i\theta}) \sum_{n=0}^{\infty} \left(\frac{n + \nu}{\nu}\right) r^n$$
$$\times C_n^{\nu}(\cos \theta) C_n^{\nu}(\cos \phi) C_n^{\nu}(1)^{-1} (\sin \theta)^{2\nu} d\theta$$

(where f is an appropriate function on the upper half circle and C_n^ν is the ultraspherical polynomial of index ν, degree n), $0 \leqq r < 1$, $0 \leqq \phi \leqq \pi$. But the sum over n in the integrand equals

$$(1 - r^2)(1 - 2r\cos(\theta + \phi) + r^2)^{-\nu-1}{}_2F_1\left(\begin{matrix}\nu, \nu + 1 \\ 2\nu\end{matrix}; X(r, \theta, \phi)\right),$$

where

$$X(r, \theta, \phi) = 4r\sin\theta\sin\phi/(1 - 2r\cos(\theta + \phi) + r^2),$$

from the sum found in [**3**], Corollary 3.7. Now we set

$$re^{i\phi} = \operatorname{th}(\sigma + i\tau),\ e^{i\theta} = \operatorname{th}(w + i\pi/4),\ d\theta = -2\operatorname{sech}(2w)dw,$$

and get

$$X(r, \theta, \phi) = 2\sin 2\tau/(\operatorname{ch} 2(\sigma - w) + \sin 2\tau).$$

A transformation of the ${}_2F_1$ series gives the stated formula. By Gauss's sum

$${}_2F_1\left(\begin{matrix}\nu, \nu - 1 \\ 2\nu\end{matrix}; 1\right) = \Gamma(2\nu)/(\Gamma(\nu)\Gamma(\nu + 1))$$

and the ${}_2F_1$-function is increasing, resp. decreasing, on $0 \leqq X \leqq 1$, when $\nu > 1$, resp. $0 < \nu < 1$, because

$$\frac{d}{dx}{}_2F_1\left(\begin{matrix}\nu, \nu - 1 \\ 2\nu\end{matrix}; X\right) = \frac{(\nu - 1)}{2}{}_2F_1\left(\begin{matrix}\nu + 1, \nu \\ 2\nu + 1\end{matrix}; X\right).$$

We observe that for each τ with $0 \leqq \tau < \pi/4$,

$$K_{\nu\nu}(\sigma, \tau) \sim A_\tau e^{-2|\sigma|(\nu+1)} \quad \text{as } \sigma \to \pm\infty$$

(some constant A_τ depending on ν and τ). Even in the general case $K_{\alpha\beta}(\sigma, \tau)$ has exponential decay in σ. The reason for this is the meromorphic nature of

$$k_{\alpha\beta}(y, \tau)/k_{\alpha\beta}(y, \pi/4).$$

By a theorem of Titchmarsh [**19**], $k_{\alpha\beta}(y, \tau)$ is an entire function of order 1, type $|\tau|$. In particular, $k_{\alpha\beta}(y, \pi/4)$ has infinitely many zeros $\{z_j : j = 1, 2, \dots\}$ with $0 < |z_1| \leqq |z_2| \leqq |z_3| \leqq \dots$ such that

1) $\operatorname{card}\{z_j : |z_j| < r\} \sim r/2$ as $r \to \infty$;

2) $k_{\alpha\beta}(y, \pi/4) = k_{\alpha\beta}(0, \pi/4)\prod_{j=1}^{\infty}\left(\left(1 - \frac{z}{z_j}\right)e^{z/z_j}\right)$;

3) $\sum_{j=1}^{\infty}\operatorname{Re}(1/z_j)$ is absolutely convergent.

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In this situation we know more since

$$\operatorname{Re}(k_{\alpha\beta}(y, \pi/4)) > 0 \quad \text{in } |\operatorname{Im} y| \leqq 2;$$

indeed

$$\operatorname{Re}(k_{\alpha\beta}(y, \pi/4)) = \int_{-\pi/4}^{\pi/4} e^{t\operatorname{Re} y} \cos(t \operatorname{Im} y) W(t) dt$$

where $W(t) \geqq 0$ on $-\pi/4 \leqq t \leqq \pi/4$. Thus the zero set

$$\{z_j\} \subset \{y \in \mathbf{C}: |\operatorname{Im} y| > 2\}.$$

If $\alpha = \beta = \nu$ the zeros of $k_{\nu\nu}(y, \pi/4)$ are exactly the points $\pm 2i(1 + \nu + 2n)$.

By a theorem of Cartwright [**1**] there is a region of the form

$$\{y \in \mathbf{C}: |y| \leqq r_0, \text{ and } |\arg y| \leqq \theta_0 \text{ or } |\arg(-y)| \leqq \theta_0\}$$

for some $r_0 > 0$, $0 < \theta_0 < \pi/2$, that is zero-free. Thus any horizontal line ($\operatorname{Im} y =$ constant) has at most finitely many zeros of $k_{\alpha\beta}(y, \pi/4)$.

Let

$$\delta = \min\{\operatorname{Im} y : k_{\alpha\beta}(y, \pi/4) = 0, \operatorname{Im} y > 0\}.$$

From the above we know $\delta > 2$, (we conjecture $\delta \geqq 2(1 + \sqrt{\alpha\beta})$). The experimental evidence is strong that there is a unique zero $a + i\delta$, but regardless, a useful statement can be made.

3.6 THEOREM. *Let $y_j = a_j + i\delta$, $j = 1, \ldots, m$ be all the zeros y of $k_{\alpha\beta}(y, \pi/4)$ with* $\operatorname{Im} y = \delta$, *then*

$$K_{\alpha\beta}(\sigma, \tau) \sim \sum_{j=1}^{m} (A_j(\tau) e^{i\sigma a_j}) e^{-\delta\sigma} \quad \text{as } \sigma \to +\infty,$$

where the $A_j(\tau)$ are continuous complex functions of τ, (note

$$K_{\alpha\beta}(-\sigma, \tau) = \overline{K_{\alpha\beta}(\sigma, \tau)},$$

by Theorem 2.9(3); *to be used for $\sigma \to -\infty$). If for some τ, all $A_j(\tau) = 0$ then*

$$K_{\alpha\beta}(\sigma, \tau) = o(e^{-\delta|\sigma|}) \quad \text{as } \sigma \to \pm\infty.$$

Proof. For notational convenience let

$$f(y, \tau) = k_{\alpha\beta}(y, \tau)/k_{\alpha\beta}(y, \pi/4).$$

We use residue calculus to find

$$K_{\alpha\beta}(\sigma, \tau) = \frac{1}{2\pi} \int_{\mathbf{R}} f(y, \tau) e^{iy\sigma} dy.$$

Assume $\sigma > 0$, and integrate $f(y, \tau)e^{iy\sigma}$ over the contour consisting of the rectangle with vertices at $\pm R$, $\pm R + bi$ which contains all the points $a_j + i\delta$ but no other zeros of $k_{\alpha\beta}(y, \pi/4)$, (thus $b > \delta$). We know

$$f(\pm R + it, \tau) \to 0$$

exponentially as $R \to +\infty$ on $0 \leqq t \leqq b$ by Theorem 2.5 (extend to complex arguments). Let $R \to +\infty$ to obtain

$$\int_{-\infty}^{\infty} e^{iy\sigma} f(y, \tau) dy = e^{-b\sigma} \int_{-\infty}^{\infty} e^{iy\sigma} f(y + ib, \tau) dy + 2\pi i \sum_j e^{i\sigma(a_j + i\delta)} \operatorname{Res}(f(y, \tau);$$

$$y = a_j + i\delta) \text{ (the residues).}$$

The function $y \mapsto f(y + ib, \tau)$ is analytic in a neighborhood of $\mathbf{R}$ and has exponential decay as $y \to \pm\infty$, thus its Fourier transform is uniformly bounded (in σ). This part contributes $O(e^{-b\sigma}) = o(e^{-\delta\sigma})$ as $\sigma \to +\infty$ to $K_{\alpha\beta}(\sigma, \tau)$.

4. Relation to Heisenberg polynomials. We will consider the problem of expanding the kernel $K_{\alpha\beta}(\sigma, \tau)$ as a series of Heisenberg polynomials, and also the question of density of these polynomials in a weighted L^2-space.

The generating function is

$$(1 - r\bar{\zeta})^{-\alpha}(1 - r\zeta)^{-\beta} = \sum_{n=0}^{\infty} r^n C_n^{(\alpha,\beta)}(\zeta), \quad |r\zeta| < 1.$$

Recall from Section 1 that

$$D_{\alpha\beta} C_n^{(\alpha,\beta)}(\zeta) = 0.$$

Since $D_{\alpha\beta}1 = 0$ and $D'_{\alpha\beta}$ is translation-invariation we have that

$$D'_{\alpha\beta}((\operatorname{ch}(\sigma - s - i\tau))^{-\alpha} (\operatorname{ch}(\sigma - s + i\tau))^{-\beta}) = 0$$

for each fixed $s \in \mathbf{R}$ (see Proposition 1.3). Each such function (of $\sigma + i\tau \in S$) has a Fourier transform for fixed τ, and hence is reproduced by $P'_{\alpha\beta}$. That is,

$$\begin{aligned} &\int_{\mathbf{R}} (\operatorname{ch}(w - s - i\pi/4))^{-\alpha} \\ &\times (\operatorname{ch}(w - s + i\pi/4))^{-\beta} K_{\alpha\beta}(\sigma - w, \tau) dw \\ &= (\operatorname{ch}(\sigma - s - i\tau))^{-\alpha} \\ &\times (\operatorname{ch}(\sigma - s + i\tau))^{-\beta}, \quad (-\pi/4 < \tau < \pi/4). \end{aligned}$$

We can use this to find the Fourier transform of

$$h_{\alpha\beta}(\sigma, \tau) := (\mathrm{ch}(\sigma - i\tau))^{-\alpha}(\mathrm{ch}(\sigma + i\tau))^{-\beta}.$$

4.1 PROPOSITION.

$$\hat{h}_{\alpha\beta}(y, \tau) = \frac{2^{2\nu-1}}{\Gamma(2\nu)}\Gamma(\nu + iy/2)\Gamma(\nu - iy/2)k_{\alpha\beta}(y, \tau),$$

for $-\pi/4 \leqq \tau \leqq \pi/4$, $y \in \mathbf{R}$ (*or* $|\mathrm{Im}\, y| < 2\nu$).

Proof. Since the Poisson integral $P'_{\alpha\beta}$ reproduces the values of $h_{\alpha\beta}$ we have

$$\hat{h}_{\alpha\beta}(y, \tau) = \hat{h}_{\alpha\beta}(y, \pi/4)k_{\alpha\beta}(y, \tau)/k_{\alpha\beta}(y, \pi/4)$$

for $-\pi/4 < \tau < \pi/4$; in particular

$$\hat{h}_{\alpha\beta}(y, 0) = \hat{h}_{\alpha\beta}(y, \pi/4)/k_{\alpha\beta}(y, \pi/4).$$

But we can directly find

$$\hat{h}_{\alpha\beta}(y, 0) = \int_{\mathbf{R}} (\mathrm{ch}\, \sigma)^{-2\nu}e^{-iy\sigma}d\sigma = \frac{2^{2\nu-1}}{\Gamma(2\nu)}\Gamma(\nu + iy/2)\Gamma(\nu - iy/2)$$

(a standard integral). Thus

$$\hat{h}_{\alpha\beta}(y, \tau) = \hat{h}_{\alpha\beta}(y, 0)k_{\alpha\beta}(y, \pi/4)k_{\alpha\beta}(y, \tau)/k_{\alpha\beta}(y, \pi/4).$$

(Indeed $\hat{h}_{\alpha\beta}$ can be found directly by using Barnes' type integrals, which led the author to the method of solution of the differential equation for $k_{\alpha\beta}$.)

The function $k_{\alpha\beta}$ can be viewed as the generating function for $C_n^{(\alpha,\beta)}$, and allows us to find their Fourier transforms. First we outline some facts about a family of Meixner-Pollaczek polynomials (due to Pollaczek **[17]**, see also **[4]** for more details).

4.2 *Definition.* For each $A > 0$ the *Meixner-Pollaczek polynomials* $p_n(x; A)$ are given by the generating function

$$(1 - it)^{(ix-A)/2}(1 + it)^{-(ix+A)/2} = \sum_{n=0}^{\infty} t^n p_n(x; A),$$

for $|t| < 1$; equivalently

$$(\mathrm{ch}\, s)^A e^{isx} = \sum_{n=0}^{\infty} (i\, \mathrm{th}\, s)^n p_n(x; A), \quad |\mathrm{th}\, s| < 1,$$

(this family is denoted by $p_n(x; A, 0)$ in **[4]** and by $P_n^{(A/2)}(x/2; \pi/2)$ in **[17]**).

4.3 *Properties of* $p_n(x; A)$.

(1) $p_n(x; A)$ is a real polynomial in x of degree n with leading coefficient $1/n!$, and

$$p_n(-x; A) = (-1)^n p_n(x; A);$$

(2) $$(2^A/(4\pi\Gamma(A))) \int_{\mathbf{R}} p_n(x; A) p_m(x; A) |\Gamma((A + ix)/2)|^2 dx = \delta_{mn}(A)_n/n!,$$

the orthogonality relation, $m, n = 0, 1, 2, \dots$;

(3) for each fixed r with $0 < r < 1$, and $x \in \mathbf{R}$,

$$|p_n(x; A)| \leqq r^{-n}(1 - r^2)^{-A/2}\exp(|x| \arctan r);$$

(this is proved by applying Cauchy's bounds to the generating function as an analytic function in t, integrating around the circle $t = re^{i\theta}$, $0 \leqq \theta \leqq 2\pi$, and using

$$|(1 - it)^{ix/2}(1 + it)^{-ix/2}| \leqq \exp(|x| \arctan |t|), \ x \in \mathbf{R}).$$

4.4 Proposition.

$$h_{\alpha\beta}(\sigma - w, \tau) = h_{\alpha\beta}(\sigma, \tau)(\text{ch } w)^{-2\nu} \sum_{n=0}^{\infty} (\text{th } w)^n C_n^{(\alpha,\beta)}(\text{th}(\sigma + i\tau)),$$

with absolute convergence in $|\text{th } w| < 1$, $\sigma + i\tau \in S$. *Further*

$$\int_{\mathbf{R}} h_{\alpha\beta}(\sigma, \tau) C_n^{(\alpha,\beta)}(\text{th}(\sigma + i\tau)) e^{-iy\sigma} d\sigma = (-i)^n p_n(y; 2\nu) \frac{2^{2\nu-1}}{\Gamma(2\nu)} \Gamma(\nu + iy/2)\Gamma(\nu - iy/2) k_{\alpha\beta}(y, \tau),$$

for $-\pi/4 \leqq \tau \leqq \pi/4$.

Proof. We note

$$\begin{aligned} h_{\alpha\beta}(\sigma - w, \tau) &= (\text{ch}(\sigma - i\tau))^{-\alpha}(\text{ch}(\sigma + i\tau))^{-\beta} \\ &\times (\text{ch } w)^{-2\nu}(1 - \text{th } w \text{ th}(\sigma - i\tau))^{-\alpha} \\ &\times (1 - \text{th } w \text{ th}(\sigma + i\tau))^{-\beta} \\ &= h_{\alpha\beta}(\sigma, \tau)(\text{ch } w)^{-2\nu} \sum_{n=0}^{\infty} (\text{th } w)^n C_n^{(\alpha,\beta)}(\text{th}(\sigma + i\tau)), \end{aligned}$$

(generating function). Now fix w, multiply both sides by $e^{-iy\sigma}$ and integrate over $\sigma \in \mathbf{R}$. The result is

$$e^{-iyw}\hat{h}_{\alpha\beta}(y, \tau) = (\text{ch } w)^{-2\nu} \sum_{n=0}^{\infty} (\text{th } w)^n$$

$$\times \int_{\mathbf{R}} h_{\alpha\beta}(\sigma, \tau) C_n^{(\alpha,\beta)}(\mathrm{th}(\sigma + i\tau)) e^{-iy\sigma} d\sigma;$$

the summation and integration may be interchanged by the absolute convergence and the bound

$$\sum_n |\mathrm{th}\, w|^n |C_n^{(\alpha,\beta)}| \leqq (1 - |\mathrm{th}\, w\ \mathrm{th}(\sigma + i\tau)|)^{-2\nu}.$$

Thus the required integral is the coefficient of $(\mathrm{th}\, w)^n$ in the expansion of

$$\hat{h}_{\alpha\beta}(y, \tau) e^{-iyw} (\mathrm{ch}\, w)^{2\nu},$$

namely

$$(-i)^n p_n(y; 2\nu) \hat{h}_{\alpha\beta}(y, \tau);$$

see 4.1.

4.5 THEOREM. $\{h_{\alpha\beta}(\sigma, \pi/4) C_n^{(\alpha,\beta)}(\mathrm{th}(\sigma + i\pi/4)) : n \geqq 0\}$ *spans a dense set in* $L^2(\mathbf{R} + i\pi/4)$.

Proof. Let $g \in L^2(\mathbf{R})$ with

$$\int_{\mathbf{R}} g(-\sigma) h_{\alpha\beta}(\sigma, \pi/4) C_n^{(\alpha,\beta)}(\mathrm{th}(\sigma + i\pi/4)) d\sigma = 0$$

for each n. Then by the Parseval theorem

$$\int_{\mathbf{R}} \hat{g}(y) p_n(y; 2\nu) \Gamma(\nu + iy/2) \Gamma(\nu - iy/2) k_{\alpha\beta}(y, \pi/4) dy = 0$$

for each n.

Thus

$$\int_{\mathbf{R}} \hat{g}(y) p(y) W(y) dy = 0$$

for each polynomial p, where

$$W(y) = |\Gamma(\nu + iy/2)|^2 k_{\alpha\beta}(y, \pi/4) > 0$$

and

$$W(y) = O(|y|^c e^{-\pi|y|/4})$$

for some $c > 0$, by the bounds from 2.4 and 2.5. By a theorem of Hamburger (see **[7]**, p. 84), polynomials are dense in

$$L^2(\mathbf{R}, W(y)dy) \supset L^2(\mathbf{R}),$$

thus $\hat{g} = 0$.

The theorem answers a question posed by Greiner **[10]**.

We will use the orthogonality structure of the Meixner-Pollaczek polynomials to produce the biorthogonal set for $\{h_{\alpha\beta} C_n^{(\alpha,\beta)} : n \geqq 0\}$.

4.6 *Definition*. For $n = 0, 1, 2, \ldots, \alpha, \beta > 0$, let

$$\phi_{n,\alpha\beta}(w) = (i^n/(2\pi)) \int_{\mathbf{R}} (p_n(y;\, 2\nu)/k_{\alpha\beta}(y, \pi/4))e^{-iyw}dy,$$

an analytic function of w in $|\mathrm{Im}\, w| < \pi/4$. Note

$$\hat{\phi}_{n,\alpha\beta}(y) = i^n p_n(-y;\, 2\nu)/k_{\alpha\beta}(-y, \pi/4),$$

by the inversion theorem.

4.7 Proposition. *The function $\phi_{n,\alpha\beta}(w)$ is rapidly decreasing on $w \in \mathbf{R}$, and*

$$\phi_{n,\alpha\beta}(w) = O(e^{-\delta|w|}),$$

(*where* $\delta = \min\{\mathrm{Im}\, y : k_{\alpha\beta}(y, \pi/4) = 0, \mathrm{Im}\, y > 0\}$).

Proof. This follows from the asymptotic behavior of $k_{\alpha\beta}(y, \pi/4)$, and the fact that $\phi_{n,\alpha\beta}$ is the Fourier transform of a meromorphic function, and an argument similar to that of Theorem 3.6.

4.8 Theorem. *For $w \in \mathbf{R}$, $\sigma \in \mathbf{C}$ with $|\mathrm{Im}\, \sigma| < \pi/4$,*

$$K_{\alpha\beta}(\sigma - w, 0) = (\mathrm{ch}\, \sigma)^{-2\nu} \sum_{n=0}^{\infty} (\mathrm{th}\, \sigma)^n \phi_{n,\alpha\beta}(w),$$

with absolute convergence, uniform for all $w \in \mathbf{R}$ and $\sigma \in \mathbf{R}$, $|\sigma| \leqq \sigma_0$; each $\sigma_0 < \infty$.

Proof. Indeed

$$\begin{aligned}
&(\mathrm{ch}\, \sigma)^{2\nu} K_{\alpha\beta}(\sigma - w, 0) \\
&= \frac{1}{2\pi} \int_{\mathbf{R}} e^{iy(\sigma - w)} (\mathrm{ch}\, \sigma)^{2\nu} k_{\alpha\beta}(y, \pi/4)^{-1} dy \\
&= \frac{1}{2\pi} \int_{\mathbf{R}} e^{-iyw} \sum_{n=0}^{\infty} (i\, \mathrm{th}\, \sigma)^n p_n(y;\, 2\nu) k_{\alpha\beta}(y, \pi/4)^{-1} dy,
\end{aligned}$$

(using the generating function for $p_n(y;\, 2\nu)$, see 4.2). To justify the interchange of summation and integration, fix σ and let $|\mathrm{th}\, \sigma| < r < 1$. Then

$$\int_{\mathbf{R}} \sum_{n=0}^{\infty} |\mathrm{th}\, \sigma|^n |p_n(y;\, 2\nu)|\, |k_{\alpha\beta}(y, \pi/4)^{-1} dy$$

$$< (1 - r^2)^{-\nu} \int_{\mathbf{R}} \sum_{n=0}^{\infty} (\,|\mathrm{th}\, \sigma|/r)^n \exp(\,|y| \arctan r) k_{\alpha\beta}(y, \pi/4)^{-1} dy$$

$$= (1 - r^2)^{-\nu}(1 - |\text{th } \sigma|/r)^{-1} \int_{\mathbf{R}} (e^{|y|c}/k_{\alpha\beta}(y, \pi/4))dy,$$

where $c = \arctan r < \pi/4$. The integral is finite because

$$\log k_{\alpha\beta}(y, \pi/4) \sim \pi|y|/4$$

(Theorem 2.5).

4.9 THEOREM.

$$\int_{\mathbf{R}} C_n^{(\alpha,\beta)}(\text{th}(w + i\pi/4))(\text{ch}(w - i\pi/4))^{-\alpha}$$
$$\times (\text{ch}(w + i\pi/4))^{-\beta}\phi_{m,\alpha\beta}(w)dw$$
$$= \delta_{mn}(2\nu)_n/n!, \quad m, n = 0, 1, 2, \dots .$$

Proof. By the Parseval theorem the integral equals

$$\frac{1}{2\pi} \int_{\mathbf{R}} \hat{\phi}_{m,\alpha\beta}(-y)(-i)^n p_n(y; 2\nu) \frac{2^{2\nu-1}}{\Gamma(2\nu)}$$
$$\times |\Gamma(\nu + iy/2)|^2 k_{\alpha\beta}(y, \pi/4)dy$$

from 4.4. By definition of $\phi_{m,\alpha\beta}$ the integral is a multiple of

$$\int_{\mathbf{R}} p_m(y; 2\nu)p_n(y, 2\nu) |\Gamma(\nu + iy/2)|^2 dy$$

for which see 4.3(2).

The functions $\{\phi_{n,\alpha\beta}\}$ are thus the biorthogonal set for $\{h_{\alpha\beta}C_n^{(\alpha,\beta)}\}$ and themselves span a dense set in $L^2(\mathbf{R})$.

4.10 PROPOSITION. *The span of* $\{\phi_{n,\alpha\beta}:n \geqq 0\}$ *is dense in* $L^2(\mathbf{R})$.

Proof. By the Plancherel theorem we need to show $\{\hat{\phi}_{n,\alpha\beta}\}$ is dense in $L^2(\mathbf{R})$. Suppose $g \in L^2(\mathbf{R})$ and

$$\int_{\mathbf{R}} g(y)\hat{\phi}_{n,\alpha\beta}(y)dy = 0$$

for all n, then

$$\int_{\mathbf{R}} g(y)p(y)k_{\alpha\beta}(-y, \pi/4)^{-1}dy = 0$$

for each polynomial p. But the weight $k_{\alpha\beta}(-y, \pi/4)^{-1}$ satisfies the hypotheses of Hamburger's theorem and thus $g = 0$.

In the special case $\alpha = \beta = \nu$ we can show that

$$\phi_{n,\nu\nu}(w) = 2^{1-\nu}B(\nu + 1/2, 1/2)^{-1}$$
$$\times ((n + \nu)/\nu)C_n^{\nu}(\text{th } 2w)(\text{ch } 2w)^{-\nu-1}, n \geqq 0,$$

(this is implied by the classical theory of ultraspherical polynomials, but it

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can be proved directly by using generating functions and the Fourier transform of $|\Gamma(\nu + iy/2)|^2$). One should note the asymptotic behavior of $\phi_{n,\nu\nu}(w)$ as $w \to \pm\infty$, namely

$$(-1)^n A_n e^{-2|w|(\nu+1)};$$

the influence of the zeros of $k_{\nu\nu}(y, \pi/4)$ at $\pm 2i(\nu + 1)$ (constants A_n).

The Poisson kernel can be expanded in terms of $\{C_n^{(\alpha,\beta)}\}$ and $\{\phi_{m,\alpha\beta}\}$.

4.11 THEOREM.

$$K_{\alpha\beta}(\sigma - w, \tau) = (\mathrm{ch}(\sigma - i\tau))^{-\alpha}(\mathrm{ch}(\sigma + i\tau))^{-\beta}$$
$$\times \sum_{n=0}^{\infty} \frac{n!}{(2\nu)_n} C_n^{(\alpha,\beta)}(\mathrm{th}(\sigma + i\tau))\phi_{n,\alpha\beta}(w)$$

for $\sigma, w \in \mathbf{R}$, $-\pi/4 < \tau < \pi/4$; *the convergence is absolute, and uniform in every region*

$$\{(\sigma, \tau):|\mathrm{th}(\sigma + i\tau)| \leqq r\}$$

for $r < 1$, *all* $w \in \mathbf{R}$.

Proof. We showed in Theorem 4.8 that

$$\sup_{w\in\mathbf{R}} \sum_{n=0}^{\infty} r^n|\phi_{n,\alpha\beta}(w)| < \infty$$

for each r satisfying $0 < r < 1$. Further

$$|C_n^{(\alpha,\beta)}(\mathrm{th}(\sigma + i\tau))| \leqq \frac{(2\nu)_n}{n!}|\mathrm{th}(\sigma + i\tau)|^n.$$

This gives the stated convergence, and thus the sum is continuous in σ, τ and w. We must show the function defined by the right hand side of the formula equals $K_{\alpha\beta}(\sigma - w, \tau)$. We do this by proving that both sides are in L^2 as functions of w and give the same inner product with each $h_{\alpha\beta}C_m^{(\alpha,\beta)}$ (suffices by 4.5). All that is needed is to show that the summation converges in the $L^2(\mathbf{R})$-sense.

Thus fix r with $0 < r < 1$ and consider

$$\sum_{n=0}^{\infty} r^{2n} \int_{\mathbf{R}} |\phi_{n,\alpha\beta}(w)|^2 dw = \frac{1}{2\pi} \sum_{n=0}^{\infty} r^{2n} \int_{\mathbf{R}} |\hat{\phi}_{n,\alpha\beta}(y)|^2 dy$$

$$= \frac{1}{2\pi} \int_{\mathbf{R}} \sum_{n=0}^{\infty} r^{2n}|p_n(y;\, 2\nu)|^2 k_{\alpha\beta}(y, \pi/4)^{-2} dy$$

$$\leqq \frac{1}{2\pi}(1 - r_0^2)^{-2\nu}(1 - (r/r_0)^2)^{-1}$$

$$\times \int_{\mathbf{R}} \exp(2|y|\arctan r_0)k_{\alpha\beta}(y, \pi/4)^{-2}dy < \infty$$

for $r < r_0 < 1$ by the bound 4.3(3).

5. The Poisson integral for other $U(N)$- types. First assume $N \geqq 2$. For each $k, l = 0, 1, 2, \ldots$ there is an irreducible $U(N)$-module V_{kl} consisting of polynomials p in z_j, $\bar{z}_j$, $(1 \leqq j \leqq N)$ satisfying

$$p(cz) = c^k \bar{c}^l p(z), \quad (c \in \mathbf{C}, z \in \mathbf{C}^N) \quad \text{and}$$

$$\sum_{j=1}^{N} \frac{\partial^2 p}{\partial z_j \partial \bar{z}_j} = 0 \quad \text{(harmonic)}.$$

Further

$$\dim V_{kl} = \frac{(N-1)_k(N-1)_l}{k!l!} \frac{(N+k+l-1)}{(N-1)}.$$

5.1 *Definition*. For $k, l = 0, 1, 2, \ldots, \omega \in \mathbf{C}$, parameter $\lambda > -1$ let

$$R_k^{(\lambda)}(\omega) := \frac{(\lambda+1)_{k+l}}{(\lambda+1)_k(\lambda+1)_l} \omega^k \bar{\omega}^l {}_2F_1\left(\begin{matrix} -k, -l \\ -k-l-\lambda \end{matrix} ; \frac{1}{\omega\bar{\omega}} \right).$$

These are called *disk polynomials*.

The function $R_{kl}^{(N-2)}$ is the $(U(N)/U(N-1))$-spherical function for V_{kl} (see [**2**] for details). Also if $p \in V_{kl}$ then

$$L_\gamma(p(z)g(t + i|z|^2)) = 0$$

(for $(z, t) \in H_N$) if and only if

$$D_{\alpha+l,\beta+k}(g) = 0$$

(as in Section 1, $\alpha := (N-\gamma)/2$, $\beta := (N+\gamma)/2$). This result is due to Greiner [**10**] for $N = 1$ and was extended to $N \geqq 2$ in [**2**], and in another way by Korányi [**16**].

Since the Poisson integral for $D_{\alpha\beta}$ acts on the strip we will adopt a polar-strip coordinate system for $H_N \backslash \{(0, t): t \geqq 1 \text{ or } t \leqq -1\}$. Let

$$S_N := \left\{ z \in \mathbf{C}^N: \sum_{j=1}^{N} |z_j|^2 = 1 \right\}$$

and let

$$E := \{\sigma + i\tau \in \mathbf{C}: \sigma \in \mathbf{R}, 0 \leqq \tau < \pi/2\},$$

then map $S_N \times E$ into H_N by

$$(z, \sigma, \tau) \to (z(\operatorname{Im} \operatorname{th}(\sigma + i\tau))^{1/2}, \operatorname{Re} \operatorname{th}(\sigma + i\tau)).$$

The points with $0 \leqq \tau < \pi/4$ correspond to B (the unit ball), and those with $\tau = \pi/4$ to $\partial B \backslash \{ (0, \pm 1) \}$.

Proposition 1.3 in this context asserts that for $p \in V_{kl}$ the function

$$(z, \sigma, \tau) \mapsto p(z \operatorname{sgn}(\operatorname{ch}(\sigma + i\tau)))$$
$$\times (\sin 2\tau)^{(k+l)/2}(\operatorname{ch}(\sigma - i\tau))^{\alpha}(\operatorname{ch}(\sigma + i\tau))^{\beta} g(\sigma, \tau)$$

is L_{γ}-harmonic if and only if

$$D'_{\alpha+l,\beta+k} g = 0.$$

The calculation, as well as others in the sequel, use the identity

$$(\operatorname{Im} \operatorname{th}(\sigma + i\tau))^{(k+l)/2}(\operatorname{ch}(\sigma - i\tau))^{l}(\operatorname{ch}(\sigma + i\tau))^{k}$$
$$= (1/2 \sin 2\tau)^{(k+l)/2} \bar{\xi}^{l} \xi^{k},$$

where

$$\xi = \operatorname{sgn}(\operatorname{ch}(\sigma + i\tau)) \quad \text{and}$$
$$\operatorname{sgn}(\omega) := \omega/|\omega| \quad \text{for } \omega \in \mathbf{C}\backslash\{0\}.$$

We will find the Poisson integral for functions of this type defined on ∂B, $(\tau = \pi/4)$.

To use harmonic analysis on $U(N)$ let m denote the normalized $U(N)$-invariant measure on S_N, and let $\{\psi_j : 1 \leqq j \leqq \dim V_{kl}\}$ be an orthogonal basis for V_{kl} with

$$\int_{S_N} \psi_j(z)\overline{\psi_n(z)} dm(z) = \delta_{jn}/(\dim V_{kl}).$$

We consider functions on ∂B of the form

$$(5.2) \quad f(z, w) = \sum_j (\dim V_{kl}) \psi_j(z)(\operatorname{Im} \operatorname{th}(w + i\pi/4))^{(k+l)/2} f_j(w);$$

note

$$f_j(w) = (\operatorname{Im} \operatorname{th}(w + i\pi/4))^{-(k+l)/2} \int_{S_N} f(z, w)\overline{\psi_j(z)} dm(z).$$

5.3 *Definition.*

$$P_{\alpha\beta,kl}[f](z, \sigma, \tau) := (\operatorname{ch}(\sigma - i\tau))^{\alpha}$$
$$\times (\operatorname{ch}(\sigma + i\tau))^{\beta} \int_{S_N} \int_{\mathbf{R}} f(z', w)$$
$$\times (\operatorname{ch}(w - i\pi/4))^{-\alpha}$$
$$\times (\operatorname{ch}(w + i\pi/4))^{-\beta} \cdot K_{\alpha+l,\beta+k}(\sigma - w, \tau)$$
$$\times R_{kl}^{(N-2)}(\operatorname{sgn}(\operatorname{ch}(\sigma + i\tau)/\operatorname{ch}(w + i\pi/4))$$
$$\times \langle z, z' \rangle) dw dm(z').$$

5.4 THEOREM.

$$L_\gamma P_{\alpha\beta,kl}[f] = 0 \quad \textit{and}$$

$$P_{\alpha\beta,kl}[f](z, \sigma, \tau) \to f(z, \sigma) \quad \textit{as } \tau \to (\pi/4)_-$$

is an L^2-sense (*namely, for* $(\text{ch } 2\sigma)^{-N} d\sigma dm$), *for f of the form* (5.2).

Proof. Convergence follows from properties of $P_{\alpha+l,\beta+k}$ (Theorem 3.4) applied to V_{kl}-valued functions on the strip (and V_{kl} is finite-dimensional). So it suffices to show that the definition actually implements the Poisson integral from 3.3. Indeed applying $P_{\alpha+l,\beta+k}$ to the given f (as in 5.2) we obtain

$$\begin{aligned}
&(\text{ch}(\sigma - i\tau))^\alpha(\text{ch}(\sigma + i\tau))^\beta(\sin 2\tau)^{(k+l)/2} \\
&\times \sum_j (\dim V_{kl})\psi_j(\text{sgn}(\text{ch}(\sigma + i\tau))z) \\
&\times \int_{S_N}\int_{\mathbf{R}} f(z', w)(\text{ch}(w - i\pi/4))^{-\alpha}(\text{ch}(w + i\pi/4))^{-\beta} \\
&\times \overline{\psi}_j(\text{sgn}(\text{ch}(w + i\pi/4))z')K_{\alpha+l,\beta+k}(\sigma - w, \tau)dwdm(z').
\end{aligned}$$

(Note we are using the homogeneity properties of $\psi_j \in V_{kl}$.) Combine this with the spherical function identity

$$\sum_j (\dim V_{kl})\psi_j(z)\overline{\psi}_j(z') = R_{kl}^{(N-2)}(\langle z, z'\rangle)$$

to establish the formula of 5.3.

For $N = 1$ these formulas still hold, but only V_{0l}, V_{k0} occur, and

$$R_{k0}^{(-1)}(\omega) = \omega^k, \quad R_{0l}^{(-1)}(\omega) = \overline{\omega}^l.$$

To define the Poisson integral for all measurable functions on ∂B satisfying

$$\int_{S_N}\int_{\mathbf{R}} |f(z, w)|^2(\text{ch } 2w)^{-N}dwdm(z) < \infty,$$

we would need to bound

$$\sup_{y\in\mathbf{R}}(\sin 2\tau)^{(k+l)/2}(k_{\alpha+l,\beta+k}(y, \tau)/k_{\alpha+l,\beta+k}(y, \pi/4))$$

over all k, l; each $\tau < \pi/4$. This remains to be done.

6. Further problems.

6.1. Complex values of α, β. We mean the situation $\alpha + \beta = 2\nu > 0$ but α, β allowed to be complex. The original definition (1.4) for $k_{\alpha\beta}(y, \tau)$ is meaningful for all $\alpha \in \mathbf{C}$, but with restricted τ, while the formula of Theorem 1.5 requires $\text{Re } \alpha > 0$ and $\text{Re } \beta > 0$. We can, however, use Proposition 4.1 as a definition for all $\alpha \in \mathbf{C}$:

$$(6.2)\quad k_{\alpha\beta}(y,\tau) = 2^{1-2\nu}\Gamma(2\nu)(\Gamma(\nu + iy/2)\Gamma(\nu - iy/2))^{-1} \times \int_{\mathbf{R}} e^{-iy\sigma}(\mathrm{ch}(\sigma - i\tau))^{-\alpha}(\mathrm{ch}(\sigma + i\tau))^{-\beta}d\sigma, \quad (-\pi/4 \leqq \tau \leqq \pi/4).$$

We will show that $k_{\alpha\beta}(y, \pi/4)$ has real zeros when $\alpha < 0$ or $\beta < 0$ so that the Dirichlet problem cannot be solved in general. However

$$\{h_{\alpha\beta}C_n^{(\alpha,\beta)}:n \geqq 0\}$$

is still dense in L^2 (in other words, density of boundary values of L_γ-harmonics does not imply extendibility).

Here is a sketch of the density argument: for $\alpha \in \mathbf{C}$ and $g \in L^2(\mathbf{R})$ such that

$$\int_{\mathbf{R}} g(-\sigma)h_{\alpha\beta}(\sigma, \pi/4)C_n^{(\alpha,\beta)}(\mathrm{th}(\sigma + i\pi/4))d\sigma = 0$$

for each n we deduce that

$$\int_{\mathbf{R}} g(-\sigma)h_{\alpha\beta}(\sigma - w, \pi/4)d\sigma = 0$$

for all $w \in \mathbf{R}$ by using Proposition 4.4 and the bound

$$|C_n^{(\alpha,\beta)}(\zeta)| \leqq |\zeta|^n(|\alpha| + |\beta|)_n/n!;$$

but then

$$\hat{g}(y)\hat{h}_{\alpha\beta}(y, \pi/4) = 0$$

and $\hat{h}_{\alpha\beta}$ is analytic in a neighborhood of $\mathbf{R}$, thus $\hat{g} = 0$.

First we consider the exceptional (non-hypoelliptic) values of γ, namely $\pm(N + 2m)$, $m = 0, 1, 2, \ldots$. Then α or $\beta = -m$. Because of the symmetry

$$k_{\beta\alpha}(y, \tau) = k_{\alpha\beta}(-y, \tau)$$

we will discuss only $\alpha = -m$. From (1.4) we see that

$$k_{\alpha\beta}(y, \tau) = e^{-y\tau}p(y, \tau)$$

where p is a polynomial of degree m in y so that

$$k_{\alpha\beta}(y, \tau) \rightarrow 0 \quad \text{as } y \rightarrow +\infty.$$

In fact we can show

$$k_{\alpha\beta}(y, \pi/4) = (-1)^m e^{-\pi y/4}\left(\frac{m!}{(2\nu)_m}\right)p_m(y; 2\nu);$$

and this has m real zeros because p_m is one of a family of orthogonal polynomials.

We now proceed to non-integral values of $\alpha < 0$. For this we establish a three-term recurrence for $k_{\alpha\beta}$.

6.3 PROPOSITION. *For any α, β with $\alpha + \beta = 2\nu > 0$, $y \in \mathbf{C}$, $-\pi/4 \leqq \tau \leqq \pi/4$,*

$$\beta k_{\alpha-1,\beta+1}(y, \tau) - \alpha k_{\alpha+1,\beta-1}(y, \tau)$$
$$= ((\beta - \alpha)\cos 2\tau - y \sin 2\tau)k_{\alpha\beta}(y, \tau).$$

Proof. It suffices to establish this identity for $\hat{h}_{\alpha\beta}$ by Proposition 4.1. The identity follows from

$$y\hat{h}_{\alpha\beta}(y, \tau) = -i \int_{\mathbf{R}} e^{-iy\sigma}\frac{\partial}{\partial\sigma} (h_{\alpha\beta}(\sigma, \tau))d\sigma.$$

6.4 THEOREM. *For α, $\beta \in \mathbf{R}$, $\alpha + \beta = 2\nu \geqq 1$,*

$$k_{\alpha\beta}(y, \pi/4) \sim \frac{\Gamma(2\nu)}{\Gamma(\alpha)}y^{-\beta}e^{\pi y/4} \quad \text{as } y \to +\infty;$$

$$k_{\alpha\beta}(y, \pi/4) \sim \frac{\Gamma(2\nu)}{\Gamma(\beta)}|y|^{-\alpha}e^{-\pi y/4} \quad \text{as } y \to -\infty.$$

Proof. This has been established in 2.6 for $\alpha > 0$, and $\beta > 0$, and for α or $\beta = 0, -1, -2, \dots$ at the beginning of this section. We introduce auxiliary functions for $\epsilon = 1$ or -1 by

$$f_{\alpha\beta,\epsilon}(y) := \epsilon B(\alpha, \beta)^{-1} \int_0^{\epsilon\pi/4} e^{yt}\left(\sin\left(\frac{\pi}{4} + t\right)\right)^{\alpha-1}$$
$$\times \left(\sin\left(\frac{\pi}{4} - t\right)\right)^{\beta-1} dt.$$

The integral is valid when $\beta > 0$ and $\epsilon = 1$, or $\alpha > 0$ and $\epsilon = -1$; but we use the identity

$$k_{\alpha\beta}(y, \pi/4) = f_{\alpha\beta,1}(y) + f_{\alpha\beta,-1}(y)$$

to define $f_{\alpha\beta,\epsilon}$ for all values of α, β satisfying $\alpha + \beta > 0$, (at least one of the integrals must work). We claim for $\epsilon = \pm 1$ and $\alpha + \beta \geqq 1$, $\alpha \in \mathbf{R}$, that

$$(6.5) \quad \beta f_{\alpha-1,\beta+1,\epsilon}(y) - \alpha f_{\alpha+1,\beta-1,\epsilon}(y) + yf_{\alpha\beta,\epsilon}(y) = -\epsilon 2^{1-\nu}/B(\alpha, \beta).$$

Indeed one of $\alpha \geqq 1$ or $\beta \geqq 1$ must hold, say $\beta \geqq 1$, then the identity can be verified for $\epsilon = 1$ by integration by parts. Then (6.5) must hold for $\epsilon = -1$ by using 6.3 with $\tau = \pi/4$.

Now by Watson's lemma

$$f_{\alpha\beta,1}(y) \sim -\frac{\Gamma(2\nu)}{\Gamma(\alpha)} y^{-\beta}e^{\pi y/4}$$

as $y \to +\infty$, when $\beta > 0$. Also

$$f_{\alpha\beta,-1}(y) = O(1) \quad \text{as } y \to \infty \text{ when } \alpha > 0.$$

We prove the theorem for the values $\alpha = \alpha_0 - n$, $n = 1, 2, 3, \dots$, for fixed α_0 with $0 < \alpha_0 < 1$. Then

$$k_{\alpha_0 - n,\beta_0 + n}(y, \pi/4) = f_{\alpha_0 - n,\beta_0 + n,1}(y) + f_{\alpha_0 - n,\beta_0 + n,-1}(y),$$

and

$$f_{\alpha_0 - n,\beta_0 + n,1}(y) \sim \frac{\Gamma(2\nu)}{\Gamma(\alpha_0 - n)} y^{-\beta_0 - n} e^{\pi y/4}$$

as $y \to +\infty$. We claim

$$f_{\alpha_0 - n,\beta_0 + n,-1}(y) = O(y^n)$$

as $y \to +\infty$. We prove this inductively from the recurrence (6.5) which starts with $f_{\alpha_0,\beta_0,-1}(y)$ and $f_{\alpha_0+1,\beta_0-1,-1}(y)$ both of which are $O(1)$ as $y \to \infty$.

For $y \to -\infty$ we will again use the recurrence starting with $k_{\alpha_0\beta_0}$ and k_{α_0+1,β_0-1}. Both of these satisfy the prescribed asymptotic relationship; if $\beta_0 < 1$ then use the above argument with α, β reversed. Inductively we show that the dominant term as $y \to -\infty$ in $k_{\alpha_0 - n,\beta_0 + n}(y, \pi/4)$ is

$$\frac{(-y)^n}{(\beta_0)_n} \frac{\Gamma(2\nu)}{\Gamma(\beta_0)} |y|^{-\alpha_0} e^{\pi|y|/4} = \frac{\Gamma(2\nu)}{\Gamma(\beta_0 + n)} |y|^{n-\alpha_0} e^{\pi|y|/4}.$$

6.6 THEOREM. *Let* $\alpha_0 + \beta_0 = 2\nu \geqq 1$ *and* $0 < \alpha_0 < 1$, *then*

$$k_{\alpha_0 - n,\beta_0 + n}(y, \pi/4)$$

has at least one real zero, $n = 1, 2, 3, \dots$.

Proof. Let

$$q_n(y) := k_{\alpha_0 - n,\beta_0 + n}(y, \pi/4).$$

By Theorem 6.5, $q_n(y) \to \infty$ as $y \to -\infty$ for $n = 1, 2, \dots$. Further

$$q_1(y) \sim (\alpha_0 - 1) \frac{\Gamma(2\nu)}{\Gamma(\alpha_0)} y^{-\beta_0 - 1} e^{\pi y/4}$$

as $y \to +\infty$, that is $q_1(y) \to -\infty$. We inductively construct a real sequence $\{y_n\}$ such that

$$q_n(y_n) = 0, \; y_{n+1} < y_n, \quad \text{and}$$

$$y_n = \inf\{y \in \mathbf{R} : q_n(y) = 0\}.$$

Since q_1 changes sign on $\mathbf{R}$ we assert the existence of y_1. Also $q_0(y) > 0$ for all $y \in \mathbf{R}$. Assuming that $y_1, \dots, y_m$ have been constructed the recurrence 6.3 shows that

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$$(\beta_0 + m)q_{m+1}(y_m) = (\alpha_0 - m)q_{m-1}(y_m),$$

that is

$$\operatorname{sgn}(q_{m+1}(y_m)) = -\operatorname{sgn}(q_{m-1}(y_m)) \quad (m \geqq 1).$$

But by hypothesis $y_m < y_{m-1}$ and $q_{m-1}(y) \to +\infty$ as $y \to -\infty$ thus $q_{m-1}(y_m) > 0$, and $q_{m+1}(y_m) < 0$. Thus q_{m+1} has a sign change in $(-\infty, y_m)$.

6.7 COROLLARY. *If $N \geqq 1$ and $\gamma \leqq -N$ or $\gamma \geqq N$ then the Dirichlet problem for L_γ on the ball B in H_N can not be solved for all L^2 boundary values.*

Proof. For these parameter values $k_{\alpha\beta}(y, \pi/4)$ has real zeros (or $\to 0$ at $\pm\infty$ when $\alpha = 0$ or $\beta = 0$). Thus the Poisson kernel $K_{\alpha\beta}(\sigma, 0)$ can not exist, or else $1/k_{\alpha\beta}(y, \pi/4)$ would be a Fourier transform.

We will leave the problem of zeros of $K_{\alpha\beta}(y, \pi/4)$ for complex α open. Theorem 6.4 is still valid for $y \to \pm\infty$, but there are technical difficulties in bounding $1/k_{\alpha\beta}(y, \pi/4)$ above.

6.8 Convergence properties of $P_{\alpha\beta}$ for continuous functions. One needs to find bounds on

$$\int_{\mathbf{R}} |K_{\alpha\beta}(\sigma, \tau)|d\sigma \quad \text{in } 0 \leqq \tau < \pi/4;$$

one would expect

$$\int_{\mathbf{R}} |K_{\alpha\beta}(\sigma, \tau)|d\sigma \leqq \int_{\mathbf{R}} |K_{\alpha\beta}(\sigma, 0)|d\sigma$$

in many cases; (guess: $2\nu \geqq 1$). Such bounds should lead to a maximum principle; technical difficulties comes from the singularity of $D_{\alpha\beta}$ on $\tau = 0$, and the unboundedness of the strip.

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